The fine print of BFKL dynamics

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Forward Physics and QCD at the LHC and EIC
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Nice surprise entering the building: Don Quijote

Bad Honnef | Alcalá de Henares

Sancho vs Don Quijote $\longleftrightarrow$ Phenomenology vs Theory

Briefly touch several subjects

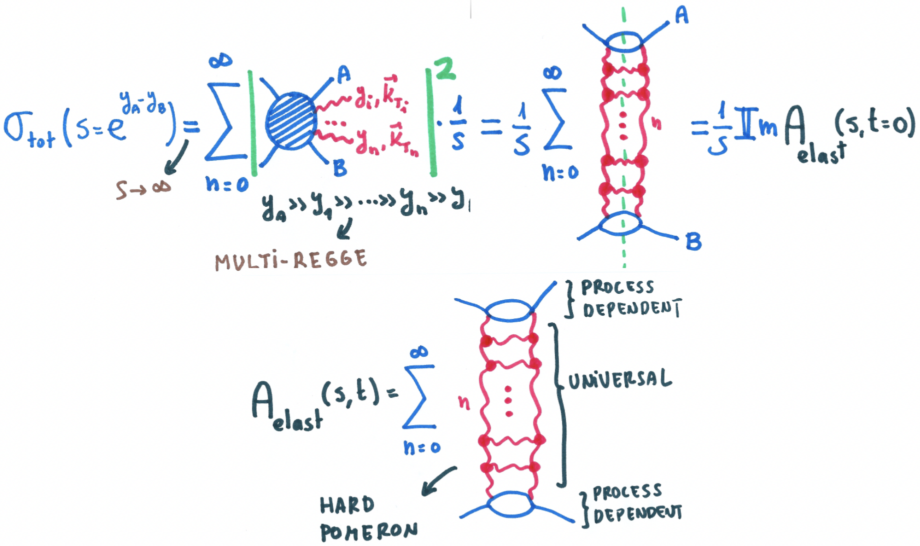
Phenomenology:

- DIS
- Jet Production
- Pomeron & Odderon

Theory:

- QCD
- N=4 SUSY
- Gravity and supergravity

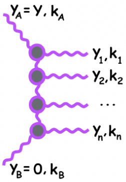
Set up for the BFKL formalism



Monte Carlo Event Generator (with G. Chachamis)

$$\sigma(Q_1, Q_2, Y) = \int d^2\vec{k}_A d^2\vec{k}_B \underbrace{\phi_A(Q_1, \vec{k}_A)\, \phi_B(Q_2, \vec{k}_B)}_{\text{PROCESS-DEPENDENT}} \underbrace{f(\vec{k}_A, \vec{k}_B, Y)}_{\text{UNIVERSAL}}$$

$$f\left(\vec{k}_A, \vec{k}_B, Y\right) = \sum_n \left| \quad \right|^2$$

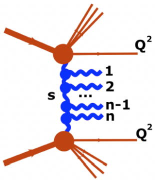


$$= e^{\omega(\vec{k}_A)Y} \left\{ \delta^{(2)}\left(\vec{k}_A - \vec{k}_B\right) + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\alpha_s N_c}{\pi} \int d^2\vec{k}_i \frac{\theta\left(k_i^2 - \lambda^2\right)}{\pi k_i^2} \right.$$

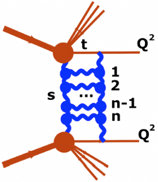
$$\left. \times \int_0^{y_{i-1}} dy_i e^{\left(\omega\left(\vec{k}_A + \sum_{l=1}^{i} \vec{k}_l\right) - \omega\left(\vec{k}_A + \sum_{l=1}^{i-1} \vec{k}_l\right)\right) y_i} \delta^{(2)}\left(\vec{k}_A + \sum_{l=1}^{n} \vec{k}_l - \vec{k}_B\right) \right\}$$

BFKLex: Monte Carlo implementation of full NLL BFKL

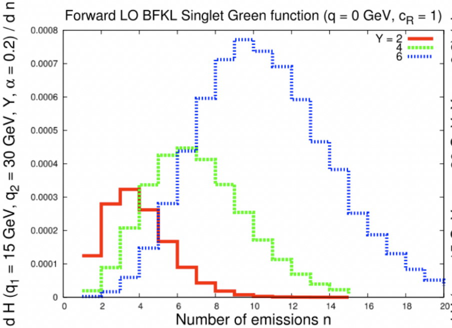
Valid both for Multiparticle production & Diffraction

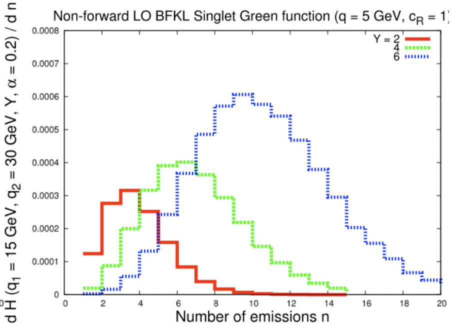


$q = 0$ Cut Pomeron: # emissions?

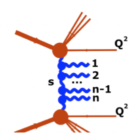


$q \neq 0$ Pomeron: # rungs?





Multiparticle production - Old Chew-Pignotti



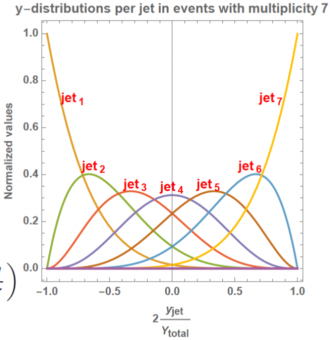
Multiperipheral models

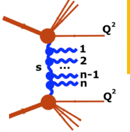
[1968] Chew-Pignotti [1971] DeTar [2020] Bethencourt-Chachamis-SV

$$\sigma_{N+2} = \alpha^{N+2} \int_0^Y \prod_{i=1}^{N+1} dz_i \delta\left(Y - \sum_{s=1}^{N+1} z_s\right)$$
$$= \alpha^{N+2} \int_{-\frac{Y}{2}}^{\frac{Y}{2}} dy_N \int_{-\frac{Y}{2}}^{y_N} dy_{N-1} \cdots \int_{-\frac{Y}{2}}^{y_3} dy_2 \int_{-\frac{Y}{2}}^{y_2} dy_1 = \alpha^2 \frac{(\alpha Y)^N}{N!} .$$

$$\frac{d\sigma_{N+2}^{(l)}}{dy_l} = \alpha^{N+2} \int_0^Y \prod_{i=1}^{N+1} dz_i \, \delta\left(Y - \sum_{s=1}^{N+1} z_s\right) \delta\left(y_l + \frac{Y}{2} - \sum_{j=1}^{l} z_j\right)$$

$$= \alpha^{N+2} \frac{\left(\frac{Y}{2} - y_l\right)^{N-l}}{(N-l)!} \frac{\left(y_l + \frac{Y}{2}\right)^{l-1}}{(l-1)!} = \alpha^{N+2} \left(\frac{Y}{2}\right)^{N-1} \sum_{s=0}^{N-1} c_{N+2,2}^{(l)} T_s \left(2\frac{y_l}{Y}\right)$$

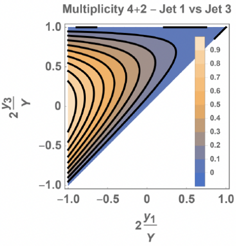


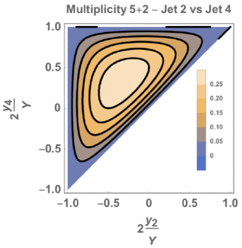


Correlations in rapidity

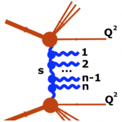
$$\frac{d^2\sigma^{(l,m)}_{N+2}}{dy_l dy_m} = \alpha^{N+2}\left(\frac{Y}{2}\right)^{N-2} \frac{(1-x_l)^{N-l}}{(N-l)!}\frac{(x_l-x_m)^{l-m-1}}{(l-m-1)!}\frac{(1+x_m)^{m-1}}{(m-1)!}$$

$$= \alpha^{N+2}\left(\frac{Y}{2}\right)^{N-2} \sum_{r,s=0}^{N-2} \mathcal{D}^{(l,m)}_{N+2,r,s} T_r(x_l)\, T_s(x_m)$$

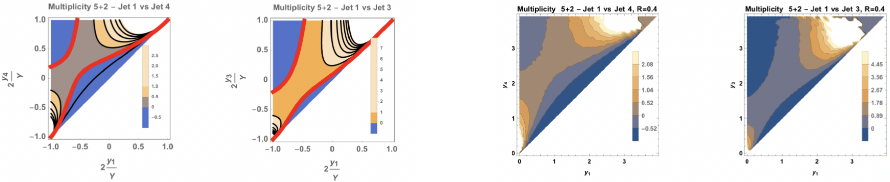




Multiparticle production - Old Chew-Pignotti



$$\mathcal{R}_{N+2}(x_l, x_m) = \sigma_{N+2} \frac{\frac{d^2\sigma^{(l,m)}_{N+2}}{dy_l dy_m}}{\frac{d\sigma^{(l)}_{N+2}}{dy_l}\frac{d\sigma^{(m)}_{N+2}}{dy_m}} - 1 = \frac{2^N}{N!}\frac{(N-m)!(l-1)!}{(l-m-1)!}\frac{(x_l - x_m)^{l-m-1}}{(1+x_l)^{l-1}(1-x_m)^{N-m}} - 1$$



BFKLex generates the same distributions

Very important to study FIXED MULTIPLICITY FINAL STATES @ LHC

[2023] [Kampshoff, Baldenegro, Chachamis, Klasen, Milhano, Royon, SV]

DIS - small x

- $n = 0$ Large correction

Origin in collinear contributions

Extend formalism beyond Regge limit [Ciafaloni, Colferai, Salam, Stasto]

$$\omega = \bar{\alpha}_s \left(2\psi(1) - \psi\left(\gamma + \frac{\omega}{2}\right) - \psi\left(1 - \gamma + \frac{\omega}{2}\right)\right)$$

Resummation in terms of a Bessel funcion [SV]

$$\omega \simeq \int_0^1 \frac{dx}{1-x} \left\{ \left(x^{\gamma-1} + x^{-\gamma}\right) \sqrt{\frac{2\bar{\alpha}_s}{\ln^2 x}} J_1 \left(\sqrt{2\bar{\alpha}_s \ln^2 x} \right) - 2\bar{\alpha}_s \right\}$$

in transverse momentum space

$$\sum_{n=1}^{\infty} \frac{(-\bar{\alpha}_s)^n}{2^n n!(n+1)!} \ln^{2n} \frac{\vec{k}_A^2}{\left(\vec{k}_A + \vec{k}_i\right)^2}$$

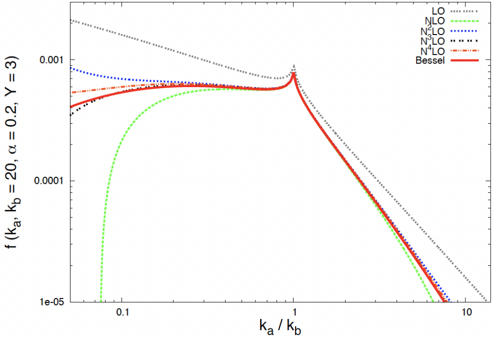
Investigating origin in HE effective action [SV]

DIS - small x

Implemented in BFKLex

Impact factors control the kinematical region:

$$\sigma = \int d^2\vec{k}_A d^2\vec{k}_B \underbrace{\phi_A(Q_1,\vec{k}_A)\,\phi_B(Q_2,\vec{k}_B)}_{\text{PROCESS-DEPENDENT}} \underbrace{f(\vec{k}_A,\vec{k}_B,Y)}_{\text{UNIVERSAL}}$$

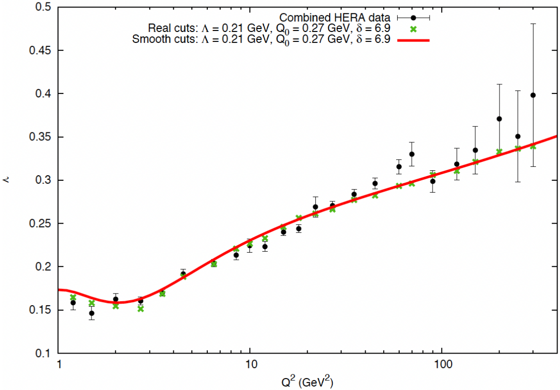


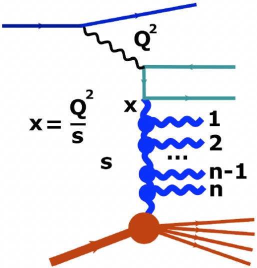
Important to go beyond the MRK limit.

For original BFKL we need "δ-like" impact factors $\phi_{A,B}$ & $Q_1 \simeq Q_2$

DIS - small x

How far can we get analytically?:
Small x DIS: $F_2(x, Q^2) \simeq x^{-\lambda(Q^2)}$
A NLL Multi-Regge approach fits data well [Hentschinski,Salas,SV]

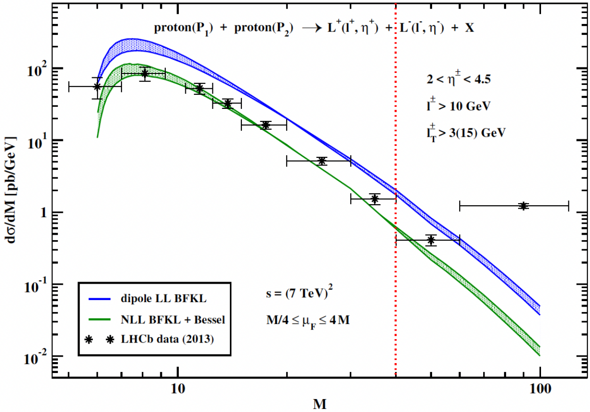


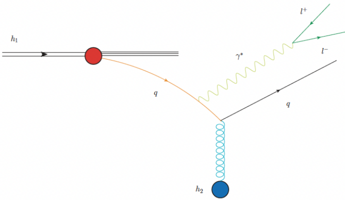


Transition from a perturbative to a non-perturbative Pomeron not well understood. Need more exclusive observables: LHC is the playground now.

DIS - small x

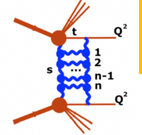
Forward Drell-Yan production at LHC [Celiberto,Gordo,SV]
The same unintegrated gluon density works well for current data





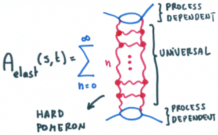
Previous analysis by [Brzeminski,Motyka,Sadzikowski,Stebel]
We work with BFKL at NLL plus collinear corrections

Diffraction - BFKL Pomeron



Conformal Spins [2022] [Chachamis, SV]

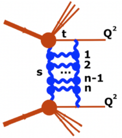
Critical review of Lipatov [86], Navelet-Peschaski [97]



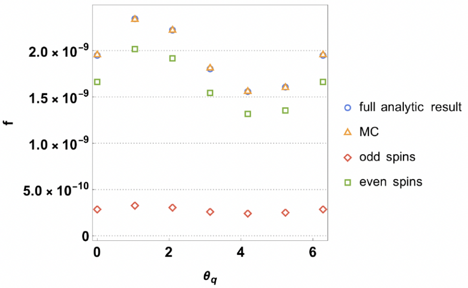
$$
\begin{aligned}
\omega f_\omega(k_a,k_b,q) &= \frac{\delta^{(2)}(k_a-k_b)}{k_a^2(q-k_a)^2} \\
&+\frac{\bar{\alpha}_s}{2\pi}\int d^2k\left\{\left[\frac{(q-k)^2}{(k-k_a)^2(q-k_a)^2}+\frac{k^2}{(k-k_a)^2k_a^2}-\frac{q^2}{k_a^2(q-k_a)^2}\right]f_\omega(k,k_b,q)\right. \\
&\left.-\left[\frac{k_a^2}{k^2+(k_a-k)^2}+\frac{(q-k_a)^2}{(q-k)^2+(k_a-k)^2}\right]\frac{f_\omega(k_a,k_b,q)}{(k-k_a)^2}\right\}
\end{aligned}
$$

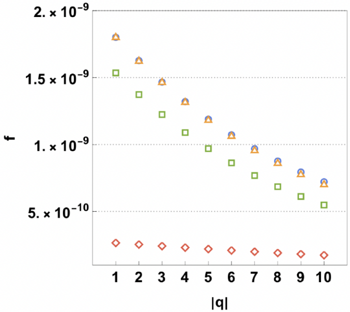
$$
\begin{aligned}
f\left(k_a,k_b,q,Y\right) &= \left(\frac{\lambda^2}{k_a^2}\frac{\lambda^2}{(k_a-q)^2}\right)^{\frac{\bar{\alpha}_s}{2}Y}\left\{\frac{\delta^{(2)}(k_a-k_b)}{k_a^2(q-k_a)^2}+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\right. \\
&\times\int d^2k_i\frac{\theta\left(k_i^2-\lambda^2\right)}{\pi k_i^2}\xi\left(k_a+\sum_{l=1}^{i-1}k_l,k_i,q\right)\int_0^{y_{i-1}}dy_i\left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l\right)^2}\right)^{\frac{\bar{\alpha}_s}{2}y_i} \\
&\left.\times\left(\frac{\left(k_a+\sum_{l=1}^{i-1}k_l-q\right)^2}{\left(k_a+\sum_{l=1}^{i}k_l-q\right)^2}\right)^{\frac{\bar{\alpha}_s}{2}y_i}\frac{\delta^{(2)}\left(\sum_{l=1}^{n}k_l+k_a-k_b\right)}{\left(k_a+\sum_{l=1}^{n}k_l\right)^2\left(k_a+\sum_{l=1}^{n}k_l-q\right)^2}\right\}
\end{aligned}
$$

Diffraction - BFKL Pomeron



New representations of $SL(2, C)$ invariant Wave Function which can be compared to Monte Carlo numerical result.





Both even and odd conformal spins contribute to Wave Function.
Relevant for possible impact factors with azimuthal angle dependence.

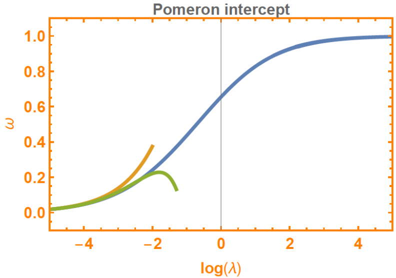
Gravity - Regge limit

- Links between QCD and gravity: AdS/CFT & BCJ color-kinematics
- Both can be studied in the high energy limit
  Graviton = dual of pomeron at strong coupling [Polchinski-Strassler]

$$j-1=\omega=\lambda\left(2\psi(1)-\psi\left(\gamma+\frac{\omega}{2}\right)-\psi\left(1-\gamma+\frac{\omega}{2}\right)\right)$$ [*Ciafaloni, Colferai, Salam, Stasto*]

$$\lambda\ll 1\ \Rightarrow\ \omega=\lambda\chi_0(\gamma)+\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^n n!(n+1)!}\frac{\lambda^{n+1}}{(\gamma+m)^{2n+1}}+\gamma\rightarrow 1-\gamma\right)$$ [*SV*]

$$\lambda\gg 1,\gamma=\frac{1}{2},\text{fixed}\,\omega\ \Rightarrow\ \frac{\omega}{\lambda}\ \rightarrow\ 0\ =\ 2\left(\psi(1)-\psi\left(\frac{1+\omega}{2}\right)\right)\ \Rightarrow\ \omega\rightarrow 1, j\rightarrow 2$$ [*Stasto*]



[Costa,Goncalves,Penedones]

[Kotikov,Lipatov]

[Bartels,Schomerus,Sprenger]

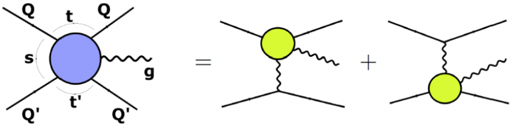
[Brower, Costa, Djuric, Raben, Tan]

[Gromov, Levkovich-Maslyuk,Sizov]

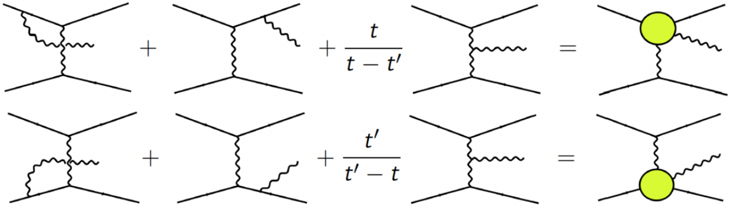
Lipatov high energy effective action is useful

Gravity - Regge limit (w Vazquez-Mozo)

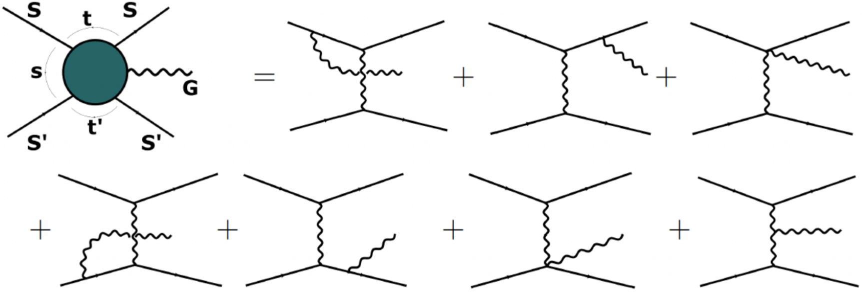
We redid Lev's calculation using standard Feynman rules [Serna,Vazquez-Mozo,SV]
In QCD we have



where the amplitude is the sum of two gauge invariant parts

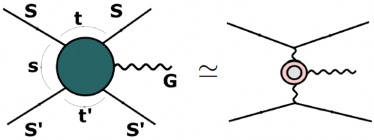


The Closest Calculation in Einstein-Hilbert Gravity:

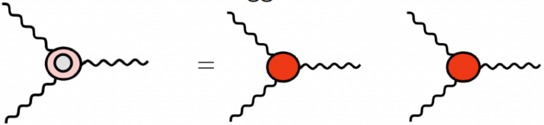


Gravity - Regge limit (w Vazquez-Mozo)

Using the same Sudakov expansion and Multi-Regge kinematics:



Universal Reggeized G - Reggeized G - G Effective Vertex (Lipatov):



$$+ \quad 4\beta_1\alpha_2 \left\{ \frac{p^\mu p^\nu}{\beta_2^2} + \frac{q^\mu q^\nu}{\alpha_1^2} + \frac{p^\mu q^\nu + q^\mu p^\nu}{\alpha_1\beta_2} \right\}$$

Subtraction Term to Fullfil Steinman Relations (no simultaneous singularities in overlapping channels).

Gravity - Regge limit (w Bartels, Lipatov)

Four-graviton amplitudes in N-SUGRA $\boxed{\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\rm Born} \mathcal{M}_{4,(N)}}$
(N = number of gravitinos)

At 1-loop three contributions

$$\begin{aligned}
\mathcal{M}^{(1)}_{4,(N=8)} &= \underbrace{\alpha\, t \ln\left(\frac{-s}{-t}\right) \ln\left(\frac{-u}{-t}\right)}_{\text{Double Logs}} \\
&+ \underbrace{\alpha \frac{t}{2} \ln\left(\frac{-t}{\lambda^2}\right)\left(\ln\left(\frac{-s}{-t}\right) + \ln\left(\frac{-u}{-t}\right)\right)}_{\text{Trajectory}} \\
&- \underbrace{\alpha \frac{(s-u)}{2} \ln\left(\frac{-t}{\lambda^2}\right) \ln\left(\frac{-s}{-u}\right)}_{\text{Eikonal}}
\end{aligned}$$

Gravity - Regge limit (w Bartels, Lipatov)

Regge limit $u \simeq -s$

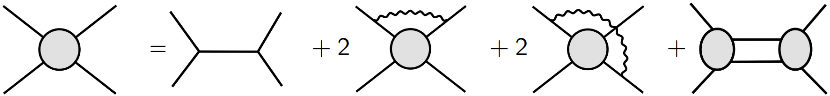
$$\mathcal{M}^{(1)}_{4,(N=8)} \simeq \underbrace{(\alpha\, t) \ln^2\left(\frac{s}{-t}\right)}_{\text{Double Logs}} + \underbrace{(\alpha\, t) \ln\left(\frac{-t}{\lambda^2}\right) \ln\left(\frac{s}{-t}\right)}_{\text{Trajectory}} + i\,\pi\, \underbrace{(\alpha\, s) \ln\left(\frac{-t}{\lambda^2}\right)}_{\text{Eikonal}}$$

Evaluate Double Logs to all orders? Use

$$\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\rm Born} \left(\frac{s}{-t}\right)^{\alpha\, t\, \ln\left(\frac{-t}{\lambda^2}\right)} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\,\omega}{2\pi i} \left(\frac{s}{-t}\right)^{\omega} \frac{f_\omega^{(N)}}{\omega}$$

Gravity - Regge limit (w Bartels, Lipatov)

Origin in the diagrams



with (1) virtual gravitons with lowest energy
(2) t-channel graviton/gravitino pairs with lowest energy

Associated equation

$$f_\omega^{(N)} = 1 - (\alpha\, t)\frac{d}{d\,\omega}\left(\frac{f_\omega^{(N)}}{\omega}\right) + (\alpha\, t)\left(\frac{N-6}{2}\right)\left(\frac{f_\omega^{(N)}}{\omega}\right)^2$$

Perturbative solution

$$\begin{aligned} f_\omega^{(N)} &= 1 + (\alpha\, t)\frac{(N-4)}{2\omega^2} + (\alpha\, t)^2\frac{(N-4)(N-3)}{2\omega^4} \\ &- (\alpha\, t)^3\frac{(N-4)\left(5N^2-26N+36\right)}{8\omega^6} + \dots \end{aligned}$$

Gravity - Regge limit (w Bartels, Lipatov)

2-loop agreement in $N = 4, \ldots, 8$ SUGRA with BCJ [Boucher-Veronneau, Dixon].
3-loop agreement in $N = 8$ SUGRA [Henn,Mistlberger].

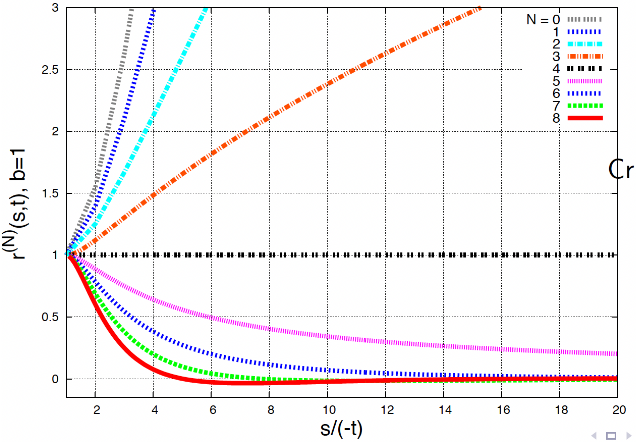
All-orders predictions *e.g.* $N = 8$:

$$\mathcal{A}_{4,(N=8)} = \mathcal{A}_4^{\rm Born} \left(\frac{-t}{\lambda^2}\right)^{\alpha t \left(\ln\left(\frac{s}{-t}\right) + i\pi\left(\frac{s}{t}\right)\right)}$$
$$\times \left\{ 1 + 2\left(\frac{\alpha\, t}{2}\right) \ln^2\left(\frac{s}{-t}\right) + \frac{5}{3}\left(\frac{\alpha\, t}{2}\right)^2 \ln^4\left(\frac{s}{-t}\right) \right.$$
$$+ \frac{37}{45}\left(\frac{\alpha\, t}{2}\right)^3 \ln^6\left(\frac{s}{-t}\right) + \frac{353}{1260}\left(\frac{\alpha\, t}{2}\right)^4 \ln^8\left(\frac{s}{-t}\right)$$
$$\left. + \frac{583}{8100}\left(\frac{\alpha\, t}{2}\right)^5 \ln^{10}\left(\frac{s}{-t}\right) + \ldots \right\}$$

to be compared with other calculations in the future.

Gravity - Regge limit (w Bartels, Lipatov)

Resummation [Bartels,Lipatov,SV] [SV]:
Solution to a Schrödinger equation in terms of parabolic cylinder functions



$N = 4$ SUGRA:
Critical boundary theory
Finite ($N > 4$)
Non-finite ($N < 4$)
at high energies

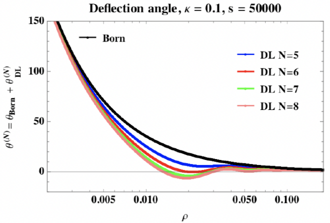
Gravity - Regge limit (SV)

Subleading correction to N-SUGRA eikonal phase [SV]

DLs reduce gravitational strength: "Graviton Deflection Angle"

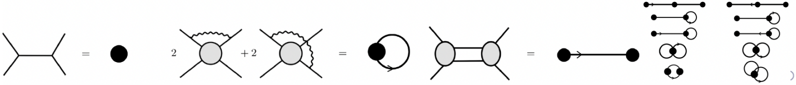
$$
\begin{aligned}
\mathcal{C}^{(4)} &= (1,0,0,0,0,0,0,\dots)\,, \\
\mathcal{C}^{(5)} &= \left(1,\frac{1}{2},1,\frac{31}{8},\frac{91}{4},\frac{2873}{16},\frac{14243}{8},\dots\right), \\
\mathcal{C}^{(6)} &= (1,1,3,15,105,945,10395,\dots)\,, \\
\mathcal{C}^{(7)} &= \left(1,\frac{3}{2},6,\frac{297}{8},306,\frac{50139}{16},38286,\dots\right), \\
\mathcal{C}^{(8)} &= (1,2,10,74,706,8162,110410\dots)\,.
\end{aligned}
$$

$$
\theta_{\rm DL}^{(N)}(\rho,s) = -\theta_{\rm Born}(\rho,s)\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{n(-\alpha s)^m\mathcal{C}^{(N)}_{m-n}}{(n!)^2 m^{2(m-n)+1}}\left(\frac{\rho^2}{4\alpha}\right)^n
$$



N-SUGRA: DLs resummation for inelastic amplitudes

N=8 calculate full amplitude to all orders?



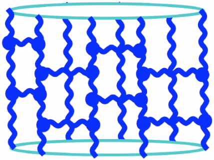
Supersymmetry - Regge limit

- $N = 4$ SYM important part of QCD amplitudes
- Experience in HE QCD very useful in SYM
- Principle of maximal transcendentality [Kotikov, Lipatov] [Dixon]
- Steinman relations [Bartels, Lipatov, SV] [Dixon, Caron-Huot ...]
- Integrability [Lipatov] [Faddeev, Korchemsky ...] [Kazakov, Gromov, Sizov ...]
- Work to calculate BFKL in QCD and SUSY at higher orders

  [Fadin] [Balitsky] [Caron-Huot, Gromov ...] [Del Duca, Gardi, Magnea ...]

Supersymmetry - Regge limit

Amplitudes in Generalized LLA
interesting structure in QCD
and $N = 4$ SYM at high energies



Equivalent to closed spin chain
(Heisenberg ferromagnet)

[Lipatov] [Faddeev, Korchemsky] [Janik, Wosiek, Kotanski, Derkachov, Manashov ...]

Monte Carlo approach to solve the system [Chachamis,SV]

Work to map this to the integrability structure

3 reggeized gluons in t-channel color singlet:
ODDERON

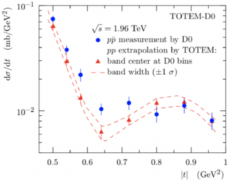


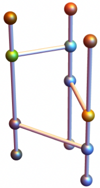
FIG. 6: Comparison between the D0 $p\bar{p}$ measurement at 1.96 TeV and the extrapolated TOTEM pp cross section, rescaled to match the OP of the D0 measurement. The dashed lines show the 1σ uncertainty band.

Drives the difference in
pp and $p\bar{p}$ total cross sections

[DØ and TOTEM Collaborations 2020 (arXiv:2012.03981)].

Bartels-Kwiecinski-Praszalowicz (BKP) equation:
Asymptotic intercept one [Bartels, Lipatov, Vacca]

$$
\begin{aligned}
&(\omega - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3))\, f_\omega(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \\
&\quad \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\, \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\, \delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\
&\quad + \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\
&\quad + \int d^2\mathbf{k}\, \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f_\omega(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k}) \\
&\quad + \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2, \mathbf{p}_3 - \mathbf{k})
\end{aligned}
$$



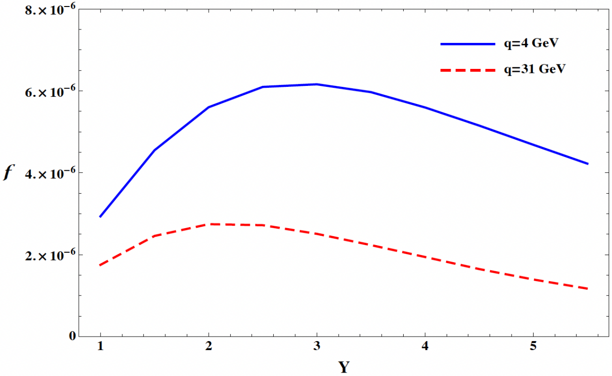
Squared Lipatov's emission vertex:

$$
\xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k}) = \frac{\alpha_s N_c}{4} \frac{\theta(\mathbf{k}^2 - \lambda^2)}{\pi^2 \mathbf{k}^2} \left(1 + \frac{(\mathbf{p}_1 + \mathbf{k})^2 \mathbf{p}_2^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \mathbf{k}^2}{\mathbf{p}_1^2 (\mathbf{k} - \mathbf{p}_2)^2} \right)
$$

Gluon Regge trajectory: $\omega(\mathbf{p}) = -\frac{\bar{\alpha}_s}{2} \ln \frac{\mathbf{p}^2}{\lambda^2}$

Supersymmetry - Regge limit

Our solution contains all the previous ones [Chachamis,SV]
which are projected out when integrating over different impact factors

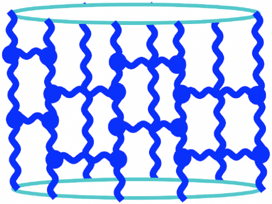


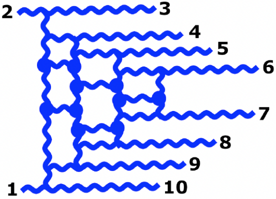
Work in pheno and formal applications

Supersymmetry - Regge limit

Similar diagrams with Regge cuts in $N = 4$ SYM MHV [Bartels,Lipatov,SV]

Complicated contributions at higher orders and more external particles:





Equivalent to an open spin chain [Lipatov]
MC methods also work here [Chachamis,SV]
Consider octet exchange with 3 reggeized gluons:

MHV 8-point amplitude

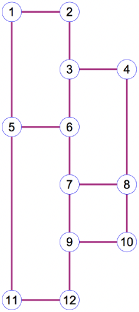
Supersymmetry - Regge limit

Graph complexity

Number of possible spanning trees in the graph.
(Cross all nodes without loops)

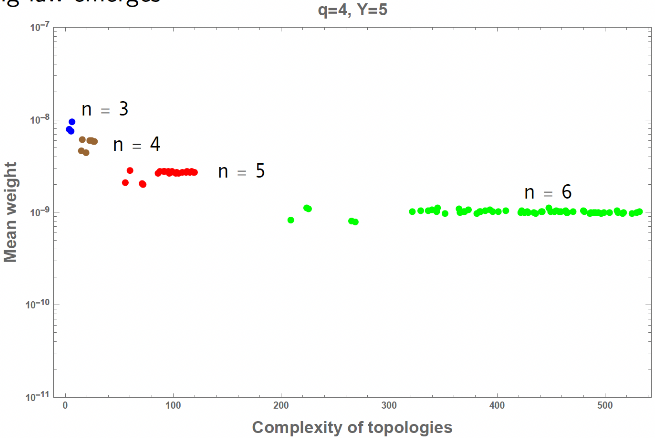
Consider Laplacian matrix L of a graph with 6 rungs

$$L = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2
\end{pmatrix}$$



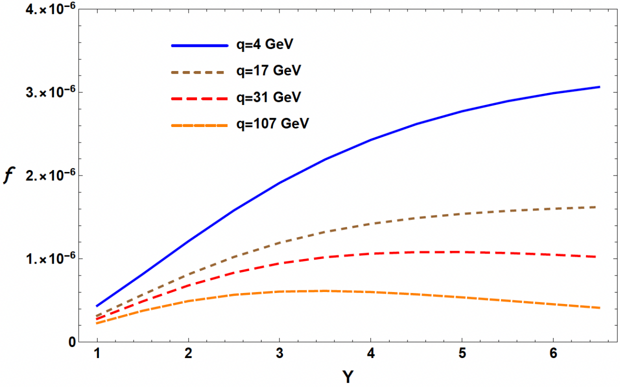
Matrix Tree theorem, Kirchoff:
The determinant of any principal minor of L = Complexity of the graph

Mean weight per Complexity Class in the reggeized gluon net
A scaling law emerges



Origin in the underlying integrability?

Supersymmetry - Regge limit



This is the solution to the "Adjoint Pomeron" (open spin chain)
Important in the all-order structure of $N = 4$ SYM amplitudes

Interpretation within integrability?

Supersymmetry - Regge limit

Work to obtain BFKL eqn to all orders

First step in this direction in $N = 4$ SYM [Bartels,Lipatov,SV]

Use BDS [Bern,Dixon,Smirnov] ansatz for planar amplitudes
Proposed for all loops and number of external legs
It needs to be improved

MHV amplitudes, color ordered and normalized by Born:

$$\log \mathcal{M}_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right)$$

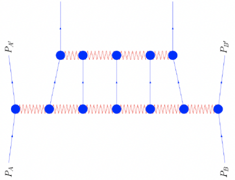
$d = 4 - 2\epsilon$, $l =$ number of loops, $n =$ number of legs

Supersymmetry - Regge limit

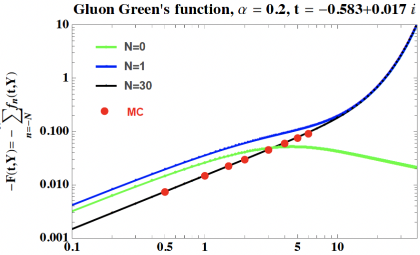
Origin: presence of a phase well known in HE QCD [Bartels,Lipatov,SV]

$$\frac{M_{2\to4}}{\Gamma(t_1)\Gamma(t_3)} = \underbrace{e^{i\pi\frac{\gamma_K(a)}{4}\left(-\frac{1}{\epsilon}+\log\Omega\right)}}_{\text{Regge cut}} \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma_{RRP} \left(\frac{-s_2}{\mu^4}\right)^{\omega(t_2)} \Gamma_{RRP} \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}$$

Solution: complete BDS with Regge cuts from 2 loops onwards



Calculated at LL [Bartels,Lipatov,SV] and NLL [Fadin,Lipatov]



Work to relate it to integrability

conformal blocks [Chachamis,SV]

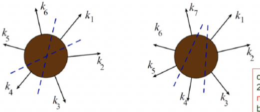
Supersymmetry - Regge limit

Our conditions for the correct analytic structure are important to solve them to all orders and arbitrary number of external legs

Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:



can't apply to 2 particle cuts in massless case because they are not independent

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

← Violated by ABDK and BDS ansatz!

L. Dixon   From 2 to 7 Loops in planar N=4 SYM   QMUL - 2019.11.07   35

Master Table

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. All functions	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,3692?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear limit	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^*)$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2F^2\ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all T^2F^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined   Also have MHV at $L=7$

L. Dixon   From 2 to 7 Loops in planar N=4 SYM   QMUL - 2019.11.07   38

Summary

Bad Honnef

Alcalá de Henares

Sancho vs Don Quijote $\longleftrightarrow$ Phenomenology vs Theory