Gravitational self-force effects on orbits around a non-rotating black hole

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- The Lorenz gauge versus the "true harmonic" gauge.

- Gauge invariant quantities for describing "circular" geodesics in $g^{\mathsf{Schw}}_{ab} + h_{ab}$.

- Actual results for gravitational self-force effects on circular orbits in the Schwarzschild geometry.

- A scheme for doing second order metric perturbations with a point mass (small, black hole) source.

The true harmonic gauge versus the Lorenz gauge

With the Schwarzschild geometry there are four different scalar fields

$$T = t, \quad X = (r_{\mathsf{Schw}} - m)\sin\theta\,\cos\phi, \quad Y = (r_{\mathsf{Schw}} - m)\sin\theta\,\sin\phi, \quad Z = (r_{\mathsf{Schw}} - m)\cos\theta,$$

for which

$$\nabla^a\nabla_a T = \nabla^a\nabla_a X = \nabla^a\nabla_a Y = \nabla^a\nabla_a Z = 0.$$

These four scalar fields may be used for *harmonic* coordinates, but for the time being we continue to use Schwarzschild coordinates. We consider an arbitrary perturbation of the metric and seek a gauge transformation which results in these same scalar fields being harmonic functions in the perturbed metric $g_{ab} + h_{ab}$.

This is surprisingly simple to do. The condition that a scalar field X be harmonic in $g_{ab} + h_{ab}$ is that $\nabla^a_{(g+h)}\nabla_a^{(g+h)} X = 0$. Three other, similar equations for T, Y and Z should also hold. Together, these give us the four gauge conditions for the *true-harmonic* gauge.

With no details being given, we define $\bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h$, where $h = g^{ab} h_{ab}$.

The true-harmonic gauge condition for the perturbed geometry is then

$$\nabla_a (\bar{h}^{ab} \nabla_b X) = 0.$$

Note that the gauge condition is *covariant* and distinct from what is usually described as the *Lorenz* gauge condition, $\nabla_a \bar{h}^{ab} = 0$.

Under a gauge transformation,

$$\bar{h}^{ab}_{\text{new}} = \bar{h}^{ab}_{\text{old}} - \nabla^a \xi^b - \nabla^b \xi^a + g^{ab} \nabla^c \xi_c.$$

Thus, with an initial metric perturbation h^{old}_{ab}, the gauge vector ξ^a to transform to the true-harmonic gauge must satisfy

$$\nabla_a (\bar{h}^{ab}_{\text{old}} \nabla_b X) = \left(\nabla_a \nabla^a \xi^b + R^b{}_c \xi^c \right) \nabla_b X + 2 (\nabla^a \xi^b) \nabla_a \nabla_b X,$$

as well as three other similar equations for T, Y, and Z.

Gauge Invariance

"The perturbation in some quantity is the difference between the value it has at a point in the physical (perturbed) spacetime and the value at the *corresponding point* in the background spacetime. A gauge transformation induces a coordinate transformation in the physical spacetime, but it also changes the point in the background spacetime corresponding to a given point in the physical spacetime. Thus, even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will not be invariant under gauge transformations if the quantity is nonzero and position dependent in the background spacetime." (Bardeen 1980)

A gauge transformation is a small change in coordinates, $x^a_{\text{new}} = x^a + \xi^a$, with $\xi^a = O(\mu)$ which changes the metric perturbation, $h^{\text{new}}_{ab} = h_{ab} - 2\nabla_{(a}\xi_{b)} + O(\mu^2)$. But, for an arbitrary ξ^a, the tensor $2\nabla_{(a}\xi_{b)}$ is explicitly a homogeneous solution of the perturbed Einstein equations, and the perturbed Einstein tensor is therefore invariant under such a gauge transformation.

Gauge invariant quantities for "circular" geodesics in $g^{\mathsf{Schw}}_{ab} + h^{\mathsf{R}}_{ab}$

Self-force analysis implies that a point mass μ moves along a geodesic of the perturbed metric $g^{\mathsf{Schw}}_{ab} + h_{ab}$, where $h_{ab} \equiv h^{\mathsf{R}}_{ab}$ is $\mathscr{C}^1$. This geodesic equation is

$$\frac{du_a}{ds} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a}(g_{bc} + h_{bc})$$

Let $R(s)$ be r for the particle, and define

$$u_t = -E, \qquad u_\phi = J \qquad \text{and} \qquad u^r = \dot{R},$$
$$u^a = \left(\frac{E + u^b h_{tb}}{1 - 2M/r}, \dot{R}, 0, \frac{J - u^b h_{\phi b}}{r^2} \right),$$

E and J are similar to the particle's energy and angular momentum per unit rest mass.

The components of the geodesic equation

$$
\begin{aligned}
\frac{dE}{ds} &= -\frac{1}{2}u^a u^b \frac{\partial h_{ab}}{\partial t} \\
\frac{dJ}{ds} &= \frac{1}{2}u^a u^b \frac{\partial h_{ab}}{\partial \phi} \\
\frac{d}{ds}\left(\frac{r\dot{R}}{r-2M} + u^a h_{ar}\right) &= \frac{1}{2}u^a u^b \frac{\partial}{\partial r}(g_{ab} + h_{ab})
\end{aligned}
$$

Assume that $\ddot{R} = O(h^2)$ and that $\dot{R} = O(h)$ —this is consistent with quasi-circular evolution.
The normalization of u^a is a first integral of the geodesic equation,

$$
1 = \frac{E^2}{1-2M/r} - \frac{J^2}{r^2} + u^a u^b h_{ab}.
$$

Symmetries for "circular" orbits

Neither $\partial/\partial t$ nor $\partial/\partial\phi$ is a Killing vector of $g_{ab}+h_{ab}$, but the combination, $k^a\frac{\partial}{\partial x^a} = \partial/\partial t + \Omega\partial/\partial\phi$ is a Killing vector, $\mathscr{L}_k h_{ab} = 0$, and u^a is tangent to a trajectory of k^a. Thus, at a "circular" orbit $u^a\partial_a h_{bc} = 0$ in Schwarzschild coordinates.

The "circular" orbits of the perturbed geometry are obtained from the r-component of the geodesic equation, and the normalization condition, and the facts that

$$\dot{E} \sim O(h) \quad \dot{J} \sim O(h)$$

A gauge transformation

Only a gauge transformation in the radial coordinate $r_{\text{new}} = r_{\text{old}} + \xi^r$ induces a change in

$$\Delta(u^a u^b \partial_r h_{ab})_\mu = -\frac{6M}{r^2(r-3M)}\xi^r$$

evaluated at the particle. Also, $u^a u^b h_{ab}|_\mu$ is invariant under any gauge transformation. These facts imply that the quantities below are gauge invariant.

Gauge invariant quantities

Consequences of the geodesic equation are

$$
\begin{aligned}
(u^t)^2 &= \left(\frac{dT}{ds}\right)^2 = \frac{(E + u^b h_{tb})^2}{(1-2M/r)^2} \\
&= \frac{r}{r-3M}\left(1 + u^a u^b h_{ab} - \frac{r}{2} u^a u^b \partial_r h_{ab}\right)
\end{aligned}
$$

$$
\begin{aligned}
(u^\phi)^2 &= \left(\frac{d\Phi}{ds}\right)^2 = \frac{1}{r^4}(J - u^b h_{\phi b})^2 \\
&= \frac{r-2M}{r(r-3M)}\left[\frac{M(1+u^a u^b h_{ab})}{r(r-2M)} - \frac{1}{2} r u^a u^b \partial_r h_{ab}\right]
\end{aligned}
$$

$$
\Omega^2 = \left(\frac{u^\phi}{u^t}\right)^2 = \frac{M}{r^3} - \frac{r-3M}{2r^2} u^a u^b \partial_r h_{ab}
$$

$$
\frac{dE}{dt} = -\frac{1}{2}\sqrt{1-3M/r}\; u^a u^b \partial_t h_{ab} \qquad \frac{dJ}{dt} = \frac{1}{2}\sqrt{1-3M/r}\; u^a u^b \partial_\phi h_{ab}
$$

Consider

$$
\begin{aligned}
(E-\Omega J)\frac{dT}{ds} &= E\frac{dT}{ds} - J\frac{d\Phi}{ds} \\
&= \frac{E(E+u^b h_{tb})}{1-2M/r} - \frac{J}{r^2}(J - u^b h_{\phi b}) \\
&= \frac{E^2}{1-2M/r} - \frac{J}{r^2} + \frac{E u^b h_{tb}}{1-2M/r} + \frac{J}{r^2} u^b h_{\phi b} \\
&= 1 - u^a u^b h_{ab} + u^a u^b h_{ab} = 1
\end{aligned}
$$

Therefore

$$
k^a u_a = E - \Omega J = (dT/ds)^{-1} = (u^t)^{-1}
$$

is also gauge invariant.

Physical interpretation of the gauge invariant u^t

Let a light source be near the small mass μ. Let the tangent vector to an affinely parameterized null geodesic of a photon from this light source be v^a. The energy $\mathcal{E}_{\mathsf{em}}$ of the photon, as emitted near μ, is proportional to $u^a v_a$, so the ratio of the energies as measured by an observer and as emitted is

$$\frac{\mathcal{E}_{\mathsf{ob}}}{\mathcal{E}_{\mathsf{em}}} = \frac{u^a v_a|_{\mathsf{ob}}}{u^a v_a|_{\mathsf{em}}}$$

With k^a a Killing vector field, $k^a v_a$ is constant along the path of the photon. At emission, $u^a_{\mathsf{em}} \propto k^a$ so that $u^a_{\mathsf{em}} = u^t k^a|_{\mathsf{em}}$. Let the photon be observed at a large distance away from the black hole along the z-axis.

It follows that

$$\frac{\mathcal{E}_{\rm ob}}{\mathcal{E}_{\rm em}} = \frac{u^a v_a|_{\rm ob}}{u^a v_a|_{\rm em}}, \quad \text{with } u^t_\infty = 1 \text{ this becomes}$$

$$= \frac{v_t^\infty}{u^t (k^a v_a)_{\rm em}} = \frac{v_t^\infty}{u^t (k^a v_a)^\infty}, \quad \text{because } k^a u_a = \text{constant along the geodesic,}$$

$$= \frac{v_t^\infty}{u^t (v_t^\infty + \Omega v_\phi^\infty)} = \frac{1}{u^t} - \frac{\Omega v_\phi^\infty}{u^t (v_t^\infty + \Omega v_\phi^\infty)}$$

$$= \frac{1}{u^t}, \quad \text{because } v_\phi^\infty = 0 \text{ at a large distance along the } z\text{-axis.}$$

Thus, the gauge invariant $u^t = 1/(E - \Omega J)$ gives the redshift of a photon, emitted from μ, when the photon is observed on the z-axis at a large distance.

A surprising (to me) fact

- An arbitrary metric perturbation of a spherically symmetric background spacetime retains some residual spherical symmetry:

- The perturbed spacetime may be foliated by a *unique* family $\{\Sigma\}$ of two-spheres, which are individually spherically symmetric, even while the spacetime as a whole is not. There is a gauge with $h_{\theta\theta} = h_{\theta\phi} = h_{\phi\phi} = 0$.

- The existence of the two-spheres in the perturbed geometry, permits the geometrical definition of a scalar field R from the area of each Σ.

- The gauge where $r_{\mathsf{schw}} = \mathsf{R}$ is called the *Easy Gauge.* The metric perturbations in the EZ gauge may be interpreted as being "gauge invariant" in the same manner that Moncrief showed that the Regge-Wheeler-gauge metric perturbations could be described as being "gauge invariant." The relationship between the EZ gauge variables and the Regge-Wheeler-Moncrief variables is not just algebraic, but also involves differentiation. As a result, the form of the perturbed Einstein equations in the EZ gauge differs from that in the Regge-Wheeler gauge. There are geometrical interpretations of all of the EZ gauge-invariant scalars in terms of the lapse and shift using a foliation of the 3-geometry in terms of $\{\Sigma\}$.

Definitions of two "radial R" quantities

The orbital frequency of a circular, Newtonian binary of masses M and μ is

$$\Omega^2 = \frac{M+\mu}{r^3}$$

where r is the *separation* between M and μ. For a general-relativistic, extreme mass ratio binary we define R_Ω by

$$\Omega^2 = \frac{M+\mu}{R_\Omega^3} \quad \text{defines } R_\Omega.$$

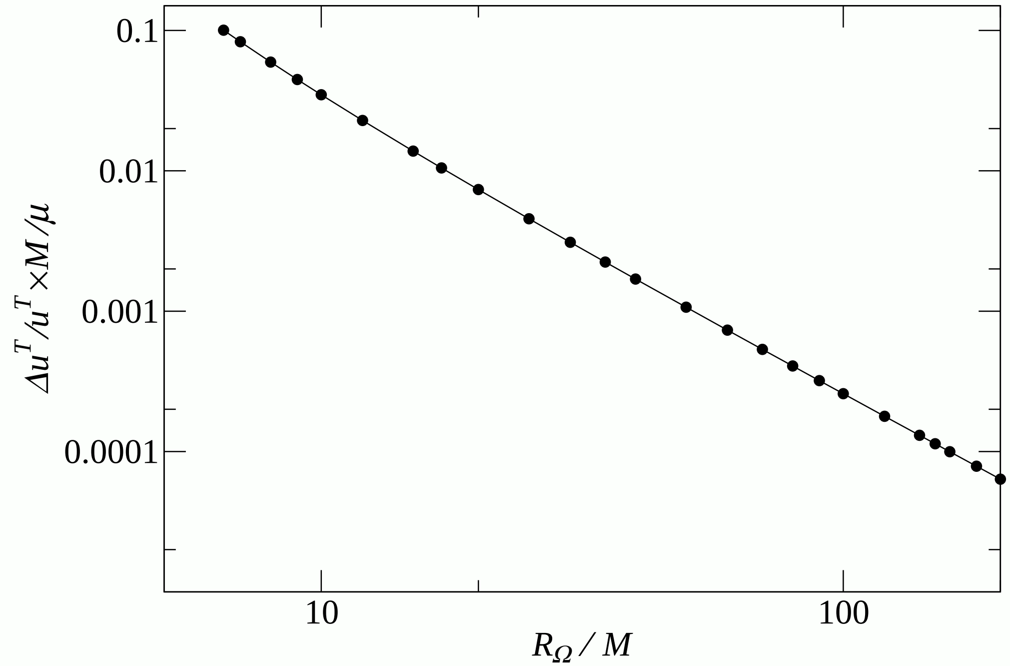
In a circular Newtonian binary, the radius of the orbit of μ is

$$\text{distance to center of mass} = \text{separation}/(1+\mu/M).$$

In the extreme mass ratio limit, this becomes

$$\text{distance to center of mass} = R_\Omega(1-\mu/M).$$

An actual well-defined consequence of the gravitational self-force



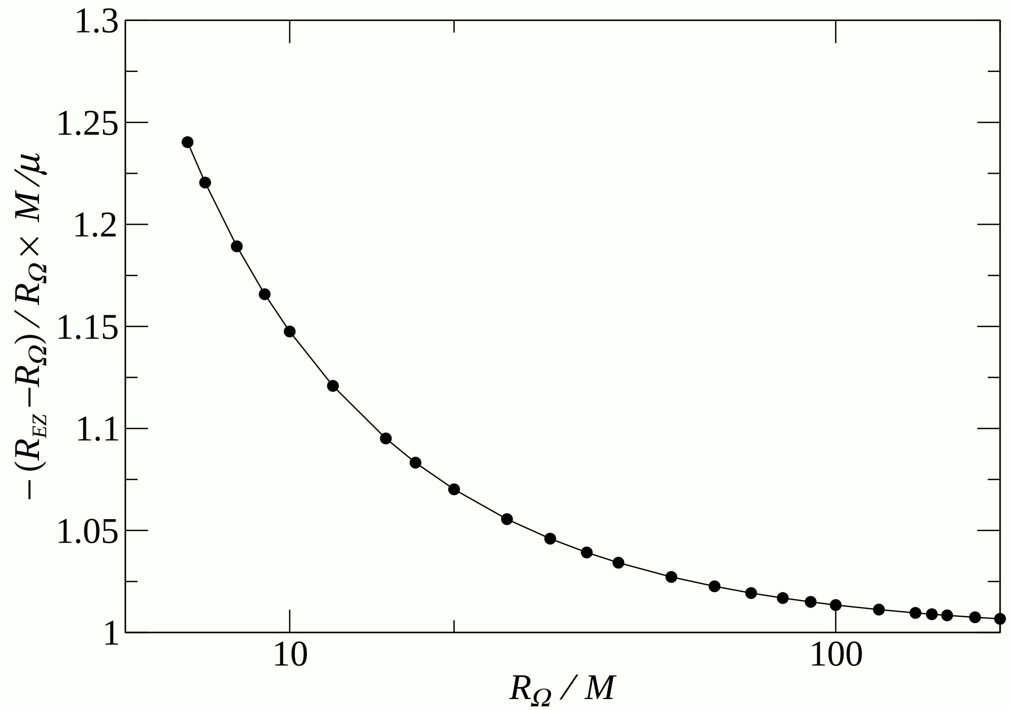
- The redshift of a photon from μ is

$$\frac{\mathscr{E}_{\mathsf{ob}}}{\mathscr{E}_{\mathsf{em}}} = \frac{1}{u^t}$$

when observed at a large distance along the z-axis.

- I have not yet calculated this redshift in the post-Newtonian approximation.

A second well-defined consequence of the gravitational self-force



- At large R_Ω, the areal radius of the geometrical two-spheres

$$R_{\mathrm{EZ}} \approx R_\Omega(1 - \mu/M) = \text{Newtonian distance to the center of mass}$$

- R_{EZ} does not (yet?) appear to be physically observable.
- I have not yet calculated R_{EZ} in the post-Newtonian approximation.

Second order perturbation theory with a point mass, schematically

Define the parts of the Einstein tensor of various orders in h by

$$G(g+h) = G^{(1)}(g,h) + G^{(2)}(g,h) + G^{(3)}(g,h) + \dots$$

$G^{(1)}(g,h)$ looks like a wave operator on h; $G^{(2)}(g,h)$ looks like "$\nabla h \nabla h$" or "$h \nabla\nabla h$".

At second order solve

$$G^{(1)}(g,h) + G^{(2)}(g,h) = 8\pi T \quad \text{or} \quad G^{(1)}(g,h^{\mathsf{R}}+h^{\mathsf{S}}) + G^{(2)}(g,h^{\mathsf{R}}+h^{\mathsf{S}}) = 8\pi T$$

by using

$$G^{(1)}(g,h^{\mathsf{R}}) = -G^{(2)}(g,h^{\mathsf{R}}) - [G^{(1)}(g+h^{\mathsf{R}},h^{\mathsf{S}}) - G^{(2)}(g,h^{\mathsf{S}}) + 8\pi T]$$

If we know h^{S} well enough then

$$[G^{(1)}(g+h^{\mathsf{R}},h^{\mathsf{S}}) - G^{(2)}(g,h^{\mathsf{S}}) + 8\pi T] = \mathscr{O}(\mu r/\mathscr{R}^4) = \mathscr{C}^0$$

- The numerical solution for h^{R} will be $\mathscr{C}^2$
- Except for being continuous but non-differentiable at the point mass, the source is relatively smooth.

$$
\begin{array}{ccccccccccccc}
g & \sim & & & & & & & & & & & \\
\eta & \& & 0 & \& & {}_2H' & \& & {}_3H' & \& & {}_4H' & \& & \cdots & = g^0 + h_{\mathsf{R}} \\
\mu/r & \& & \mu/\mathscr{R} & \& & \mu r/\mathscr{R}^2 & \& & \mu r^2/\mathscr{R}^3 & \& & \mu r^3/\mathscr{R}^4 & \& & \cdots & = h_{\mathsf{S}}^{\mu} \\
\mu^2/r^2 & \& & \mu^2/r\mathscr{R} & \& & \mu^2/\mathscr{R}^2 & \& & \mu^2 r/\mathscr{R}^3 & \& & \mu^2 r^2/\mathscr{R}^4 & \& & \cdots & = h_{\mathsf{S}}^{\mu^2} \\
\mu^3/r^3 & \& & \mu^3/r^2\mathscr{R} & \& & \mu^3/r\mathscr{R}^2 & \& & \mu^3/\mathscr{R}^3 & \& & \mu^3 r/\mathscr{R}^4 & \& & \cdots & = h_{\mathsf{S}}^{\mu^3} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & & \\
\hline
g^{\mathsf{Schw}} & & 0 & & {}_2h' & & {}_3h' & & {}_4h' & & & &
\end{array}
\qquad (1)
$$

- The point mass moves along a geodesic of $g + h^{\mathsf{R}}$

- At higher order, as long as the motion is geodesic in $g + h^{\mathsf{R}}$, the formulation of higher order perturbation theory is relatively straightforward.

Conclusions

- Other conservative self-force effects will be studied for slightly non-circular orbits of Schwarzschild. These include the self-force effects on the precession of the perihelion of an orbit, and on the orbital frequency of the innermost stable circular orbit.

- There seems to be no fundamental difficulty in doing second order perturbation theory.

- Second order calculations will certainly be able to provide improved wave-forms for LISA and also be able to test convergence of the post-Newtonian approximation.