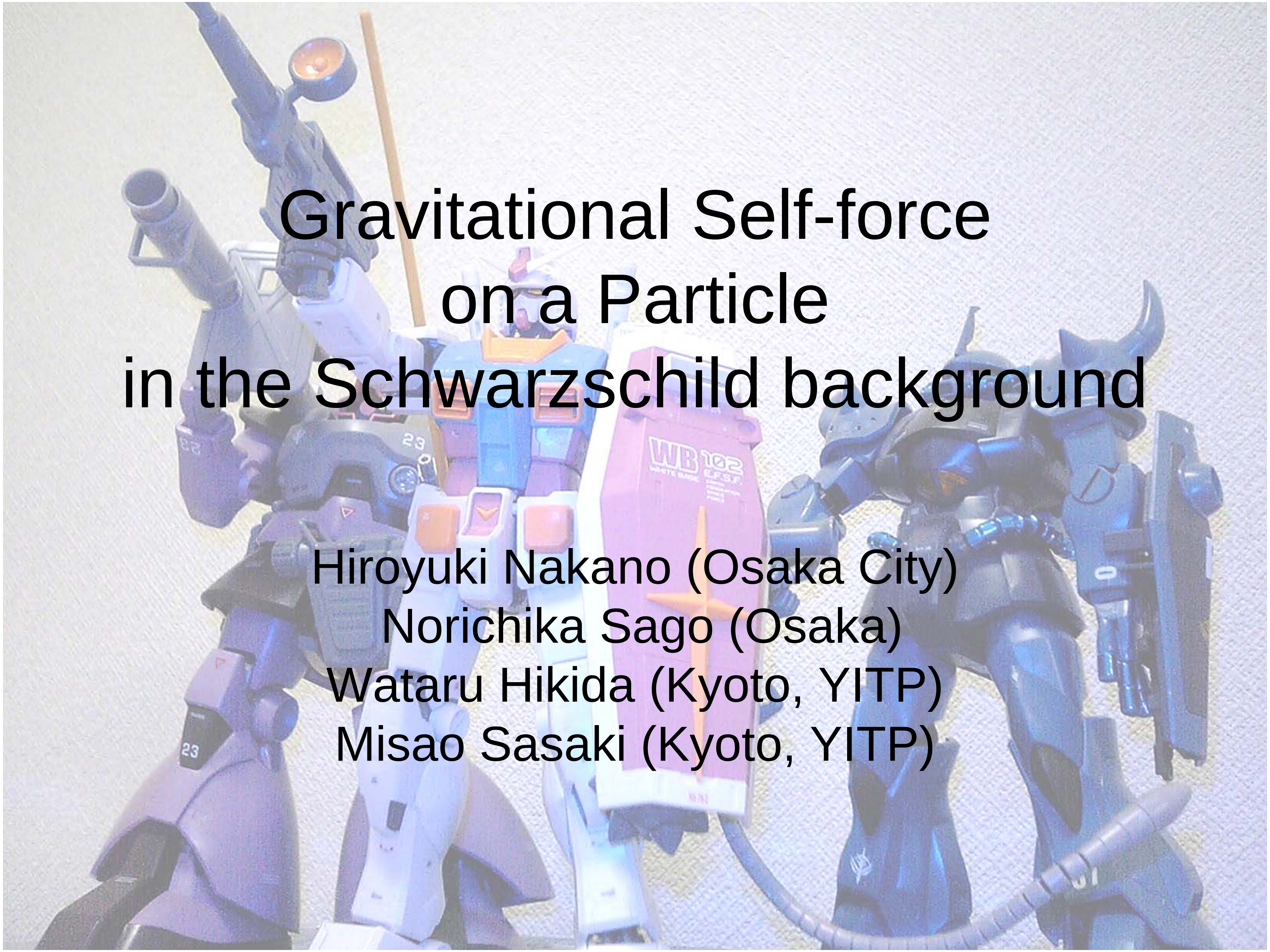

Gravitational Self-force on a Particle in the Schwarzschild background

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1. Introduction

Laser Interferometer Space Antenna (LISA)
One of the most promising wave source
Super massive black hole - Compact object binaries

Black hole perturbation:
Black hole background (mass: M)
+ Perturbation [point particle] μ
(mass:)

$$g_{\mu\nu} = g^{(\mathrm{b})}_{\mu\nu} + h^{\mathrm{full}}_{\mu\nu} \,.$$

We want to know the precise particle motion
which includes the self-force in a black hole space-time.

Point particle $\longrightarrow$ Self-force diverge $\longrightarrow$ Need regularization

MiSaTaQuWa force

Mino, Sasaki and Tanaka ('97), Quinn and Wald ('97)
Detweiler and Whiting ('03) under Lorenz (L) gauge

$$h_{\mu\nu}^{\mathrm{full,L}} = h_{\mu\nu}^{\mathrm{S(dir),L}} + h_{\mu\nu}^{\mathrm{R(tail),L}} ,$$

Regularized gravitational self-force (reaction force)

$$\mu \frac{D^2 z^\mu(\tau)}{d\tau^2} = F^\mu(z)$$

$$F^\mu = -\frac{\mu}{2}(g_{(\mathrm{b})}^{\mu\nu} + u^\mu u^\nu)\left(2h_{\nu\beta;\alpha}^{\mathrm{R,L}} - h_{\alpha\beta;\nu}^{\mathrm{R,L}}\right) u^\alpha u^\beta ,$$

$\{u^\alpha\}$: Four velocity of a particle

``R-part'': Homogeneous solution of linearized Einstein equation,
Depend on the history and the global structure of a space-time.
It is difficult to obtain this directly.

Regularization

Regularization: Subtract the singular part.

$$h_{\mu\nu}^{\mathrm{R,L}} = h_{\mu\nu}^{\mathrm{full,L}} - h_{\mu\nu}^{\mathrm{S,L}} \,.$$

\`\`S-part'': Possible to calculate around the particle location.
(under the Lorenz gauge condition)

\`\`Full'': Regge-Wheeler-Zerilli formalism, Teukolsky formalism
(not the Lorenz gauge)

1) Subtraction problem: How do we subtract singular part?
We use the spherical harmonics expansion.

2) Gauge problem: Do we treat the gauge difference?
We consider an appropriate gauge transformation.

Strategy

* Schwarzschild background

Regge-Wheeler-Zerilli formalism for full metric perturbation

Sec. 2. Solution of the gauge problem: Finite gauge transformation

Sec. 3. Standard form for the regularization parameters (Singular part)

Sec. 4. New analytic regularization ($\tilde{S} + \tilde{R}$ decomposition)
[Hikida's talk]
$\longrightarrow$ Metric perturbation, Self-force

Sec. 5. Regularized self-force

Sec. 6. Discussion

2. Gauge Problem

We consider the Schwarzschild background.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Full metric perturbation is calculated by

Regge-Wheeler-Zerilli formalism
under the Regge-Wheeler gauge condition.

We want to consider the regularization under the RW gauge analytically.

* R-part of the metric perturbation is the (homogeneous) solution
  of the linearized Einstein equation.

Finite gauge transformation: Gauge transformation for R-part

$$x_\mu^{\mathrm{L}} \rightarrow x_\mu^{\mathrm{RW}} = x_\mu^{\mathrm{L}} + \xi_\mu^{\mathrm{L}\rightarrow\mathrm{RW}}[h_{\alpha\beta}^{\mathrm{R,L}}]$$

Finite gauge transformation

Gauge transformation for R-part: $$\boxed{x_\mu^{\rm L} \;\rightarrow\; x_\mu^{\rm RW} = x_\mu^{\rm L} + \xi_\mu^{\rm L\rightarrow RW}\left[h_{\alpha\beta}^{\rm R,L}\right]}$$

*** We can define the (regularized) self-force under the RW gauge.**

$$\boxed{\begin{aligned}
F_\alpha^{\rm RW}(\tau) &= \lim_{x\rightarrow z(\tau)} F_\alpha\left[h^{\rm R,RW}\right] \\
&= \lim_{x\rightarrow z(\tau)} F_\alpha\left[h^{\rm R,L} - 2\,\nabla\xi^{\rm L\rightarrow RW}\left[h^{\rm R,L}\right]\right](x) \\
&= \lim_{x\rightarrow z(\tau)} F_\alpha\left[h^{\rm full,L} - h^{\rm S,L} - 2\,\nabla\xi^{\rm L\rightarrow RW}\left[h^{\rm full,L} - h^{\rm S,L}\right]\right](x) \\
&= \lim_{x\rightarrow z(\tau)} F_\alpha\left[h^{\rm full,L} - 2\,\nabla\xi^{\rm L\rightarrow RW}\left[h^{\rm full,L}\right]\right. \\
&\qquad\qquad \left. - h^{\rm S,L} + 2\,\nabla\xi^{\rm L\rightarrow RW}\left[h^{\rm S,L}\right]\right](x) \\
&= \lim_{x\rightarrow z(\tau)} \Big(F_\alpha\left[h^{\rm full,RW}\right](x) \\
&\qquad\qquad - F_\alpha\left[h^{\rm S,L} - 2\,\nabla\xi^{\rm L\rightarrow RW}\left[h^{\rm S,L}\right]\right](x)\Big)\,,
\end{aligned}}$$

3. Singular Part

3-1. S-part under the L gauge

S-part of the metric perturbation: calculated around the particle location.

$$\bar{h}^{\mathrm{S,L}}_{\mu\nu}(x) \;=\; 4\mu\left[\frac{\bar{g}_{\mu\alpha}(x,z_{\mathrm{ret}})\bar{g}_{\nu\beta}(x,z_{\mathrm{ret}})u^{\alpha}(\tau_{\mathrm{ret}})u^{\beta}(\tau_{\mathrm{ret}})}{\sigma_{;\gamma}(x,z_{\mathrm{ret}})u^{\gamma}(\tau_{\mathrm{ret}})}\right]$$
$$+2\mu(\tau_{\mathrm{adv}}-\tau_{\mathrm{ret}})\bar{g}_{\mu}{}^{\alpha}(x,z_{\mathrm{ret}})\bar{g}_{\nu}{}^{\beta}(x,z_{\mathrm{ret}})R_{\gamma\alpha\delta\beta}(z_{\mathrm{ret}})u^{\gamma}(\tau_{\mathrm{ret}})u^{\delta}(\tau_{\mathrm{ret}})$$
$$+O(y^2)\,,$$

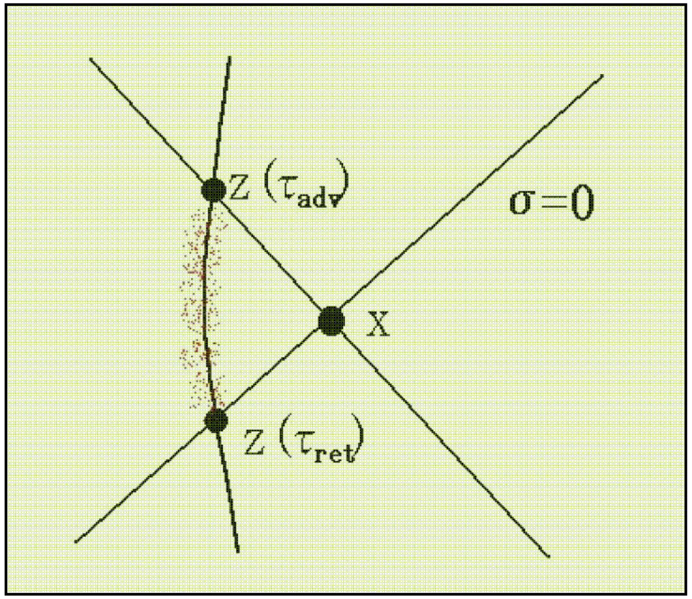
$\sigma(x,z)$: Bi-scalar of half the squared geodesic distance

$\bar{g}_{\alpha\beta}(x,z)$: Parallel displacement bi-vector

$\tau_{\mathrm{ret}}(x)$: Retarded time for x

y : Coordinate difference between x and z_0 (small)

z_0: Location of the particle



Local coordinate expansion

Metric component: around the particle location.

$$h^{\mathrm{S,H}}_{\alpha\beta} = \mu \sum_{m,n,p,q,r} C^{m,n,p,q,r}_{\alpha\beta} \frac{T^m R^n \Theta^p \Phi^q}{\epsilon^r} + O(y^2)$$

Small quantities:

$$\epsilon := \left(r_0^2 + r^2 - 2\, r_0\, r\, \cos\Theta \cos\Phi\right)^{1/2} ,$$
$$T := t - t_0 \,, \qquad R := r - r_0 \,,$$
$$\Theta := \theta - \frac{\pi}{2} \,, \qquad \Phi := \phi - \phi_0 \,.$$

Harmonics expansion: (Example)

$$\frac{1}{\epsilon} = \sum_{\ell m} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^{\ell} Y^*_{\ell m}(\Omega_0) Y_{\ell m}(\Omega)$$

Mode sum regularization

S-part of the self-force under the Lorenz gauge:

$$\boxed{F^{\mu(\pm)}_{\mathrm{S,L}}\Big|_{\ell} = \pm A^{\mu} L + B^{\mu} + D^{\mu}_{\ell}\,, \;\; L = \ell + \frac{1}{2}\,.}$$

A^{μ} -term: Quadratic divergence
B^{μ} -term: Linear divergence

Standard form

* These terms are independent of ℓ .

D^{μ}_{ℓ} -term: Remaining finite contribution

$$D^{\mu}_{\ell} = \frac{d^{\mu}}{L^2-1} + \frac{e^{\mu}}{(L^2-1)(L^2-4)} + \frac{f^{\mu}}{(L^2-1)(L^2-4)(L^2-9)} + \cdots .$$

* D^{μ}_{ℓ}-term vanishes after summing over ℓ modes from $\ell = 0$ to ∞ .

We can consider regularization if we don’t know the exact S-part.

Regularization under the L gauge

Regularization by using the S-force with the standard form:

$$
\begin{aligned}
F^{\mu}_{\mathrm{R,L}} &= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - F^{\mu}_{\mathrm{S,L}}\Big|_{\ell}\right) \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - A^{\mu}L - B^{\mu} - D^{\mu}_{\ell}\right) \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - A^{\mu}L - B^{\mu}\right) + \sum_{\ell=0,1} D^{\mu}_{\ell} \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - A^{\mu}L - B^{\mu}\right) + \sum_{\ell=0,1}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - A^{\mu}L - B^{\mu}\right) \\
&= \boxed{\sum_{\ell\geq 0}\left(F^{\mu}_{\mathrm{full,L}}\Big|_{\ell} - A^{\mu}L - B^{\mu}\right)}\,. \qquad (\text{ for } r > r_0 \text{)}
\end{aligned}
$$

The R-force is derived from the homogeneous metric perturbation.

* We need to calculate the $\ell = 0$ and 1 modes.

3-2. *About the RW gauge*

Regge-Wheeler gauge condition:

* Some coefficients of the tensor harmonics expansion = 0

$$\boxed{h_{2\ell m}^{\mathrm{S,RW}} = h_{0\ell m}^{(e)\mathrm{S,RW}} = h_{1\ell m}^{(e)\mathrm{S,RW}} = G_{\ell m}^{\mathrm{S,RW}} = 0\,.}$$

Tensor harmonics

$$\mathbf{a}_{\ell m}{}^{(0)} = \begin{pmatrix} Y_{\ell m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{a}_{\ell m}{}^{(1)} = (i/\sqrt{2}) \begin{pmatrix} 0 & Y_{\ell m} & 0 & 0 \\ Sym & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{a}_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{b}_{\ell m}{}^{(0)} = ir[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{b}_{\ell m} = r[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix},$$

$$\mathbf{c}_{\ell m}{}^{(0)} = r[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{c}_{\ell m} = ir[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix},$$

$$\mathbf{d}_{\ell m} = ir^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta)X_{\ell m} & \sin\theta W_{\ell m} \\ 0 & 0 & Sym & \sin\theta X_{\ell m} \end{pmatrix},$$

$$\mathbf{g}_{\ell m} = (r^2/\sqrt{2}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m} & 0 \\ 0 & 0 & 0 & \sin^2\theta Y_{\ell m} \end{pmatrix},$$

$$\mathbf{f}_{\ell m} = r^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{\ell m} & X_{\ell m} \\ 0 & 0 & Sym & -\sin^2\theta W_{\ell m} \end{pmatrix}.$$

Metric perturbation

$$\begin{aligned} \boldsymbol{h} = \sum_{\ell m} \Bigg[& f(r) H_{0\ell m}(t,r)\boldsymbol{a}_{\ell m}^{(0)} - i\sqrt{2} H_{1\ell m}(t,r)\boldsymbol{a}_{\ell m}^{(1)} + \frac{1}{f(r)} H_{2\ell m}(t,r)\boldsymbol{a}_{\ell m} \\ & - \frac{i}{r}\sqrt{2\ell(\ell+1)}\, h_{0\ell m}^{(e)}(t,r)\boldsymbol{b}_{\ell m}^{(0)} + \frac{1}{r}\sqrt{2\ell(\ell+1)}\, h_{1\ell m}^{(e)}(t,r)\boldsymbol{b}_{\ell m} \\ & + \sqrt{\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)}\, G_{\ell m}(t,r)\boldsymbol{f}_{\ell m} + \left(\sqrt{2}K_{\ell m}(t,r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}(t,r)\right)\boldsymbol{g}_{\ell m} \\ & - \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\ell m}(t,r)\boldsymbol{c}_{\ell m}^{(0)} + \frac{i\sqrt{2\ell(\ell+1)}}{r} h_{1\ell m}(t,r)\boldsymbol{c}_{\ell m} \\ & + \frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2} h_{2\ell m}(t,r)\boldsymbol{d}_{\ell m} \Bigg] \quad , \end{aligned}$$

f (r)= 1-2M/r

$$\begin{aligned} X_{\ell m} &= 2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta} - \cot\theta\right) Y_{\ell m}\,, \\ W_{\ell m} &= \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{\ell m}\,. \end{aligned}$$

Gauge transformation

Generator of a gauge transformation from the L gauge to the RW gauge:

$$\xi_\mu^{(\mathrm{odd})} = \sum_{\ell m} \Lambda_{\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)\left\{0, 0, \frac{-1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi), \sin\theta\partial_\theta Y_{\ell m}(\theta,\phi)\right\},$$

$$\xi_\mu^{(\mathrm{even})} = \sum_{\ell m}\Big\{M_{0\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)Y_{\ell m}(\theta,\phi), M_{1\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)Y_{\ell m}(\theta,\phi),$$

$$M_{2\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)\partial_\theta Y_{\ell m}(\theta,\phi), M_{2\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)\partial_\phi Y_{\ell m}(\theta,\phi)\Big\}.$$

$$\boxed{h_{2\ell m}^{\mathrm{S,RW}} = h_{0\ell m}^{(e)\mathrm{S,RW}} = h_{1\ell m}^{(e)\mathrm{S,RW}} = G_{\ell m}^{\mathrm{S,RW}} = 0\,.}$$

Solution:

$$\begin{aligned}
\Lambda_{\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r) &= \frac{i}{2}h_{2\ell m}^{\mathrm{S,L}}(t,\,r)\,,\\
M_{2\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r) &= -\frac{r^2}{2}G_{\ell m}^{\mathrm{S,L}}(t,\,r)\,,\\
M_{0\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r) &= -h_{0\ell m}^{(\mathrm{e})\mathrm{S,L}}(t,\,r) - \partial_t M_{2\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)\,,\\
M_{1\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r) &= -h_{1\ell m}^{(\mathrm{e})\mathrm{S,L}}(t,\,r) - r^2\,\partial_r\left(\frac{M_{2\ell m}^{\mathrm{S,L}\to\mathrm{RW}}(t,\,r)}{r^2}\right).
\end{aligned}$$

* It is not necessary to calculate any integration with respect to time or radial coordinates.

Gauge transformation (cont.)

odd:
$$\Lambda_{\ell m}^{\rm S,L\to RW}(t,\,r) = \frac{i}{2} h_{2\ell m}^{\rm S,L}(t,\,r)\,,$$

even:
$$\begin{aligned}
M_{2\ell m}^{\rm S,L\to RW}(t,\,r) &= -\frac{r^2}{2} G_{\ell m}^{\rm S,L}(t,\,r)\,,\\
M_{0\ell m}^{\rm S,L\to RW}(t,\,r) &= -h_{0\ell m}^{\rm (e)S,L}(t,\,r) - \partial_t M_{2\ell m}^{\rm S,L\to RW}(t,\,r)\,,\\
M_{1\ell m}^{\rm S,L\to RW}(t,\,r) &= -h_{1\ell m}^{\rm (e)S,L}(t,\,r) - r^2\,\partial_r \left(\frac{M_{2\ell m}^{\rm S,L\to RW}(t,\,r)}{r^2}\right).
\end{aligned}$$

Metric perturbation under the RW gauge:

odd:
$$\begin{aligned}
h_{0\ell m}^{\rm S,RW}(t,\,r) &= h_{0\ell m}^{\rm S,L}(t,\,r) + \partial_t\,\Lambda_{\ell m}^{\rm S,L\to RW}(t,\,r)\,,\\
h_{1\ell m}^{\rm S,RW}(t,\,r) &= h_{1\ell m}^{\rm S,L}(t,\,r) + r^2\,\partial_r \left(\frac{\Lambda_{\ell m}^{\rm S,L\to RW}(t,\,r)}{r^2}\right),
\end{aligned}$$

even:
$$\begin{aligned}
H_{0\ell m}^{\rm S,RW}(t,\,r) &= H_{0\ell m}^{\rm S,L}(t,\,r) + \frac{2\,r}{r-2\,M}\left[\partial_t\,M_{0\ell m}^{\rm S,L\to RW}(t,\,r) - \frac{M(r-2\,M)}{r^3} M_{1\ell m}^{\rm S,L\to RW}(t,\,r)\right],\\
H_{1\ell m}^{\rm S,RW}(t,\,r) &= H_{1\ell m}^{\rm S,L}(t,\,r) + \left[\partial_t\,M_{1\ell m}^{\rm S,L\to RW}(t,\,r) + \partial_r\,M_{0\ell m}^{\rm S,L\to RW}(t,\,r) - \frac{2\,M}{r(r-2\,M)} M_{0\ell m}^{\rm S,L\to RW}(t,\,r)\right],\\
H_{2\ell m}^{\rm S,RW}(t,\,r) &= H_{2\ell m}^{\rm S,L}(t,\,r) + \frac{2(r-2\,M)}{r}\left[\partial_r\,M_{1\ell m}^{\rm S,L\to RW}(t,\,r) + \frac{M}{r(r-2\,M)} M_{1\ell m}^{\rm S,L\to RW}(t,\,r)\right],\\
K_{\ell m}^{\rm S,RW}(t,\,r) &= K_{\ell m}^{\rm S,L}(t,\,r) + \frac{2(r-2\,M)}{r^2} M_{1\ell m}^{\rm S,L\to RW}(t,\,r)\,,
\end{aligned}$$

* The above calculation does not include factors $1/\ell(\ell+1)$ and $1/(\ell-1)\ell(\ell+1)(\ell+2)$.

[Interruption factor for the standard form]

Does the standard form recovers under the RW gauge?

Standard form: $$D_\ell^\mu = \frac{d^\mu}{(2\ell-1)(2\ell+3)} + \frac{e^\mu}{(2\ell-1)(2\ell+3)(2\ell-3)(2\ell+5)} + \cdots .$$

At first glance, (last year)
S-part of the metric perturbation under the L gauge:

$$\cancel{h_{0\ell m}^{(\mathrm{e})\mathrm{S,H}}(t,r) \sim \frac{1}{\ell(\ell+1)}},$$
$$\cancel{h_{1\ell m}^{(\mathrm{e})\mathrm{S,H}}(t,r) \sim \frac{1}{\ell(\ell+1)}},$$
$$\cancel{G_{\ell m}^{\mathrm{S,H}}(t,r) \sim \frac{1}{\ell(\ell+1)(\ell-1)(\ell+2)}}.$$

These factor does not vanish in the self-force calculation under the RW gauge.

$\longrightarrow$ D-term does not vanish?

Standard form recovers under the RW gauge.

Tensor harmonics expansion:

Spherical symmetry of the Schwarzschild space-time

e.g.) Even parity $(t\theta)$ -component of the metric perturbation $\longrightarrow$ $\partial_\theta Y_{\ell m}$

Tensor harmonics expansion of the S-part under the L gauge

$$\begin{aligned}\epsilon &= \left(r_0^2 + r^2 - 2\,r_0\,r\,\cos\Theta\cos\Phi\right)^{1/2} \\ &= \sum_{\ell m} [\mathrm{S.\,F.\ of\ } D\text{–term}](r) Y_{\ell m}(\theta,\,\phi) Y^*_{\ell m}(\pi/2,\,\phi_0)\,,\end{aligned}$$

Corresponding quantity

$$\partial_\theta \epsilon = \sum_{\ell m} [\mathrm{S.\,F.\ of\ } D\text{–term}](r) \partial_\theta Y_{\ell m}(\theta,\,\phi) Y^*_{\ell m}(\theta_0,\,\phi_0)\,.$$

$$h^{(\mathrm{even})}_{t\theta} = \sum_{\ell m} h^{(\mathrm{e})}_{0\ell m}(t,r) \partial_\theta Y_{\ell m}(\theta,\,\phi)\,.$$

This fact is trivial. (We can not obtain from the local quantity.)

* All components have no interruption factor ($1/\ell(\ell+1)$ etc.).

Regularization under the RW gauge

Standard form recovers under the RW gauge.

$\longrightarrow$ $D^{\mu}_{\ell,\mathrm{RW}}$-term vanishes after summing over ℓ modes.

(The S-force is obtained in the form of the local coordinate expansion.)

Same regularization procedure as the L gauge:

$$
\begin{aligned}
F^{\mu}_{\mathrm{R,RW}} &= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - F^{\mu}_{\mathrm{S,RW}}\Big|_{\ell}\right) \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - A^{\mu}_{\mathrm{RW}}L - B^{\mu}_{\mathrm{RW}} - D^{\mu}_{\ell,\mathrm{RW}}\right) \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - A^{\mu}_{\mathrm{RW}}L - B^{\mu}_{\mathrm{RW}}\right) + \sum_{\ell=0,1} D^{\mu}_{\ell,\mathrm{RW}} \\
&= \sum_{\ell\geq 2}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - A^{\mu}_{\mathrm{RW}}L - B^{\mu}_{\mathrm{RW}}\right) + \sum_{\ell=0,1}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - A^{\mu}_{\mathrm{RW}}L - B^{\mu}_{\mathrm{RW}}\right) \\
&= \sum_{\ell\geq 0}\left(F^{\mu}_{\mathrm{full,RW}}\Big|_{\ell} - A^{\mu}_{\mathrm{RW}}L - B^{\mu}_{\mathrm{RW}}\right) .
\end{aligned}
$$

* The regularization parameters for the $\ell \geq 2$ modes can be used for the $\ell = 0$ and 1 modes.

4. New Analytic Regularization

Hikida, Jhingan, Nakano, Sago, Sasaki and Tanaka ('04, '05)
[Hikida's talk]

Treatment of the Green function ($\tilde{S}+\tilde{R}$ decomposition)

$$\boxed{g^{\rm full}_{\ell m\omega}(r,r') = g^{\tilde{\rm S}}_{\ell m\omega}(r,r') + g^{\tilde{\rm R}}_{\ell m\omega}(r,r')\,,}$$

$\tilde{S}$-part force: Possible to calculate for general orbits analytically.
(if we use slow motion approximation)
* We can extract the S-part.

$\tilde{R}$-part force: Generally Need to numerical calculation.
* The ℓ mode convergence is good.

$$\begin{aligned} F^{\rm R}_\alpha &= F^{\rm full}_\alpha - F^{\rm S}_\alpha \\ &= (F^{\tilde{\rm S}}_\alpha - F^{\rm S}_\alpha) + \sum_{\ell=0}^{\ell_{\max}} F^{\tilde{\rm R}}_{\alpha\ell} \end{aligned}$$

Formulation for scalar case $\longrightarrow$ Apply to Regge-Wheeler-Zerilli function.

$\tilde{S}$*-part*

Regge-Wheeler-Zerilli formalism

$\longrightarrow$ Fourier component: include $\frac{1}{\omega}$

* Time integration is needed.

Improvement of Jhingan and Tanaka ('03)

The $\tilde{S}$- part of the Green function has only positive power of ω.

We can calculate the $\tilde{S}$ - part for general orbit.

* The $\tilde{R}$- part of the Green function has $\ln \omega$ terms.

Third post-Newtonian (3PN) order calculation

$\longrightarrow$ $\tilde{R}$- part is also written by the positive power of ω .

3PN order Calculation for general orbit

$\tilde{S}$-part force:

$$F_\ell^{t,\tilde{S}(-)} = \mu^2 u^r \Bigg[\frac{\ell}{r_0^2} + \frac{2\,(8\,\ell^3+8\,\ell^2-6\,\ell-15)\,M^2}{r_0^4\,(3+2\,\ell)\,(-1+2\,\ell)} - \frac{(\ell^2+\ell+3)\,\delta}{r_0^2\,(3+2\,\ell)\,(-1+2\,\ell)}$$
$$- \frac{(8\,\ell^3+15\,\ell^2+\ell+6)\,L_z^2}{2r_0^4\,(3+2\,\ell)\,(-1+2\,\ell)} + \frac{(8\,\ell^3+8\,\ell^2-6\,\ell-15)\,M}{r_0^3\,(3+2\,\ell)\,(-1+2\,\ell)}$$
$$- \frac{(32\,\ell^7+124\,\ell^6+36\,\ell^5+233\,\ell^4+784\,\ell^3-3447\,\ell^2-3942\,\ell+720)\,M\,L_z^2}{r_0^5\,(\ell+1)\,\ell\,(5+2\,\ell)\,(-1+2\,\ell)\,(-9+4\,\ell^2)}$$
$$- \frac{8\,(\ell^4+2\,\ell^3+4\,\ell^2+3\,\ell-45)\,\delta\,M}{r_0^3\,(5+2\,\ell)\,(-1+2\,\ell)\,(-9+4\,\ell^2)}$$
$$+ \frac{(128\,\ell^5+455\,\ell^4-50\,\ell^3+2965\,\ell^2+3702\,\ell-16020)\,L_z^4}{8\,r_0^6(-1+2\,\ell)\,(5+2\,\ell)\,(-9+4\,\ell^2)}$$
$$- \frac{3\,(\ell^4+2\,\ell^3+54\,\ell^2+53\,\ell-180)\,\delta\,L_z^2}{r_0^4(-1+2\,\ell)\,(5+2\,\ell)\,(-9+4\,\ell^2)} - \frac{3\,(\ell^4+2\,\ell^3+4\,\ell^2+3\,\ell-45)\,\delta^2}{r_0^2\,(5+2\,\ell)\,(-1+2\,\ell)\,(-9+4\,\ell^2)} \Bigg]$$

S-part force:

$$F_\ell^{t,S(-)} = \mu^2 u^r \Bigg[\left(\frac{L_z^4}{r_0^6} - \frac{2\,M\,L_z^2}{r_0^5} - \frac{L_z^2}{r_0^4} + \frac{1}{r_0^2} + \frac{2\,M}{r_0^3} + \frac{4\,M^2}{r_0^4} \right) L$$
$$- \frac{2\,M^2}{r_0^4} - \frac{3\,L_z^2}{8r_0^4} - \frac{3\,M\,L_z^2}{4\,r_0^5} + \frac{135\,L_z^4}{128r_0^6} - \frac{M}{r_0^3}$$
$$- \frac{1}{2\,r_0^2} - \frac{M\,\delta}{2\,r_0^3} - \frac{3\,L_z^2\,\delta}{16r_0^4} - \frac{3\,\delta^2}{16\,r_0^2} - \frac{\delta}{4\,r_0^2} \Bigg]$$

$$L_z = r_0^2 u^\phi$$
$$\delta = 1 - 1/E^2$$

5. Regularized gravitational self-force

3PN(v^6) order regularized gravitational self-force for general orbits:

(We consider only for the $\ell \geq 2$ modes.)

$$
\begin{aligned}
F^{t,R} &= F^{t,\tilde{S}-S} + \sum_\ell F_\ell^{t,\tilde{R}} \\
&= \frac{\mu^2}{r_0^2}\left[\frac{64\,M^3}{3\,r_0^3} + \frac{160\,M^2\,L_z^2}{3\,r_0^4} + \frac{256\,M^2\,\delta}{15\,r_0^2} - \frac{40\,M\,L_z^4}{r_0^5} + \frac{136\,M\,L_z^2\,\delta}{5\,r_0^3} + \frac{16\,M\,\delta^2}{5\,r_0}\right] \\
&\quad + \frac{\mu^2 u^r}{r_0^2}\left[-\frac{4\,M^2}{r_0^2} - \frac{11\,M\,L_z^2}{2\,r_0^3} - \frac{2\,M}{r_0} - \frac{M\,\delta}{r_0} + \frac{1095\,L_z^4}{64 r_0^4} - \frac{3\,L_z^2}{4 r_0^2} - \frac{51\,L_z^2\,\delta}{8 r_0^2}\right. \\
&\qquad \left. - \frac{\delta}{2} - \frac{3\,\delta^2}{8}\right],
\end{aligned}
$$

$$
\begin{aligned}
F^{r,R} &= F^{r,\tilde{S}-S} + \sum_\ell F_\ell^{r,\tilde{R}} \\
&= \frac{\mu^2}{r_0^2}\left[\frac{490\,M^3}{3\,r_0^3} - \frac{41\,M^3\,\pi^2}{4\,r_0^3} - \frac{68\,M^2\,L_z^2}{15\,r_0^4} + \frac{369\,L_z^2\,M^2\,\pi^2}{64\,r_0^4} + \frac{196\,M^2\,\delta}{3\,r_0^2}\right. \\
&\qquad - \frac{123\,\delta\,M^2\,\pi^2}{32\,r_0^2} - \frac{3647\,M\,L_z^4}{120 r_0^5} - \frac{15\,M\,L_z^2}{r_0^3} + \frac{186\,M\,L_z^2\,\delta}{5\,r_0^3} - \frac{2\,M}{r_0} - \frac{3\,M\,\delta^2}{4\,r_0} \\
&\qquad - \frac{M\,\delta}{r_0} - \frac{27\,L_z^2\,\delta}{4 r_0^2} - \frac{39\,L_z^2\,\delta^2}{4 r_0^2} + \frac{651\,L_z^4}{64 r_0^4} - \frac{\delta}{2} - \frac{5\,\delta^2}{8} - \frac{7065\,L_z^6}{256 r_0^6} \\
&\qquad \left. - \frac{11\,\delta^3}{16} + \frac{3\,L_z^2}{4 r_0^2} + \frac{1703\,\delta\,L_z^4}{64 r_0^4}\right] \\
&\quad + \frac{\mu^2 u^r}{r_0^2}\left[\frac{32\,M^2}{3\,r_0^2} + \frac{16\,M\,L_z^2}{r_0^3} + \frac{16\,M\,\delta}{5\,r_0}\right].
\end{aligned}
$$

Circular limit

Energy and angular momentum of a point particle

$$E = \frac{1-2M/r_0}{\sqrt{1-3M/r_0}}$$
$$L = \sqrt{\frac{M r_0}{1-3M/r_0}}$$

Contribution of only the $\ell \geq 2$ modes

$$F^{t,R} = -\frac{32\,\mu^2\,M^3}{5\,r_0^5},$$
$$F^{r,R} = -\frac{3\,\mu^2\,M}{4\,r_0^3}\left[1 - \frac{97\,M}{16\,r_0} + \frac{(164\,\pi^2 - 739)\,M^2}{192\,r_0^2}\right].$$

* We can systematically calculate higher PN order.
(same as the scalar case, 18PN or higher?)

$$\boxed{\omega = m\sqrt{\frac{M}{r_0^3}}}$$

6. Discussion

Gravitational self-force on a particle
around the Schwarzschild black hole (for general orbits)

Subtraction problem:

Spherical harmonics expansion

Mode sum regularization + New analytic regularization

($\tilde{S} + \tilde{R}$ decomposition)

Gauge problem:

Full metric perturbation can be obtained the RW gauge.

Gauge transformation for the R-part

$\longrightarrow$ Finite gauge transformation

Standard form recovers under the RW gauge.

To complete this self-force, we need the $\ell = 0$ and 1 modes.

* $\ell = 0$ and 1 odd modes satisfy the RW gauge automatically.

An appropriate choice of gauge

Full metric perturbation under the Zerilli gauge

$\longrightarrow$ Retarded causal boundary condition in the L gauge

Detweiler and Poisson ('04) Low multipole contribution in the L gauge

$\ell = 0$ mode

$\ell = 1$ odd mode $\longrightarrow$ analytically calculated

$\ell = 1$ even mode $\longrightarrow$ need numerical calculation

* S-part of the self-force (regularization parameter) for the $\ell \geq 2$ modes can be used.

Transform to the RW gauge and subtract the S-part.

Next step

* Higher order PN calculation. (Next Capra meeting by Hikida?)

  Self-force $\longrightarrow$ dissipative / conservative part

  Adiabatic radiation reaction [Sago's talk]

* Regularization for the master variable.
  (Regge-Wheeler-Zerilli function, Teukolsky function)

* Extend to Kerr background.

* Second order metric perturbation and derive the wave form.
  (gauge invariant)

  Compare and combine with the standard PN method.