Mapping spacetimes with LISA: EMRIs in a ‘quasi-Kerr’ field

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Outline of this talk:

- Motivation
- Non-Kerr metrics
- Building a ‘quasi-Kerr’ metric
- Point particle orbital motion
- Approximate waveforms
- Conclusions and future challenges

Motivation

- "*Mapping of Kerr geometry, holiodesy, etc*" : central points in every single review talk on LISA
- What does this actually mean, and how can we flesh out a "spacetime-mapping" program for LISA ?
- EMRIs: a unique probe for such an experiment. Observations by LISA could:

  (i) Provide evidence in favour of GR's 'no-hair' theorem (uniqueness of the Kerr metric)

  (ii) Reveal the true identity of the 'dark objects' in galactic nuclei (Kerr holes *vs* Boson stars, Gravastars ....)
- LISA data analysis plans should be prepared for the possibility of non-Kerr EMRIs

Non-Kerr metrics

- The metric exterior to *any* stationary and axisymmetric 'source' can be written in terms of mass and current multipole moments M_ℓ, S_ℓ. Symbolically:

$$g_{ab}^{axi/stat} \sim \sum_{n=0}^{\infty} \frac{M_{2n}}{r^{2n+1}} \mathcal{P}_1(\theta), \quad \sum_{n=1}^{\infty} \frac{S_{2n-1}}{r^{2n+1}} \mathcal{P}_2(\theta)$$

- Unless information on the nature of the source is provided, M_ℓ, S_ℓ are free parameters. For a Kerr BH, the multipole moments are locked to the lowest two, mass $M \equiv M_0$ and spin $J \equiv S_1 = aM$:

$$M_\ell + iS_\ell = M(ia)^\ell, \quad \ell = 0, 1, 2, 3, \ \ldots$$

- 'Mapping' spacetimes with LISA means to construct EMRIs waveforms which will take into account a non-Kerr multipolar structure.
- Previous work: F. Ryan (late 90s) provided basic estimates on the precision with which LISA will be able to extract the first few moments. Measuring just the first three moments M, J and M_2 is sufficient for identifying a non-Kerr object. Measuring higher moments could lead to a full identification !

Non-Kerr EMRI: not a milk run !

- Despite the elegance and generality of the $g_{ab}^{axi/stat}$ metric, there is a major problem: the spacetime is no longer Petrov-type $\mathcal{D}$ (unlike Kerr).

- As a consequence, all the 'miracles' of the Kerr spacetime are lost:
  (i) The Hamilton-Jacobi equation for point-particle motion is separable with respect to t and ϕ coordinates only $\Rightarrow E, L_z$ are still there, but the "third" integral of motion (Carter constant in Kerr) is lost. $\Rightarrow$ *complicates orbital motion, possibility of chaotic behaviour ...*
  (ii) Cannot formulate a Teukolsky-like wave equation since the NP perturbative equations for the Weyl ψ scalars do not decouple anymore.

- As an alternative one would need to solve 10 PDEs for direct metric perturbations. In the meantime a cheap approximate solution is to construct 'kludge' waveforms.

- The expansion in multipole moments is also an expansion in $1/r$ i.e. a weak-field expansion. To probe strong-field orbits, we need to include several multipoles.
  In addition taking the Kerr limit $M_\ell, S_\ell \rightarrow M_\ell^{\mathrm{Kerr}}, S_\ell^{\mathrm{Kerr}}$ of $g_{ab}^{axi/stat}$ does not lead to the Kerr metric in familiar Boyer-Lindquist coordinates.

A less detailed 'map': the quasi-Kerr metric

- Basic idea: as we are almost certain for the existence of Kerr BHs then the simplest approach to a non-Kerr program would be to assume a *slight* deviation from the Kerr metric. In other words, a 'quasi-Kerr' metric would have:

$$M,J, \quad M_\ell = M_\ell^{\rm Kerr} + \delta M_\ell, \quad S_\ell = S_\ell^{\rm Kerr} + \delta S_\ell$$

- We only consider the deviation in the quadrupole moment:

$$\delta M_2 = -\epsilon M^3, \quad \delta M_\ell = \delta S_\ell = 0 \quad \text{for} \quad \ell \geq 3$$

- Previous work: Collins & Hughes (2004) built a 'bumpy' Schwarzschild metric by adding a small amount of quadrupole moment in the form of (i) a pair of masses at the hole's north and south poles or (ii) a ring of matter on the equator. Then, they studied equatorial motion in this metric and computed periastron shift:

$$\Delta\phi \approx -\frac{3\pi\mu b^2}{Mp^2}$$

Building a 'quasi-Kerr' metric (I)

- A quasi-Kerr metric would look like:

$$g_{ab}^{qK} = g_{ab}^{\rm Kerr} + \epsilon\, h_{ab}(r,\theta) + \mathcal{O}(\delta M_{\ell\geq 4}, \delta S_{\ell\geq 3})$$

- We would also like to recover the Boyer-Lindquist Kerr metric at the $\epsilon \to 0$ limit $\Rightarrow$ enjoy separability of particle/wave equations.
- The objective is to find $h_{ab}(r,\theta)$. This can be easily achieved by using the exterior Hartle-Thorne (HT) metric:
  This metric describes the spacetime outside *any* axisymmetric-stationary body up to $\mathcal{O}(J^2)$ accuracy. Hence, this metric is fully accurate up to the first three moments M, J, Q.
  For the choice $Q = Q_{\rm Kerr} = -a^2 M$ it reduces to the $\mathcal{O}(a^2)$ Kerr metric.
- Write: $Q = Q_{\rm Kerr} - \epsilon M^3$, where ϵ is taken as the small parameter.
- Split the HT metric in the following manner:

$$g_{ab}^{HT} = g_{ab}^{HT,a^2 Kerr} + \epsilon h_{ab}^{HT} + \mathcal{O}(J^{3+})$$

Building a 'quasi-Kerr' metric (II)

- Key point: original H-T coordinates lead to *nonseparable* particle/wave equations even at the Kerr limit $\Rightarrow$ shift to familiar Boyer-Lindquist coordinates.
- Then the resulting $h_{ab}^{HT} = h_{ab}$
- The final form of our quasi-Kerr metric is:

$$g_{ab}^{qK} = g_{ab}^{\mathrm{Kerr}} + \epsilon\, h_{ab} + \mathcal{O}(\delta M_{\ell\geq 4}, \delta S_{\ell \geq 3})$$

with

$$h^{tt} = [\,f_3(r) + f_4(r)\cos^2\theta]/(1-2M/r), \quad h^{rr} = (1-2M/r)[\,f_3(r) + f_4(r)\cos^2\theta]$$

$$h^{\theta\theta} = -[\,h_3(r) + h_4(r)\cos^2\theta]/r^2, \quad h^{\phi\phi} = -[\,h_3(r) + h_4(r)\cos^2\theta]/(r^2\,\sin^2\theta)$$

$$h^{t\phi} = 0$$

- This metric is not a $1/r$ expansion $\Rightarrow$ suitable for strong field computations.
- Strictly speaking, g_{ab}^{qK} is not type-$\mathcal{D}$, but still close to be !

Hamilton-Jacobi theory in the Kerr field

- Point particle Hamiltonian: $H_0(x^a, p_b) = \frac{1}{2}\, g^{\mu\nu} p_\mu\, p_\nu$ with $p^a = dx^a/d\lambda$. The H-J equation is:

$$\frac{1}{2}\, g^{ab}\, \frac{\partial S}{\partial x^a}\, \frac{\partial S}{\partial x^b} + \frac{\partial S}{\partial \lambda} = 0$$

- Kerr metric in BL coordinates: H-J equation is fully separable
$\Rightarrow S = \frac{1}{2}\, \mu^2\, \lambda - E\, t + L_z\, \phi + S_r(r) + S_\theta(\theta)$
- Equations of motion:

$$p^a = g^{ab}\, \frac{\partial S}{\partial x^b} \quad \Rightarrow \left\{ \begin{array}{lcl} p^t &=& -g^{tt}\, E + g^{t\phi}\, L_z = V_t \\ p^r &=& \pm g^{rr}\, \Delta^{-1}\, \sqrt{R} = \pm\Sigma^{-1}\, \sqrt{R} \\ p^\theta &=& \pm\, g^{\theta\theta}\, \sqrt{\Theta} = \pm\Sigma^{-1}\, \sqrt{\Theta} \\ p^\phi &=& g^{\phi\phi}\, L_z - E\, g^{t\phi} = V_\phi \end{array} \right.$$

- Separation constants: $\alpha_k = \{-\mu^2/2, E, Q, L_z\}$.

Action-Angle variables

- Canonical transformation $\{x^a, p_b\} \rightarrow \{\beta_a, \gamma_k\}$ where $\gamma_k = \{p_t, J_i\} =$ constant
- For the actions we have:

$$J_a = \oint p_a \, dq^a \Rightarrow \begin{cases} J_r = 2 \int_{r_p}^{r_a} dr \sqrt{R}/\Delta \\ J_\theta = 2 \int_{\theta_n}^{\theta_s} d\theta \sqrt{\Theta} \\ J_\phi = 2\,\pi\, L_z \end{cases}$$

- And for the (constant) conjugate coordinates we get,

$$\beta^a = \frac{\partial S}{\partial \gamma_a} \quad \Rightarrow \quad \beta^a + \nu^a \lambda = \frac{\partial W}{\partial \gamma_a} = \frac{\partial W}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial \gamma_a} \equiv w^a, \quad W(x^a, \alpha_k) \equiv S - \lambda/2$$

  These are the equations of motion in the integrated form. Differentiating these takes us back to the previous form of equations of motion.
- Orbital frequencies:

$$M\Omega_i = \nu_i/\nu_t, \quad i = \{r, \theta, \phi\}$$

Hamilton-Jacobi theory in the quasi-Kerr metric

- Solve the H-J equation by expanding :

$$S_{r,\theta} = S_{r,\theta}^{\rm Kerr} + \epsilon\, S_{r,\theta}^{(1)} + \mathcal{O}(\epsilon^2)$$

  We get:

$$2\,r(r-2M)\,\frac{dS_r^{\rm Kerr}}{dr}\,\frac{dS_r^{(1)}}{dr} + f_3\, r\,(r-2M)\left(\frac{dS_r^{\rm Kerr}}{dr}\right)^2 + \frac{E^2\, r^3 f_3}{r-2M} - K\, h_3 =$$

$$-2\,\frac{dS_\theta^{\rm Kerr}}{d\theta}\,\frac{dS_\theta^{(1)}}{d\theta} + \cos^2\theta\, Z(r)$$

- $\Rightarrow$ non-separable at $\mathcal{O}(\epsilon)$.
- For equatorial motion, H-J equation is trivially separable and this is the case we study first.

Equatorial motion

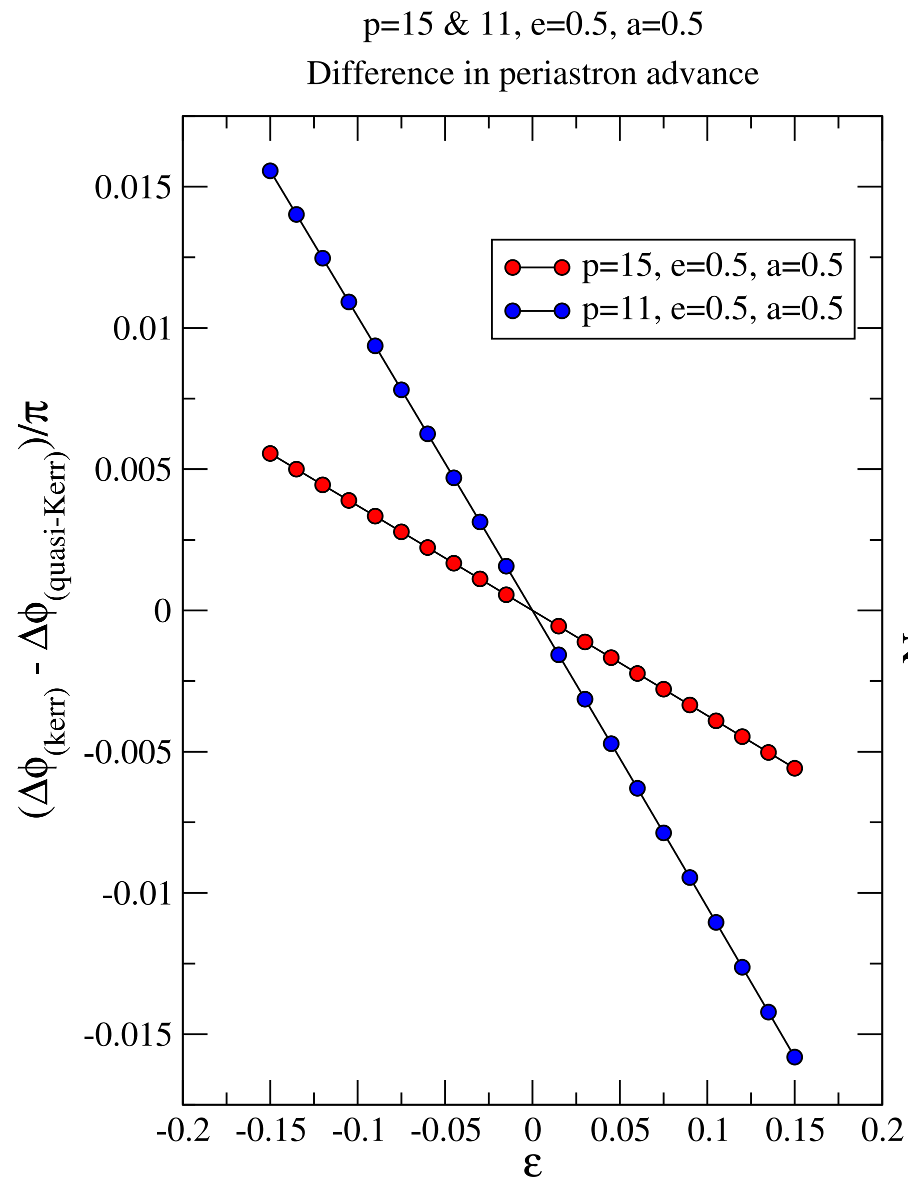
- Equatorial equations:

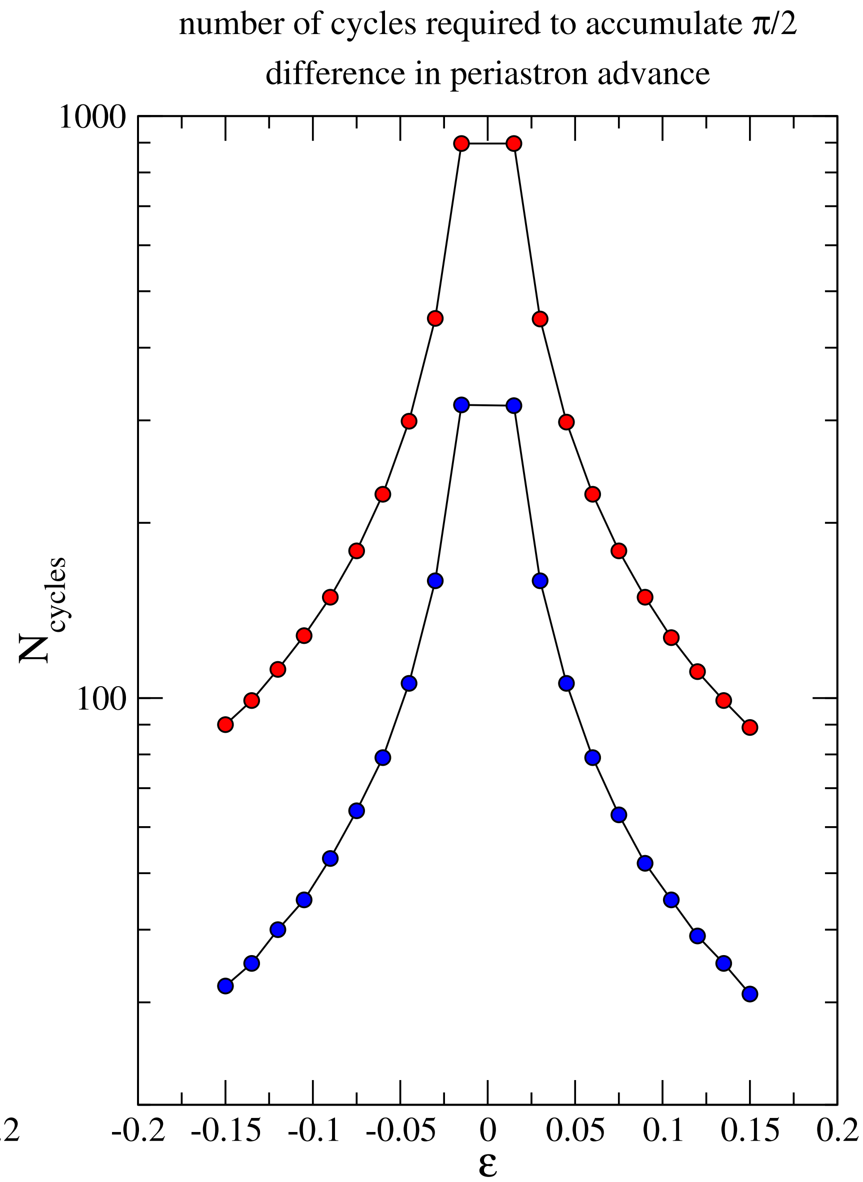
$$
\begin{aligned}
(u^r)^2 &= \left(\frac{dr}{d\tau}\right)^2 = (E^2-1) + \frac{2M}{r} - [L^2 + a^2(1-E^2)]\frac{1}{r^2} + \frac{2M}{r^3}(L-aE)^2 \\
&\quad -\epsilon\left(1-\frac{2M}{r}\right)\left[(f_3-h_3)\frac{L^2}{r^2} + f_3\right] \\
u^\phi &= \frac{d\phi}{d\tau} = \frac{1}{\Delta}\left[\frac{2M}{r}(aE-L) + L\right] - \epsilon\frac{h_3}{r^2} \\
u^t &= \frac{dt}{d\tau} = \frac{1}{\Delta}\left[E(r^2+a^2) + \frac{2Ma}{r}(Ea-L)\right] - \epsilon\left(1-\frac{2M}{r}\right)^{-1} f_3 E
\end{aligned}
$$

- Parameterise the orbit in the usual Keplerian manner:

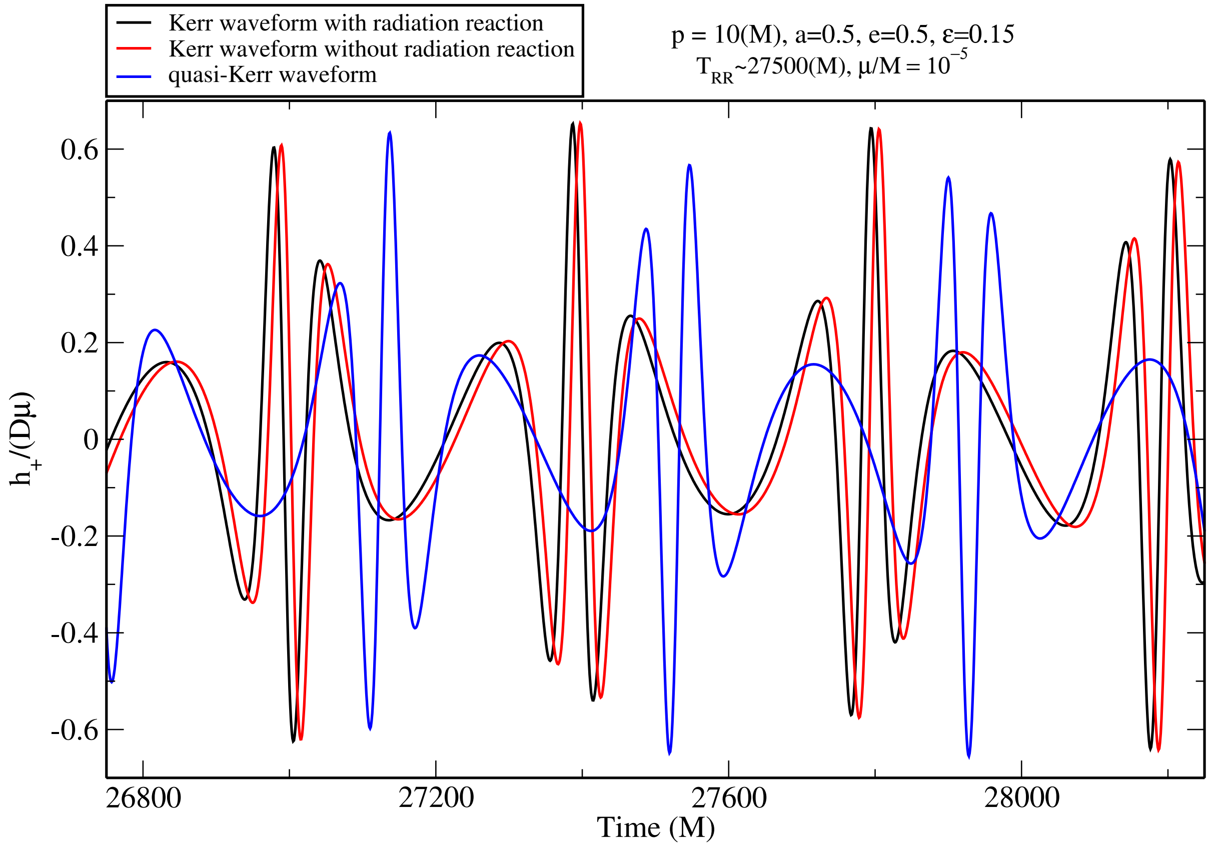
$$
r(\chi) = \frac{p}{1+e\cos\chi}, \quad \chi = 0 \to 2\pi
$$

Results: periastron advance

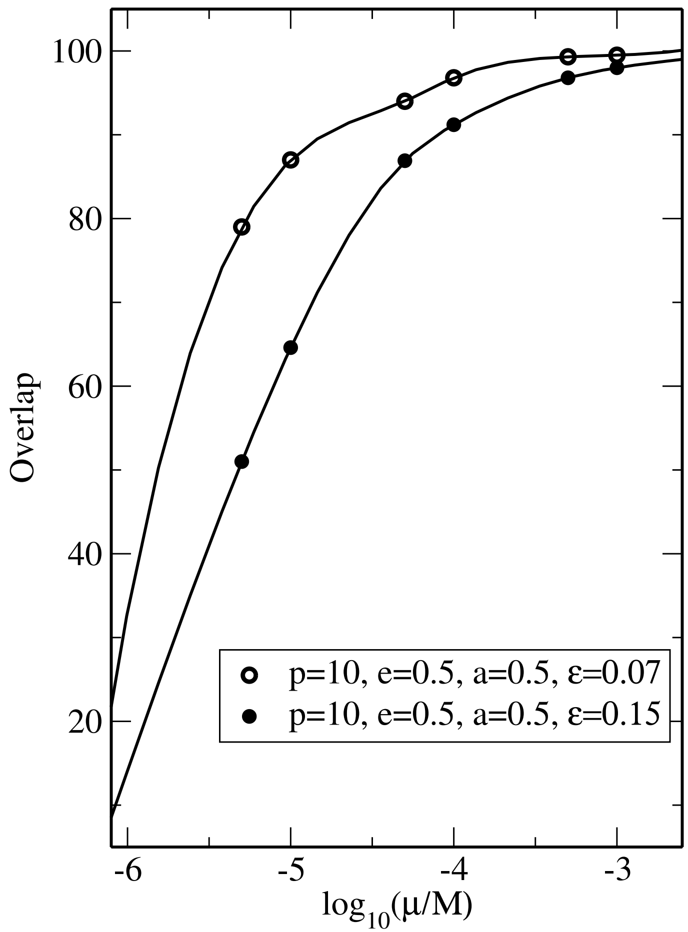


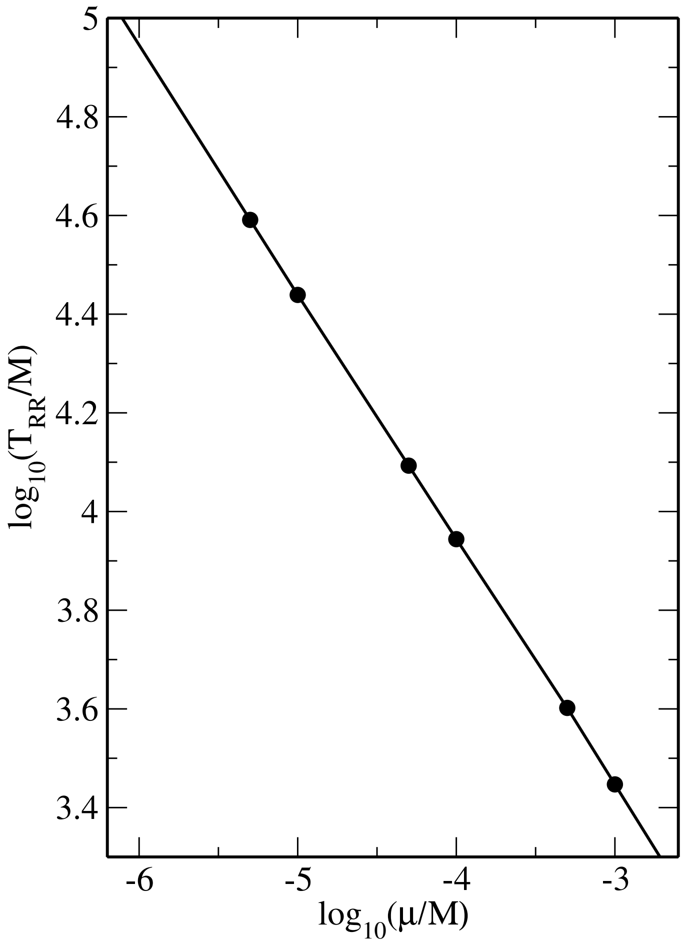


Results: ‘kludge’ waveforms



Results: overlaps





Prelude to the 'confusion problem'

- We have compared Kerr vs quasi-Kerr waveforms for the *same* orbital parameters $\{p, e\}$ and spin a/M. The resulting overlaps are low ($\sim 50 - 70\%$) after a time interval T_{RR} at which radiation reaction begins to become important.

- Emerging question: is it possible to construct a Kerr waveform with a *different* set of parameters $\{a/M, p, e\}$ that would match the original quasi-Kerr waveform (overlap $\gtrsim 95\%$) ?

- If yes, then LISA could confuse a true non-Kerr object with a Kerr BH !

- One suggestion is to compare waveforms that correspond to the same orbital frequencies Ω_r, Ω_ϕ but differ by $\delta p, \delta e \sim \mathcal{O}(\epsilon)$.

Generic orbits

- Using the fact that the quasi-Kerr H-J equation is only $\mathcal{O}(\epsilon)$ away from being separable, we can employ canonical perturbation theory and arrive to approximate equations of motion.
- Treat perturbatively the non-Kerr piece of the metric:

$$H = \frac{1}{2}\, g^{ab}_{\mathrm{Kerr}}\, p_a\, p_b + \frac{1}{2}\, \epsilon\, h^{ab}\, p_a\, p_b = H_{\mathrm{Kerr}} + \epsilon\, H_1$$

- Canonical transformation : $\{x^a, p_b\} \rightarrow \{\beta^a, \gamma_b\}$ and

$$\langle \frac{d\beta^a}{d\lambda} \rangle = \epsilon\, \langle \frac{\partial H_1}{\partial \gamma_a} \rangle, \qquad \langle \frac{d\gamma^a}{d\lambda} \rangle = -\epsilon\, \frac{\partial \langle H_1 \rangle}{\partial \beta_a}$$

  Use Kerr geodesic motion for RHS derivatives.
- Equations of motion look like: $\beta^a + [\nu^a + \langle \beta^a \rangle]\, \lambda = \frac{\partial W}{\partial \gamma_a} = \frac{\partial W}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial \gamma_a}$
- The non-Kerr character result in frequency-shifts, but there are still three integrals of motion !

Conclusions ...

- Our quasi-Kerr formalism appears as a practical tool for carrying out a spacetime-mapping program for LISA.
- For a modest quadrupole deviation, initially identical orbits ‘phase-out’ after few hundreds cycles.
- Waveform calculations (presently equatorial orbits) suggest that a Kerr waveform template with identical $\{a/M, p, e\}$ could become useless at a time span $\sim T_{RR}$.
- Possible to write approximate equations of motion for generic orbits without sacrificing the third constant.

and future challenges ...

- The ultimate goal for any non-Kerr program is the computation of rigorous EMRI waveforms.
- The question that needs to be addressed is whether the quasi-Kerr framework will allow an approximate 'Teukolsky-like' formalism.
  If not, then there is no much choice than solving (in the time-domain as a 2+1 system) the 'box' equations:

$$\Box\,\bar{h}_{ab} + 2R_{acbd}\,\bar{h}^{cd} = -16\pi\,T_{ab}, \quad \bar{h}^{ab}{}_{;b} = 0$$

- The confusion problem: do we have to worry ?
- Another interesting application of our formalism would be to estimate the change in QNMs frequencies induced by the deviation from the exact Kerr metric. This could constitute an independent check on the BH identity of the central massive object.