# Lévy α-stable model for the non-exponential low-|t| proton-proton differential cross section

based on *Universe* **2023**, *9*(8), 361 arXiv:2308.05000 and other recent results

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Outline:

Lévy α-stable generalization of the Bialas-Bzdak model

Simple Lévy α-stable model by approximations and fits to data

Relation between the parameters of the full and the simplified model

**9th Day of Femtoscopy**
**30-31 October 2023, Gyöngyös, Hungary**

# Preliminaries: ReBB model analysis of pp and p$\bar{\text{p}}$ data

- **the Real extended Bialas-Bzdak (ReBB) model describes elastic pp and p$\bar{\text{p}}$ $d\sigma/dt$ data in a statistically acceptable way (CL≥0.1%) in the kinematic region:**

$$546\ \text{GeV} \le \sqrt{s} \le 8\ \text{TeV}$$

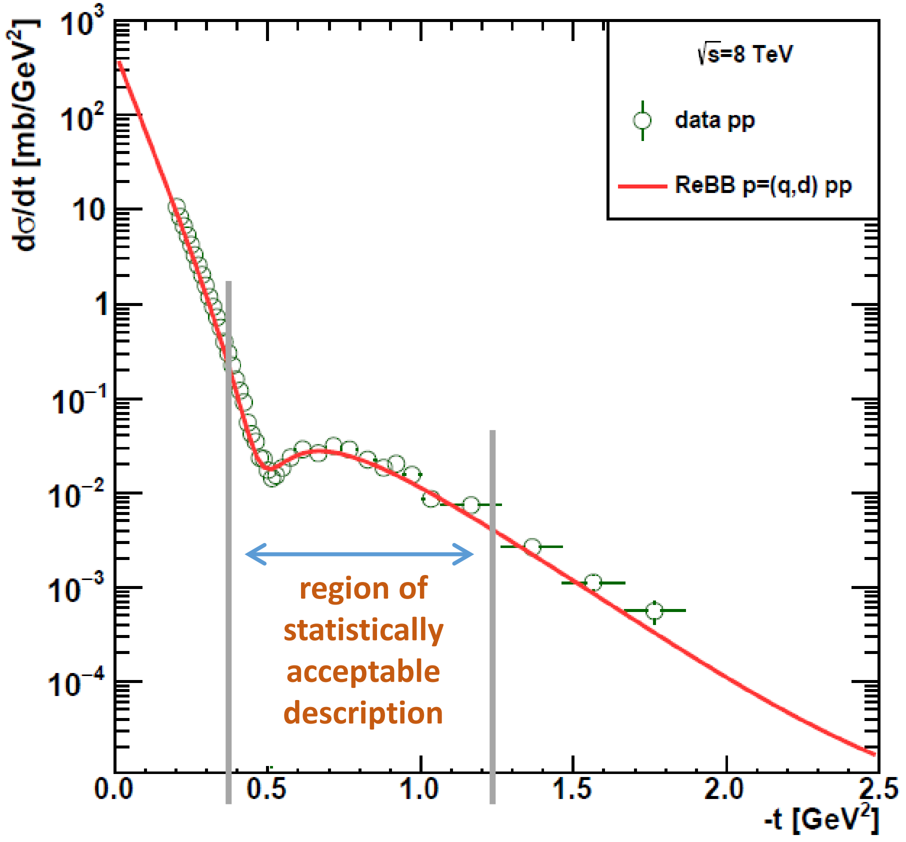
$$0.37\ \text{GeV}^2 \le -t \le 1.2\ \text{GeV}^2$$

- significant model dependent odderon signal observation

- main goal: **to improve the ReBB model to have a statistically acceptable (CL≥0.1%) description to elastic pp and p$\bar{\text{p}}$ data in a wider kinematic range**

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81***, *611 (2021)*

I. Szanyi, T. Csörgő, *Eur. Phys. J. C* ***82***, *827 (2022)*




**ReBB model description to the 8 TeV pp data**

# Unitarity and the elastic scattering amplitude

- the $S$-matrix is unitary expressing the conservation of probability

$$SS^{\dagger} = I$$

- **the unitaritiy constraint can be rewritten in impact parameter ($\vec{b}$) representation**

$$2\,\mathcal{Im}\,t_{\mathrm{el}}(s,\vec{b}) = |t_{el}(s,\vec{b})|^2 + \tilde{\sigma}_{in}(s,\vec{b})$$ ($\sqrt{s}$ is the CM energy)

- **the elastic scattering amplitude $t_{el}(s,\vec{b})$ is a solution of the unitarity equation and written in terms of the inelastic cross section $\tilde{\sigma}_{in}(s,\vec{b})$**

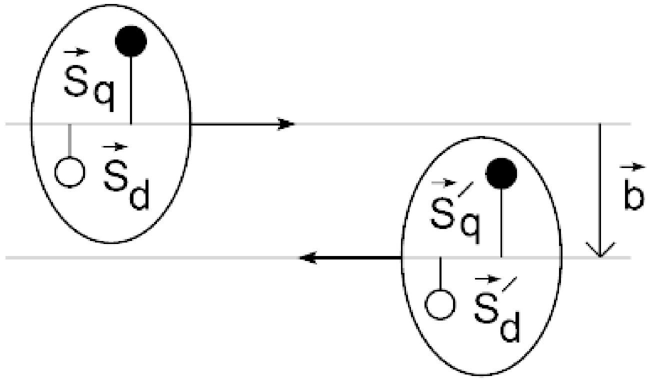
$$0 \le \tilde{\sigma}_{in}(s,\vec{b}) \le 1$$

- **at a given energy $\tilde{\sigma}_{in}(s,\vec{b})$ is the probability of inelastic scattering as fuction of $\vec{b}$ and it can be calculated by using probability calculus and R. J. Glauber's multiple diffractive scattering theory**

# The Bialas-Bzdak (BB) p=(q,d) model

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- **in the Bialas-Bzdak (BB) p=(q,d) model the proton is a bound state of a constiuent quark and constituent a diquark**
- **the probability of inelastic scattering of protons as a function of transverse positions of quarks and diquarks inside the protons ($\vec{s}_q, \vec{s}_d, \vec{s}_q', \vec{s}_d'$) and the impact parameter ($\vec{b}$) at given energy is given by a Glauber expanasion**




**Proton-proton collision in the quark-diquark model**

$$\sigma(\vec{s}_q,\vec{s}_d;\vec{s}_q',\vec{s}_d';\vec{b}) = 1 - [1-\sigma_{qq}(\vec{b}+\vec{s}_q'-\vec{s}_q)][1-\sigma_{qd}(\vec{b}+\vec{s}_d'-\vec{s}_q)] \times \times [1-\sigma_{dq}(\vec{b}+\vec{s}_q'-\vec{s}_d)][1-\sigma_{dd}(\vec{b}+\vec{s}_d'-\vec{s}_d)]$$

$\sigma_{ab}(\vec{x}) \equiv \frac{d^2\sigma_{ab}(\vec{x})}{dx^2}$ is the inelastic differential cross section (inelastic scattering probability) for the collision of two constituents

$$\tilde{\sigma}_{in}(\vec{b}) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d^2s_q d^2s_q' d^2s_d d^2s_d' \, D(\vec{s}_q,\vec{s}_d)\, D(\vec{s}_q',\vec{s}_d')\, \sigma(\vec{s}_q,\vec{s}_d;\vec{s}_q',\vec{s}_d';\vec{b})$$

$D(\vec{s}_q,\vec{s}_d)$ is the distribution of the quark-diquark distance inside a proton

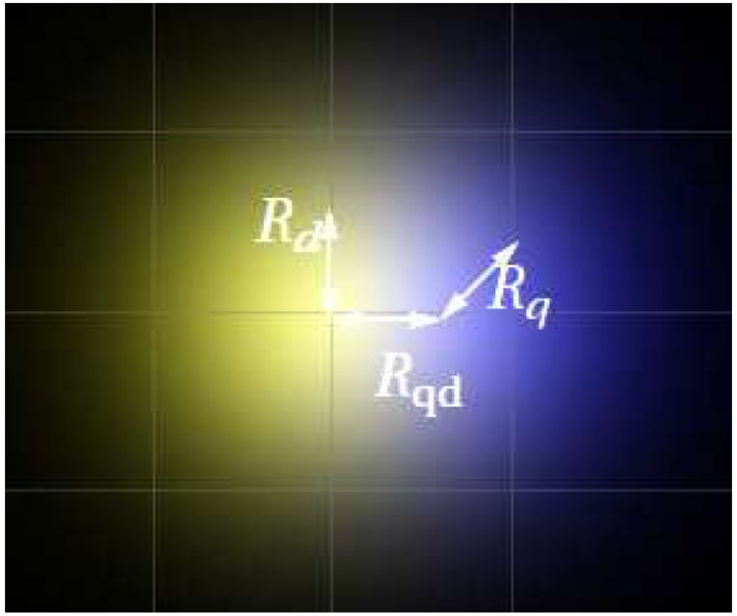
# Inelastic constituent-constituent collisons

- **the inelastic differential cross section for the collision of two constituents can be written in terms of a convolution of their parton distributions**
- **in the original BB model the parton distributions of the constiuents are Gausian distributions**

$$G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a G(\vec{r}_a|R_a/\sqrt{2}) G(\vec{x}-\vec{r}_a|R_b/\sqrt{2}) \equiv A_{ab}\pi S_{ab}^2 G(\vec{x}|S_{ab}/\sqrt{2})$$

$$\vec{x} = \vec{b} + \vec{s}'_b - \vec{s}_a \qquad S_{ab}^2 = R_a^2 + R_b^2 \qquad a,b \in \{q,d\}$$




**The picture of the proton in the quark-diquark model**

- assumption: the diquark contains twice as many partons than the quark and the colliding constituents do not shadow each other, then $\sigma_{qq}^{int}:\sigma_{qd}^{int}:\sigma_{dd}^{int} = 1:2:4$, $\sigma_{ab}^{int} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_{ab}(\vec{x})\, d^2x$
- this assumption reduces the number of free paramters by two

Free parameters by now: $R_q, R_d, A_{qq}$

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

# The quark-diquark distance

- **in the original BB model the the distribution of the quark-diquark distance follows Gaussian distribution**

$$G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

- **considering the relative distance between** the quark and diquark ($\vec{s}_q - \vec{s}_d$) one can write $D(\vec{s}_q, \vec{s}_d)$ in terms of a single Gaussian distribution:

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 G(\vec{s}_q - \vec{s}_d | R_{qd}/\sqrt{2}) \delta^2(\vec{s}_q + \lambda \vec{s}_d)$$

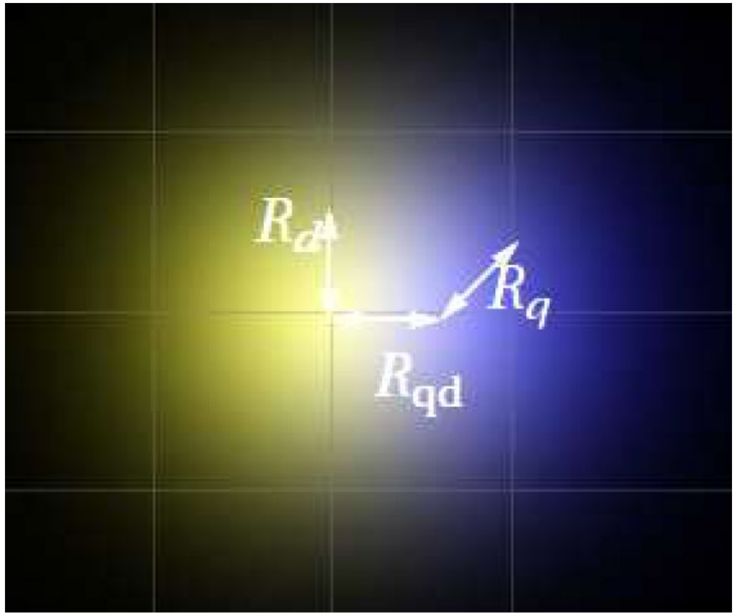
$$\lambda = m_q/m_d$$

- the Dirac $\delta$ fixes the center of the mass of the proton making the calculations easier

- $D(\vec{s}_q, \vec{s}_d)$ is normalized as $\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$

$$A_{qq} = 1 \; and \; \lambda = 1/2 \; can \; be \; fixed$$

*F. Nemes, T. Csörgő, M. Csanád, Int. J. Mod. Phys. A Vol. 30, 1550076 (2015)*




**The picture of the proton in the quark-diquark model**

Free parameters by now:
$R_q, R_d, A_{qq}, R_{qd}, \lambda$

# Real extended Bialas-Bzdak (ReBB) model

F. Nemes, T. Csörgő, M. Csanád, *Int. J. Mod. Phys. A Vol. 30, 1550076 (2015)*

- **the elastic scattering amplitude was extended with a real part respecting unitarity**

$$\tilde{t}_{el}(s,\vec{b}) = i\left(1-\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right) \rightarrow \tilde{t}_{el}(s,\vec{b}) = i\left(1-e^{i\,\alpha_R\,\tilde{\sigma}_{in}(s,\vec{b})}\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right)$$

new free parameter ($\alpha_R$)

- **statistically acceptable description (CL≥0.1%) to the elastic pp and p$\bar{\text{p}}$ $d\sigma/dt$ in the kinematic region $\mathbf{0.546\ TeV} \leq \sqrt{\mathbf{s}} \leq \mathbf{8\ TeV}$ & $\mathbf{0.37GeV^2} \leq -\mathbf{t} \leq \mathbf{1.2\ GeV^2}$**

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81***, *611 (2021)*

I. Szanyi, T. Csörgő, *Eur. Phys. J. C* ***82***, *827 (2022)*

- the strong non-exponential low-|t| pp dσ/dt measured by TOTEM at LHC and earlier efficient modelling with Lévy α-stable distribution motivates the Lévy α-stable generalization of the BB model for having a statistically acceptable descriptions in a wider kinematic range

G. Antchev et al. (TOTEM Collab.), *Nucl. Phys. B, 899, 527 (2015)*

G. Antchev et al. (TOTEM Collab.), *Eur. Phys. J. C 79, 861 (2019)*

T. Csörgő, R. Pasechnik, A. Ster, *Eur. Phys. J. C 79, 62 (2019)*

# Lévy α-stable generalized Bialas-Bzdak (LBB) model

- **the parton distributions of the constituent quark and diquark are Levy α-stable distributions and the inelastic differential cross section for the collision of two constiuents is:**

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a L(\vec{r}_a|\alpha_L, R_a/2)L(\vec{x}-\vec{r}_a|\alpha_L, R_b/2) \equiv A_{ab}\pi S_{ab}^2 L(\vec{x}|\alpha_L, S_{ab}/2)$$

$$S_{ab}^{\alpha_L} = R_a^{\alpha_L} + R_b^{\alpha_L}$$

**another new free parameter:** $\alpha_L$

- **the quark-diquark realtive distance has a Levy α-stabil distribution:**

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 L(\vec{s}_q - \vec{s}_d|\alpha_L, R_{qd}/2)\delta^2(\vec{s}_q + \lambda\vec{s}_d)$$

$$\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$$

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$$L(\vec{x}|\alpha_L, R_L) \equiv L(\vec{x}|\beta = 0, \vec{\delta} = 0, \alpha_L, R_L) = \frac{1}{(2\pi)^2}\int d^2 q\, e^{i\vec{q}^T\vec{x}} e^{-|\vec{q}^2 R_L^2|^{\alpha_L/2}}$$

**$\alpha_L$ is a new free parameter of the model and if $\alpha_L = 2$ the BB model with Gaussian distributions is recovered**

# Difficulties with LBB model

- $\tilde{\sigma}_{in}(\vec{b})$ **can be written as sum of 11 different terms that are integrals** of products of Lévy α-stable distributions

$$\tilde{\sigma}_{in}(\vec{b}) = \tilde{\sigma}_{in}^{qq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd}(\vec{b}) + \tilde{\sigma}_{in}^{dd}(\vec{b}) - [2\tilde{\sigma}_{in}^{qq,qd}(\vec{b}) + \tilde{\sigma}_{in}^{qd,dq}(\vec{b}) + \tilde{\sigma}_{in}^{qq,dd}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd,dd}(\vec{b})] \\ +[\tilde{\sigma}_{in}^{qq,qd,dq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qq,qd,dd}(\vec{b}) + \tilde{\sigma}_{in}^{dd,qd,dq}(\vec{b})] - \tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b})$$

- difficulties with the calculation of integrals of products of Lévy α-stable distributions

- the calculation is easy only if the integral can be written in a convolution form as in case of the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$

# Leading order terms in $\tilde{\sigma}_{in}$ in the LBB model

$$\begin{aligned}\tilde{\sigma}_{in}^{qq}(\vec{b}) &= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\ &\times \int d^2 s_q d^2 s'_q L\left(\vec{s}_q|\alpha_L, R_{qd*}/2\right) L\left(\vec{s}_q{}'|R_{qd*}/2\right) L\left(\vec{b} + \vec{s}_q{}' - \vec{s}_q|\left(2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\ &= \pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left(2R_{qd*}^{\alpha_L} + 2R_q^{\alpha_L}\right)^{1/\alpha_L}/2\right),\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{in}^{qd}(\vec{b}) &= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\ &\times \int d^2 s_q d^2 s'_q L\left(\vec{s}_q|R_{qd*}/2\right) L\left(\vec{s}_q{}'|R_{qd*}/2\right) L\left(\vec{b} - \lambda\vec{s}_q{}' - \vec{s}_q|\alpha_L, \left(R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\ &= 2\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left((1 + \lambda^{\alpha_L})R_{qd*}^{\alpha_L} + R_q^{\alpha_L} + R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right),\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{in}^{dd}(\vec{b}) &= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} \times \\ &\times \int d^2 s_q d^2 s'_q L\left(\vec{s}_q|R_{qd*}/2\right) L\left(\vec{s}_q{}'|R_{qd*}/2\right) L\left(\vec{b} + \lambda\left(\vec{s}_q - \vec{s}_q{}'\right)|\alpha_L, \left(2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right) \\ &= 4\pi A_{qq}\left(2R_q^{\alpha_L}\right)^{2/\alpha_L} L\left(\vec{b}|\alpha_L, \left(2\lambda^{\alpha_L}R_{qd*}^{\alpha_L} + 2R_d^{\alpha_L}\right)^{1/\alpha_L}/2\right).\end{aligned}$$

# Difficulties with LBB model fits to the data

- since multivariate Lévy α-stable distributions have forms in terms of special functions, it is hard to perform a numerical fitting procedure
- a relatively high computing capacity and improved analytic insight is needed to proceed with the full model
- **quick solution:** approximations that are valid at the low –t domain
- at low –t values, the original ReBB model had difficulties to describe the strongly non-exponential features of the experimental data on $d\sigma/dt$
- a simple model which is valid at the low –t domain easily illustrates the power of the Lévy α-stable generalization

# Simple Lévy α-stable model for low-|t| pp $d\sigma/dt$

- low-$|t|$ scattering corresponds to high-$b$ scattering and at high $b$ values $\tilde{\sigma}_{in}(s,b)$ is small
- leading order term in the Taylor expansion of the amplitude in $\tilde{\sigma}_{in}(s,b)$ dominates at low $-t$ values if $\alpha_R$ is small too

$$\tilde{t}_{el}(s,\vec{b}) = i\left(1 - e^{i\,\alpha_R(s)\,\tilde{\sigma}_{in}(s,\vec{b})}\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right) \rightarrow \tilde{t}_{el}(s,\vec{b}) = \left(\alpha_R(s) + \frac{i}{2}\right)\tilde{\sigma}_{in}(s,\vec{b})$$

- motivated by the fact that the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ have Lévy α-stable shapes in the LBB model, $\tilde{\sigma}_{in}(s,\vec{b})$ is approximated with a single Lévy α-stable shape

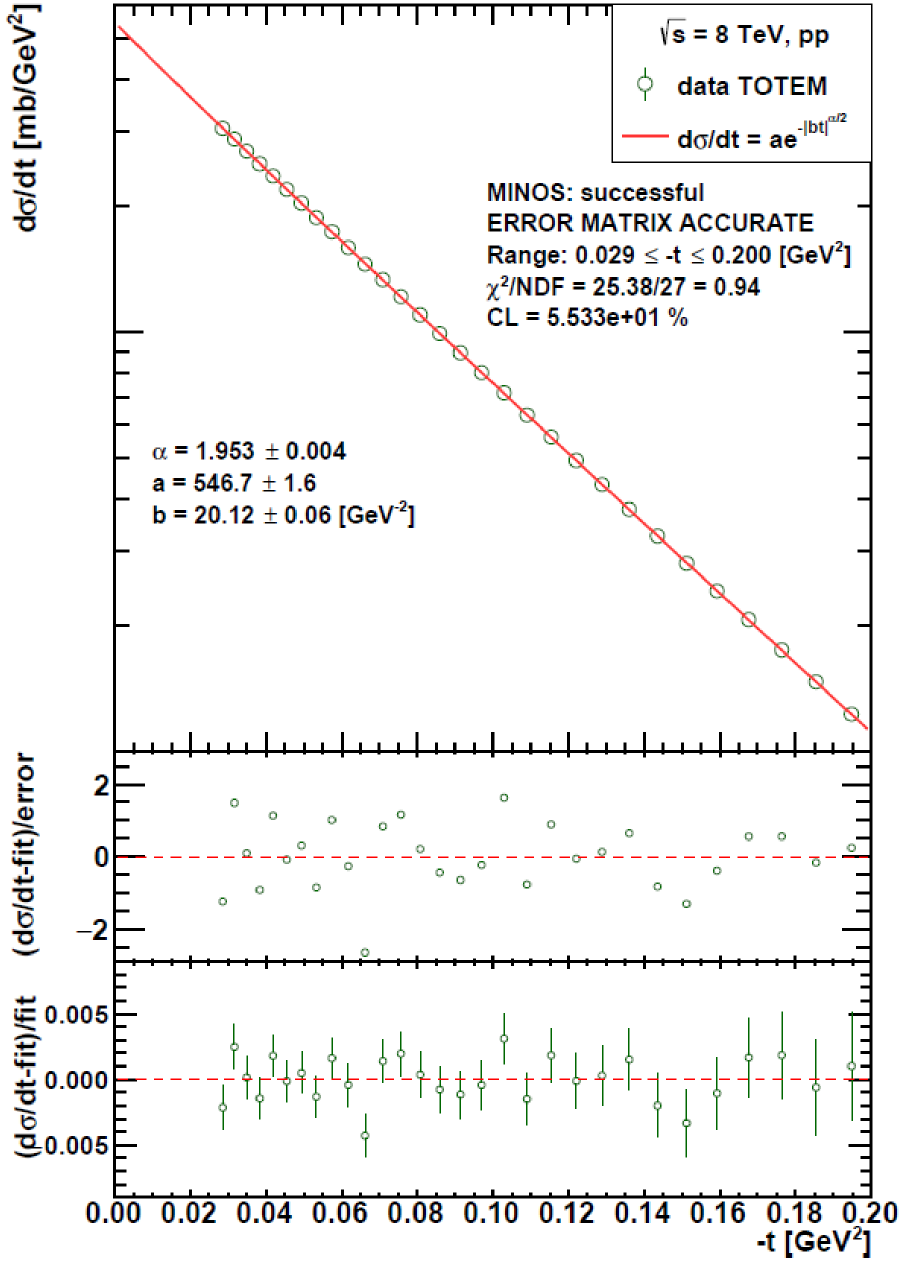
$$\tilde{\sigma}_{in}(s,\vec{b}) = \tilde{c}(s)L\left(\vec{b}|\alpha_L(s), r(s)\right)$$

- **a simple Lévy α-stable model model for low-|t| pp $d\sigma/dt$ arises**

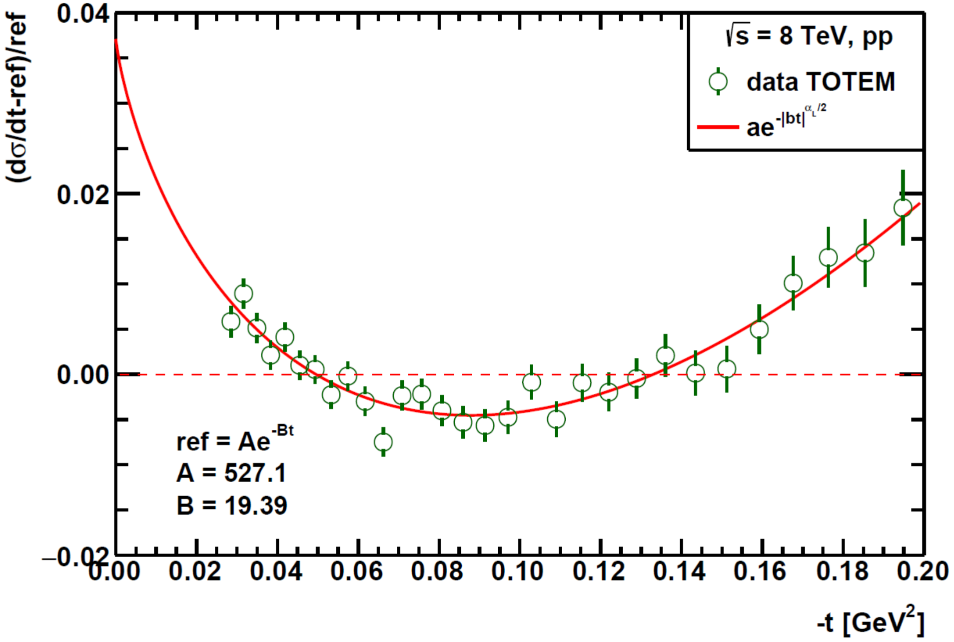
$$t_{el}(s,t) = \int d^2b\, e^{i\vec{\Delta}^T\vec{b}}\tilde{t}_{el}(s,\vec{b}),\ |\vec{\Delta}| = \sqrt{-t} \rightarrow \frac{d\sigma}{dt}(s,t) = \frac{1}{4\pi}|t_{el}(s,t)|^2 = a(s)e^{-|tb(s)|^{\alpha_L(s)/2}}$$

- the model has three adjustable parameters, $\alpha_L$, $a$, and $b$, to be determined at a given energy

# Simple Lévy α-stable model and the data




T. Csörgő, S. Hegyi, I. Szanyi, *Universe 2023, 9(8), 361*




- the non-exponential Lévy α-stable model with $\alpha_L$ = 1.953 $\pm$ 0.004 represents the LHC TOTEM $\sqrt{s} = 8$ TeV low-$|t|$ differential cross section data with a confidence level of 55% (publieshed)
- similarly good description is obtained to all the LHC data on low-|t| pp (and p$\bar{\text{p}}$) $\boldsymbol{d\sigma/dt}$

# Fits with simple Lévy α-stable model

- fits to the existing pp and p$\bar{\text{p}}$ $d\sigma/dt$ data in the kinematic range:

$$546\ \text{GeV} \leq \sqrt{s} \leq 13\ \text{TeV}$$

$$0.02\ \text{GeV}^2 \leq -t \leq 0.15\ \text{GeV}^2$$

- the CL values of the fits range between 8.8% and 96%.

- statistical, systemtic and normalization errors are taken into account using the $\chi^2$ definition developed by PHENIX Collab.

A. Adare *et al.* (PHENIX Collab.) *Phys. Rev. C 77, 064907*

| $\sqrt{s}$, GeV | $\alpha_L$ | $a$, mb/GeV$^2$ | $b$, GeV$^{-2}$ | CL, % |
|---|---|---|---|---|
| 546 | 1.93 ± 0.09 | 209 ± 15 | 15.8 ± 0.9 | 18.1 |
| 1800 | 2.0 ± 1.5 | 270 ± 24 | 16.2 ± 0.2 | 77.1 |
| 2760 | 1.600 ± 0.3 | 637 ± 25 | 28 ± 11 | 20.5 |
| 7000 (T) | 1.95 ± 0.01 | 535 ± 30 | 20.5 ± 0.2 | 8.8 |
| 7000 (A) | 1.97 ± 0.01 | 463 ± 13 | 19.8 ± 0.2 | 96.0 |
| 8000 (T1) | 1.955 ± 0.005 | 566 ± 31 | 20.09 ± 0.08 | 43.86 |
| 8000 (T2) | 1.90 ± 0.03 | 582 ± 33 | 20.9 ± 0.4 | 19.6 |
| 8000 (A) | 1.97 ± 0.01 | 480 ± 11 | 19.9 ± 0.1 | 55.8 |
| 13000 (T1) | 1.959 ± 0.006 | 677 ± 36 | 20.99 ± 0.08 | 76.5 |
| 13000 (T2) | 1.958 ± 0.003 | 648 ± 95 | 21.06 ± 0.05 | 89.1 |
| 13000 (A) | 1.968 ± 0.006 | 569 ± 17 | 20.84 ± 0.07 | 29.7 |

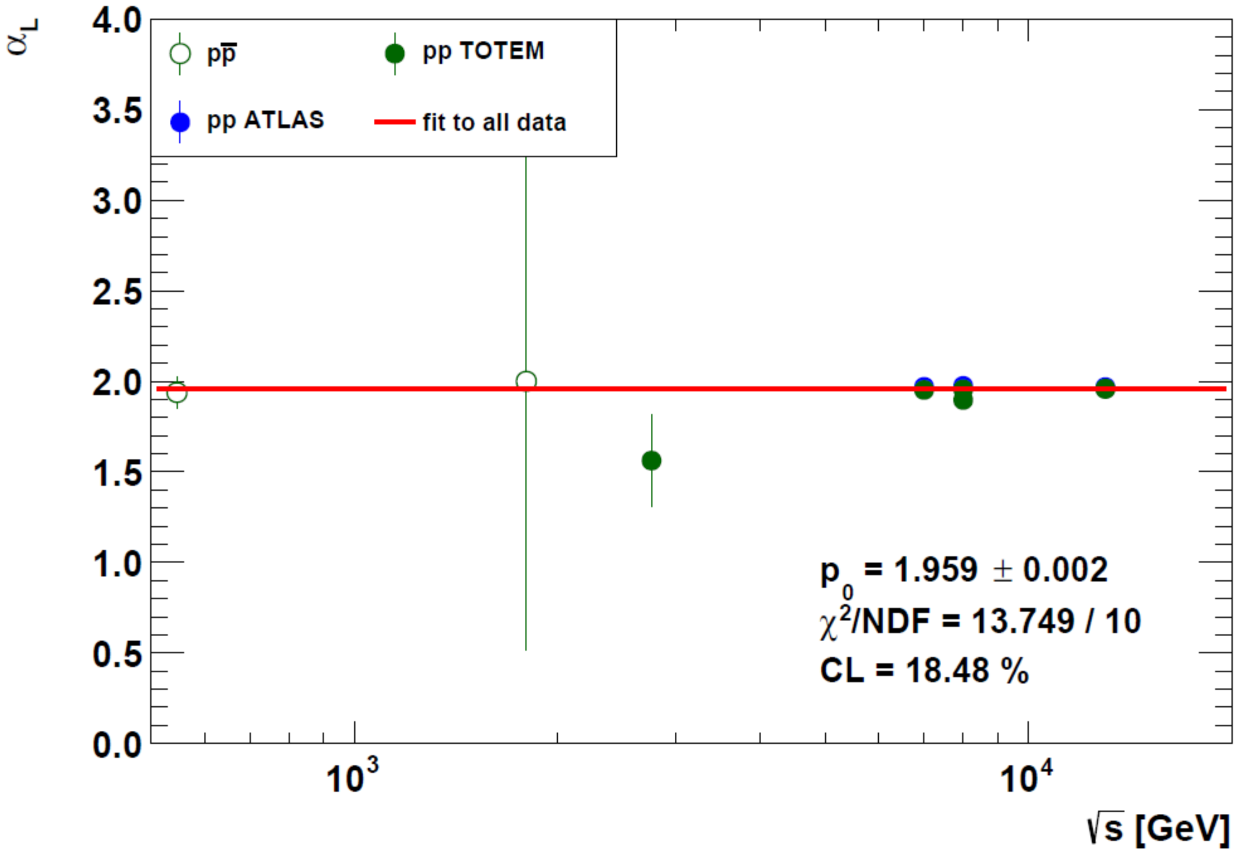
**Values of the fitted parameters of the simple Lévy-α stable model at different energies**

# Energy dependence of the $\alpha_L$ parameter

- the value of the $\alpha_L$ parameter does not depend on energy

- its value is $1.959 \pm 0.002$, i.e., slightly but in a statistical sense significantly different from 2

  $\rightarrow$ strong non-exponential behavior at low $-t$ in the differential cross section, power law tail at high-$\vec{b}$ in $\tilde{\sigma}_{in}(s, \vec{b})$




**Energy dependence of the $\alpha_L$ parameter of the simple Lévy-α stable model**

# Energy dependence of the optical point parameter

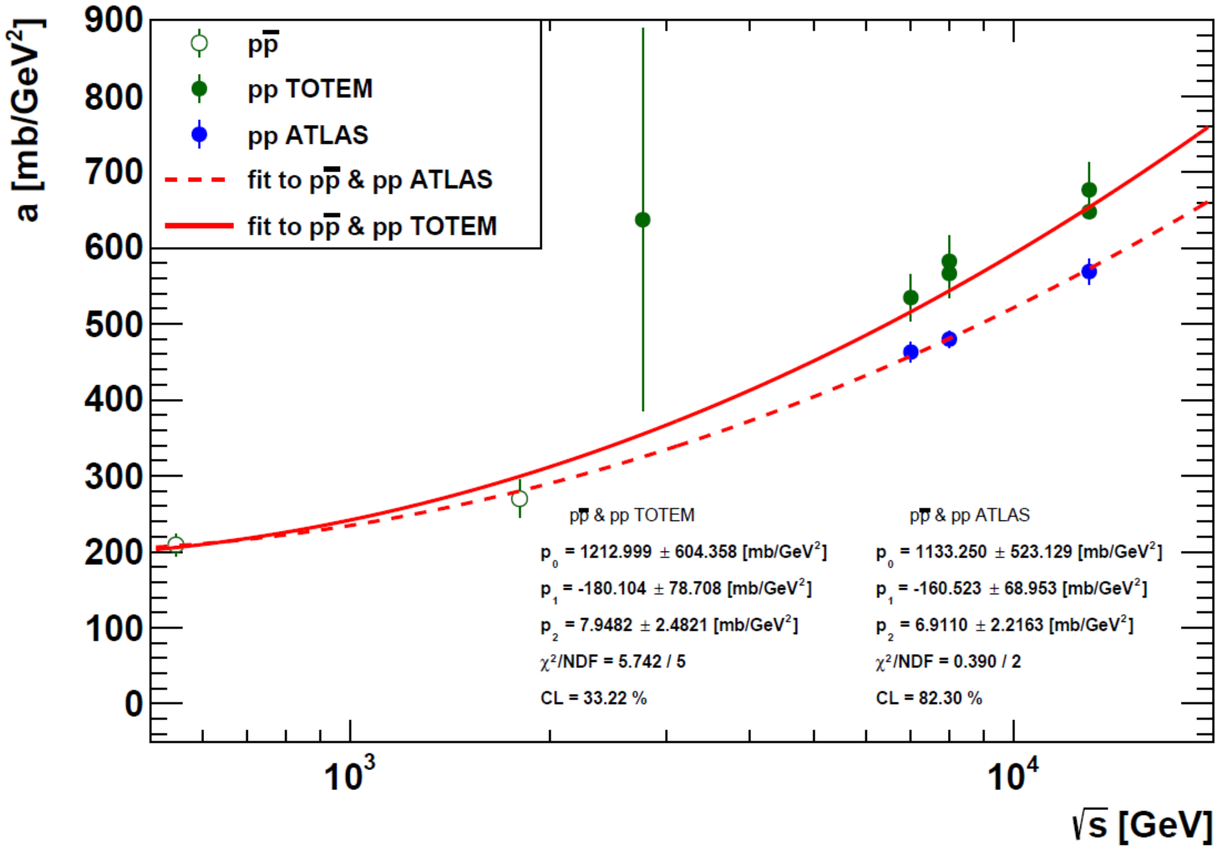
- the energy dependence of the $a$ parameter is quadratically logarithmic:

$$a(s) = p_0 + p_1\, ln \frac{s}{1\; GeV^2} + p_2\, ln^2 \frac{s}{1\; GeV^2}$$

- ATLAS and TOTEM data result sligthly different energy dependences

- reason: ATLAS and TOTEM use different methods to obtain the absolute normalization of the measurements

ATLAS Collab., *Eur. Phys. J. C 83 (2023) 441*




**Energy dependence of the $a$ parameter of the simple Lévy-α stable model**

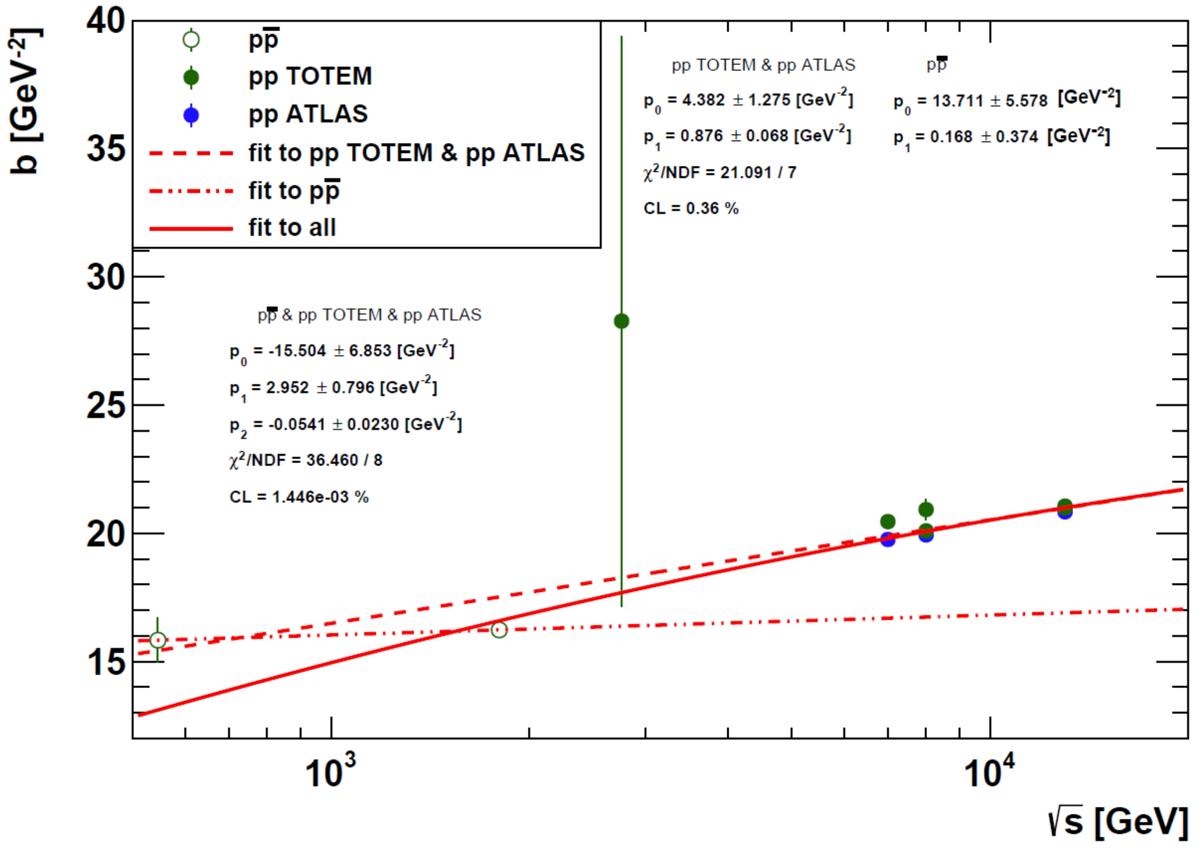
# Energy dependence of the slope parameter

- the energy dependence of the $b$ parameter for TOTEM and ATLAS data together, and for $p\bar{p}$ data alone are linearly logarithmic:

$$b(s) = p_0 + p_1\, ln \frac{s}{1\ GeV^2}$$

- the LHC pp and the lower energy $p\bar{p}$ data do not lie on the same curve
- reason: the slope parameter data have a jump in the energy dependence around 3-4 GeV

TOTEM Collab., *Eur. Phys. J. C (2019) 79:103*




**Energey dependence of the *b* parameter of the simple Lévy-α stable model**

# Simple Lévy α-stable & LBB model parameters

- parameters of the simple Levy α-stable model and the measurable quantites at $t \to 0$ can be apprixmately expressed in terms of the parameters of the LBB model
- only leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ are considered; $A_{qq} = 1$ and $\lambda = 1/2$ are fixed

$$\frac{d\sigma}{dt}(s,t=0) = \; a(s) = \frac{81}{16}\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{4/\alpha_L}\left(1+4\alpha_R^2(s)\right)$$

$$b(s) = \frac{1}{36}\left(\frac{4}{3}\right)^{2/\alpha_L(s)}\left(\left(2+2^{\alpha_L(s)}\right)R_{qd}^{\alpha_L(s)}(s) + 3^{\alpha_L(s)}\left(2R_d^{\alpha_L(s)}(s) + R_q^{\alpha_L(s)}(s)\right)\right)^{2/\alpha_L(s)}$$

(obtained after a Taylor expansion in $t^{\alpha_L/2}$)

$$\sigma_{tot}(s) = 9\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{2/\alpha_L(s)}$$

$$\rho_0(s) = \frac{Re\,t_{el}(s,t=0)}{Im\,t_{el}(s,t=0)} = 2\alpha_R$$

$$\sigma_{el}(s) = \frac{a(s)}{b(s)}\Gamma\left(\frac{2+\alpha_L(s)}{\alpha_L(s)}\right)$$

- According to the analysis of elastic pp and p$\bar{\text{p}}$ data in the energy region $0.5\text{ TeV} \le \sqrt{s} \le 13\text{ TeV}$ only $\alpha_R$ is different for pp and p$\bar{\text{p}}$ scattering

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81****, 611 (2021)*

- in the low-|t| apprximation, difference between pp and p$\bar{\text{p}}$ scattering could be seen in the data on $\boldsymbol{d\sigma/dt}$, $\boldsymbol{\rho_0}$, $\boldsymbol{a}$ (optical point), and $\boldsymbol{\sigma_{el}}$, no difference in the data on $\boldsymbol{\sigma_{tot}}$ and $\boldsymbol{b}$

# Summary

- the formal Lévy α-stable generalization of the Bialas-Bzdak model is done, the $\alpha_L = 2$ limit corresponds to the original model

- solution of difficult and complex technical (mathamatical and computational) problems is needed to fit the experimental data with the generalized model

- based on approximations a highly simplified Levy $\alpha$-stable model of the $\mathrm{pp}$ (and $\mathrm{p\bar{p}}$) differential cross section is deduced and successfully fitted to the data in the low-|t| region

- the energy dependences of the parameters of the simple model are determined; the parameters of the simple model are related to the parameters of the Lévy α-stable generalized real extended Bialas-Bzdak (LBB) model

- final conclusion: the successful fit results indicate promising prospect for the future utility of the LBB model in describing experimental data

# Thank you for your attention!

**SUPPORTED BY:**
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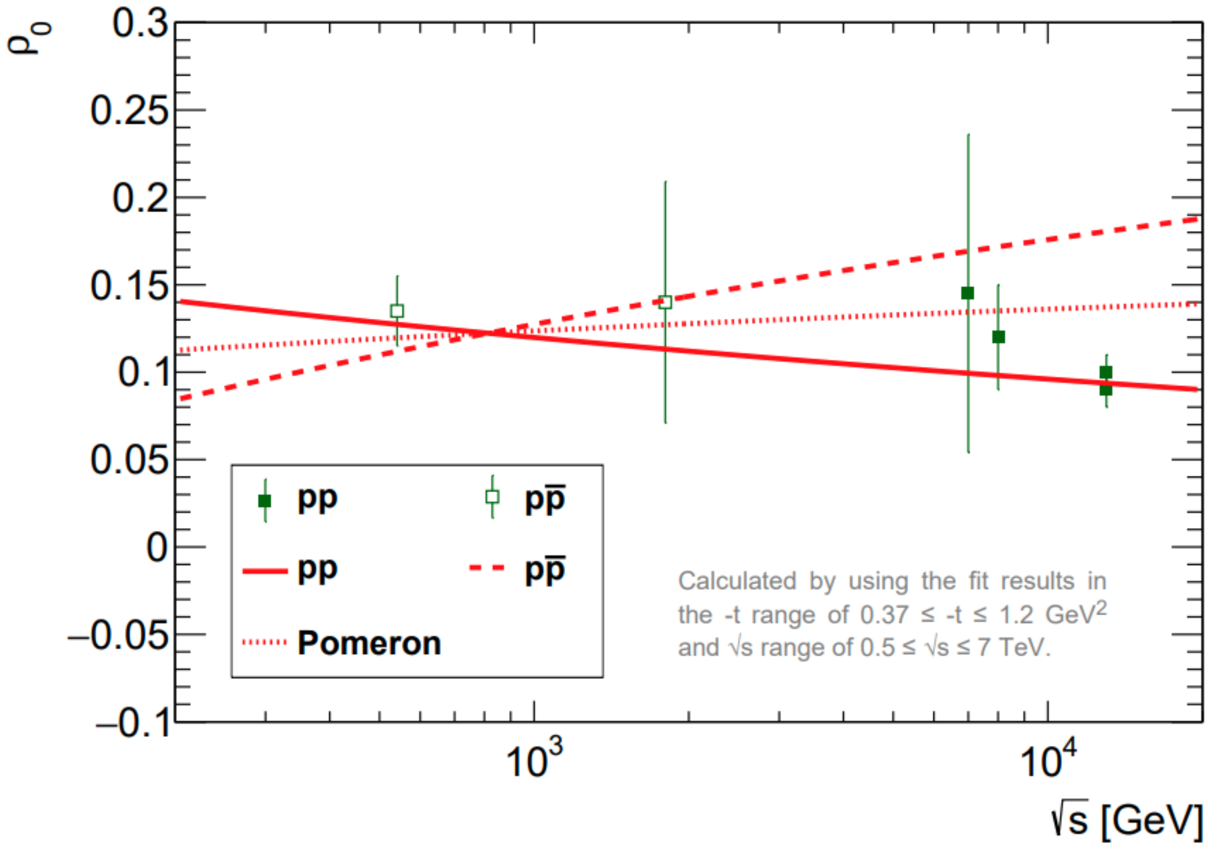
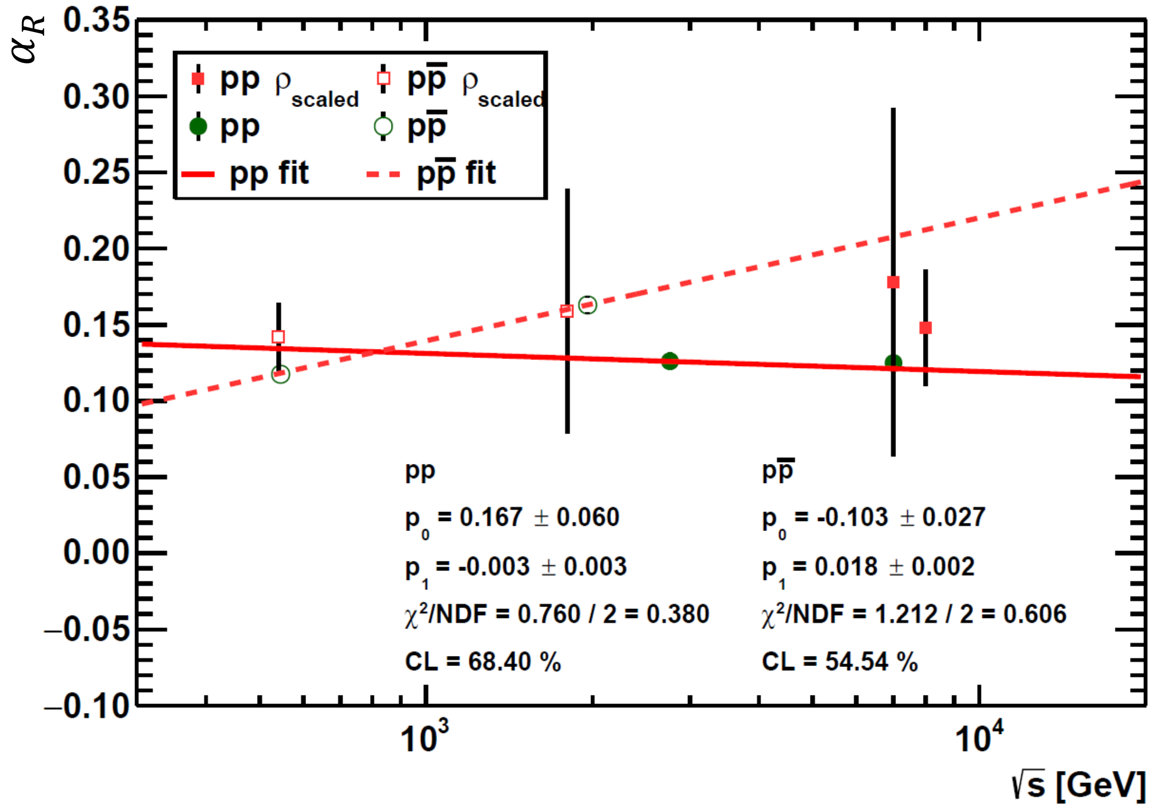
# Backup slides

# $\rho_0$ & $\alpha_R$: connection between $t = 0$ and $t \neq 0$ data

- there is a connection between the $\rho_0$ parameter and the $\alpha_R$ parameter of the ReBB model regulating the real part of the scattering amplitude and the minimum-maximum structure of the $\mathrm{d}\sigma/dt$
- $\alpha_R$ is determined by the $\mathrm{d}\sigma/dt$ data at the minimum-maximum region but at the same time specifies the value of the $\rho_0$ in the ReBB model

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81****, 611 (2021)*



# Most general term in $\tilde{\sigma}_{in}$

$$\tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b}) = \int d^2 s_q d^2 s'_q L\left(\vec{s}_q \middle| R_{qd*}/2\right) L\left(\vec{s}_q' \middle| R_{qd*}/2\right) \times \sigma_{qq}(\vec{s}_q, \vec{s}_q{}'; \vec{b}) \sigma_{qd}(\vec{s}_q, -\lambda \vec{s}_q{}'; \vec{b}) \sigma_{dq}(\vec{s}_q{}', -\lambda \vec{s}_q; \vec{b}) \sigma_{dd}(-\lambda \vec{s}_q, -\lambda \vec{s}_q{}'; \vec{b})$$

$$\sigma_{qq}(\vec{s}_q, \vec{s}_q{}'; \vec{b}) = \pi A_{qq} \left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_q{}' - \vec{s}_q | \alpha, \left(2R_q^\alpha\right)^{1/\alpha}/\sqrt{2}\right)$$

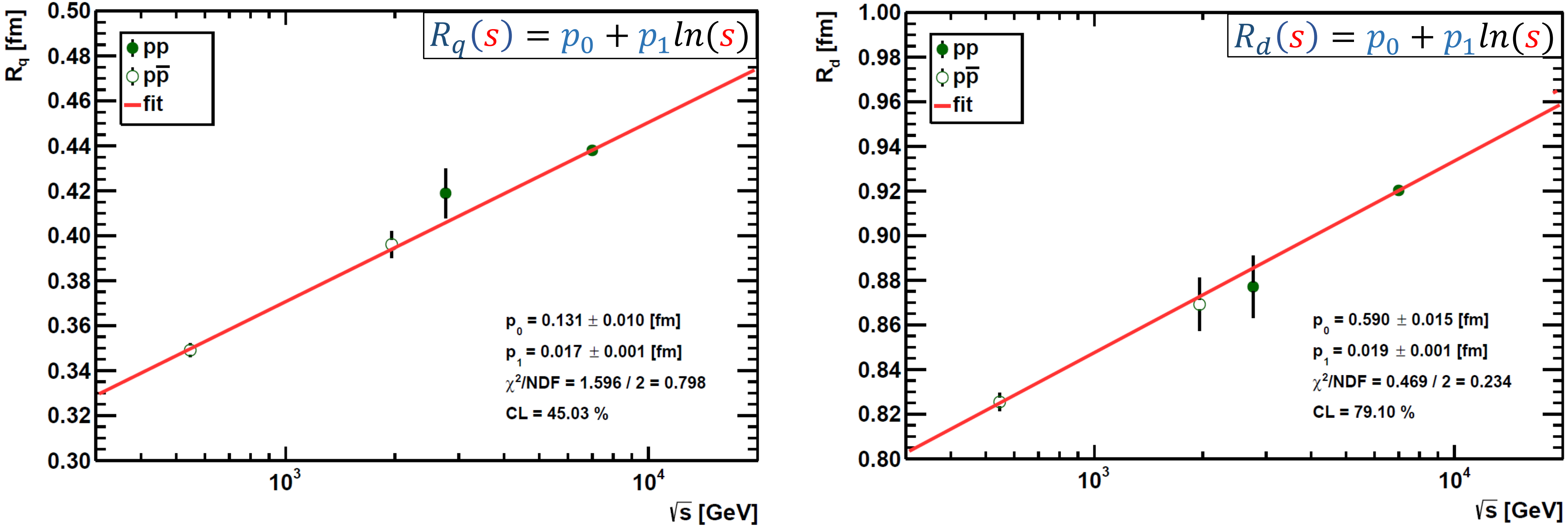
$$\sigma_{qd}(\vec{s}_q, \vec{s}_d'; \vec{b}) = 2\pi A_{qq} \left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_d' - \vec{s}_q \middle| \alpha, \left(R_q^\alpha + R_d^\alpha\right)^{1/\alpha}/\sqrt{2}\right)$$

$$\sigma_{dd}(\vec{s}_d, \vec{s}_d{}'; \vec{b}) = 4\pi A_{qq} \left(2R_q^\alpha\right)^{2/\alpha} \times L(\vec{b} + \vec{s}_d{}' - \vec{s}_d | \alpha, (2R_d^\alpha)^{1/\alpha}/2)$$

$$\sigma_{dq}(\vec{s}_d, \vec{s}_q{}'; \vec{b}) = 2\pi A_{qq} \left(2R_q^\alpha\right)^{2/\alpha} \times L\left(\vec{b} + \vec{s}_q{}' - \vec{s}_d \middle| \alpha, \left(R_q^\alpha + R_d^\alpha\right)^{1/\alpha}/2\right)$$

# Energy dependences of the ReBB model parameters

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)




The energy dependences of the scale parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$ are linear logarithmic and the same for pp and p$\bar{p}$ processes!

The energy dependence of the $\alpha$ parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and p$\bar{p}$ processes!

# Energy dependences of the ReBB model parameters

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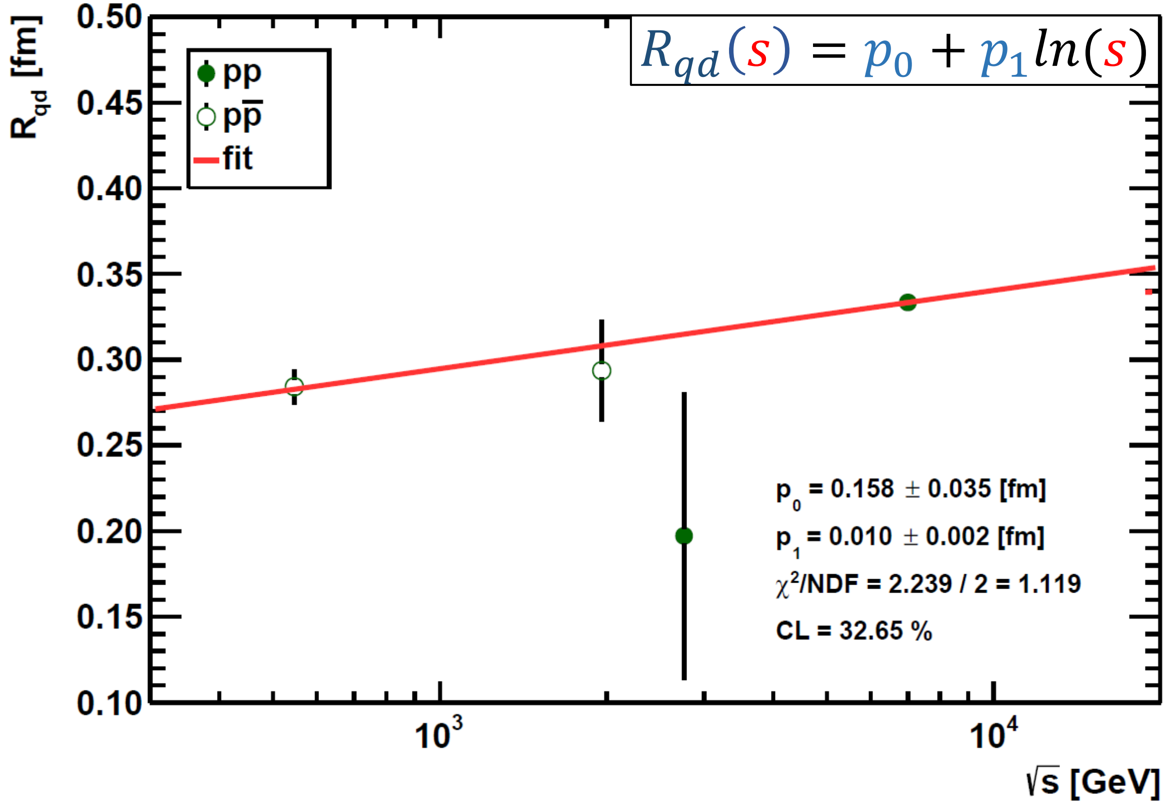


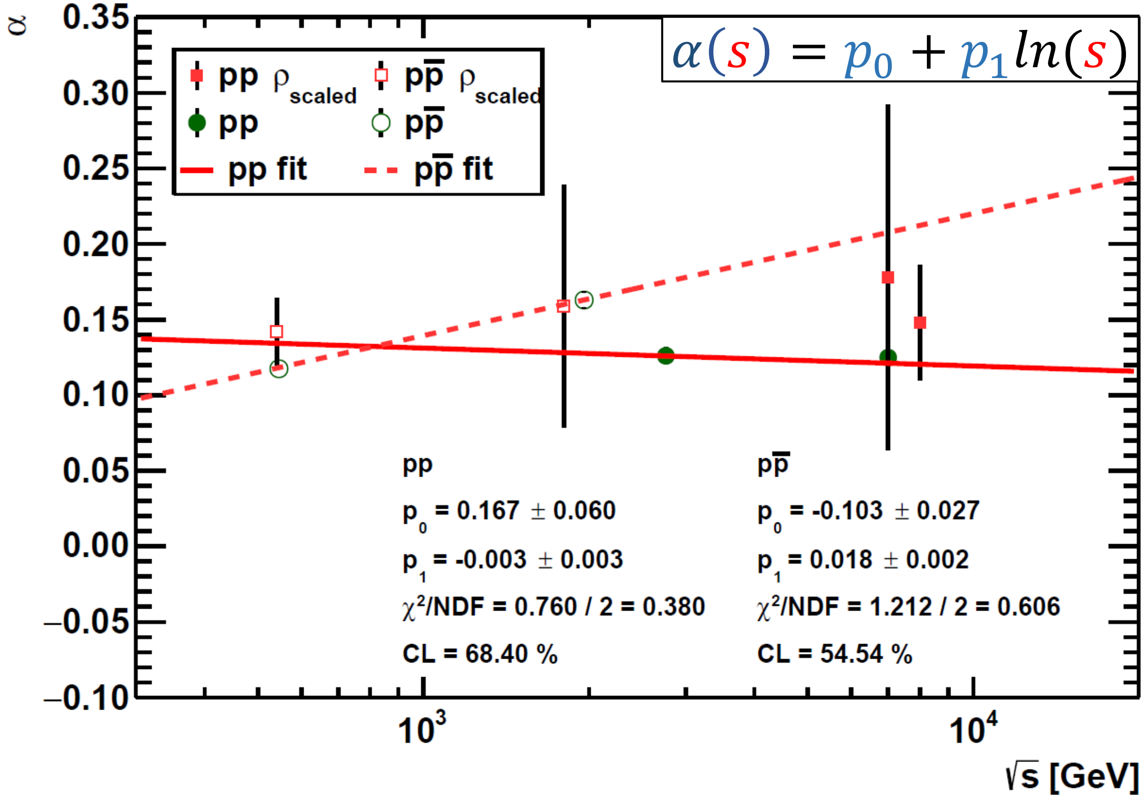


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The energy dependence of the $\alpha$ parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and p$\bar{p}$ processes!

# Fit method

- least squares fitting with the method developed by the PHENIX collaboration
- this method is **equivalent to the diagonalization of the covariance matrix** if the experimental errors are separated into three different types:
  - **type A: point-to-point varying uncorrelated statistical and systematic errors**
  - **type B: point-to-point varying 100% correlated systematic errors**
  - **type C: point-independent, overall systematic uncertainties**
- **i.e least squares fitting with:**

A. Adare *et al.* (PHENIX Collab.) Phys. Rev. C 77, 064907

$$\chi^2 = \left( \sum_{j=1}^{M} \left( \sum_{i=1}^{n_j} \frac{\left(d_{ij} + \epsilon_{bj}\tilde{\sigma}_{bij} + \epsilon_{cj} d_{ij}\sigma_{cj} - th_{ij}\right)^2}{\tilde{\sigma}_{ij}^2} \right) + \epsilon_{bj}^2 + \epsilon_{cj}^2 \right) + \left( \frac{d_{\sigma_{tot}} - th_{\sigma_{tot}}}{\delta\sigma_{tot}} \right)^2 + \left( \frac{d_{\rho_0} - th_{\rho_0}}{\delta\rho_0} \right)^2$$

$$\tilde{\sigma}_{ij}^2 = \tilde{\sigma}_{aij} \left( \frac{d_{ij} + \epsilon_{bj}\tilde{\sigma}_{bij} + \epsilon_{cj} d_{ij}\sigma_{cj}}{d_{ij}} \right)$$

$$\tilde{\sigma}_{kij} = \sqrt{\sigma_{kij}^2 + (d'_{ij}\delta_k t_{ij})^2}, \qquad k \in \{a, b\}$$

- **minimization with CERN Root MINUIT, parameter error estimation by MINOS.**

# Fit method

- the method takes into account (in $M$ separately measured $t$ ranges):
  - the $t$-dependent statistical (type $A$) and systematic (type $B$) errors (both vertical $\sigma_k$ and horizontal $\delta_k t$) → $\epsilon_b$ parameters;
  - the $t$-independent $\sigma_c$ normalization uncertainties (type $C$) → $\epsilon_c$ parameters;
  - the measured total cross-section $d_{\sigma_{tot}}$ and ratio $d_{\rho_0}$ and their total uncertainties $\delta\sigma_{tot}$ and $\delta\rho_0$.

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  - the measured total cross-section $d_{\sigma_{tot}}$ and ratio $d_{\rho_0}$ and their total uncertainties $\delta\sigma_{tot}$ and $\delta\rho_0$.

**$\epsilon_i$-s must be considered as both measurements and fit parameters not effecting the NDF** (since they have known central value of zero and known standard deviation of one)

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**The PHENIX method is validated by evaluating the χ2 from a full covariance matrix fit of the √s = 13 TeV TOTEM differential cross-section data using the Lévy expansion method from T. Csörgő, R. Pasechnik, & A. Ster, Eur. Phys. J. C 79, 62 (2019).**

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**The PHENIX method and the fit with the full covariance matrix result in the same minimum within one standard deviation of the fit parameters.**

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- **minimization with CERN Root MINUIT, parameter error estimation by MINOS.**

# Proportionality between $\rho_0(s)$ and $\alpha(s)$

$$t_{el}(s,b) = i\left(1 - e^{i\,\alpha\,\tilde{\sigma}_{in}(s,b)}\sqrt{1-\tilde{\sigma}_{in}(s,b)}\right)$$

$$\alpha\tilde{\sigma}_{in} \ll 1$$

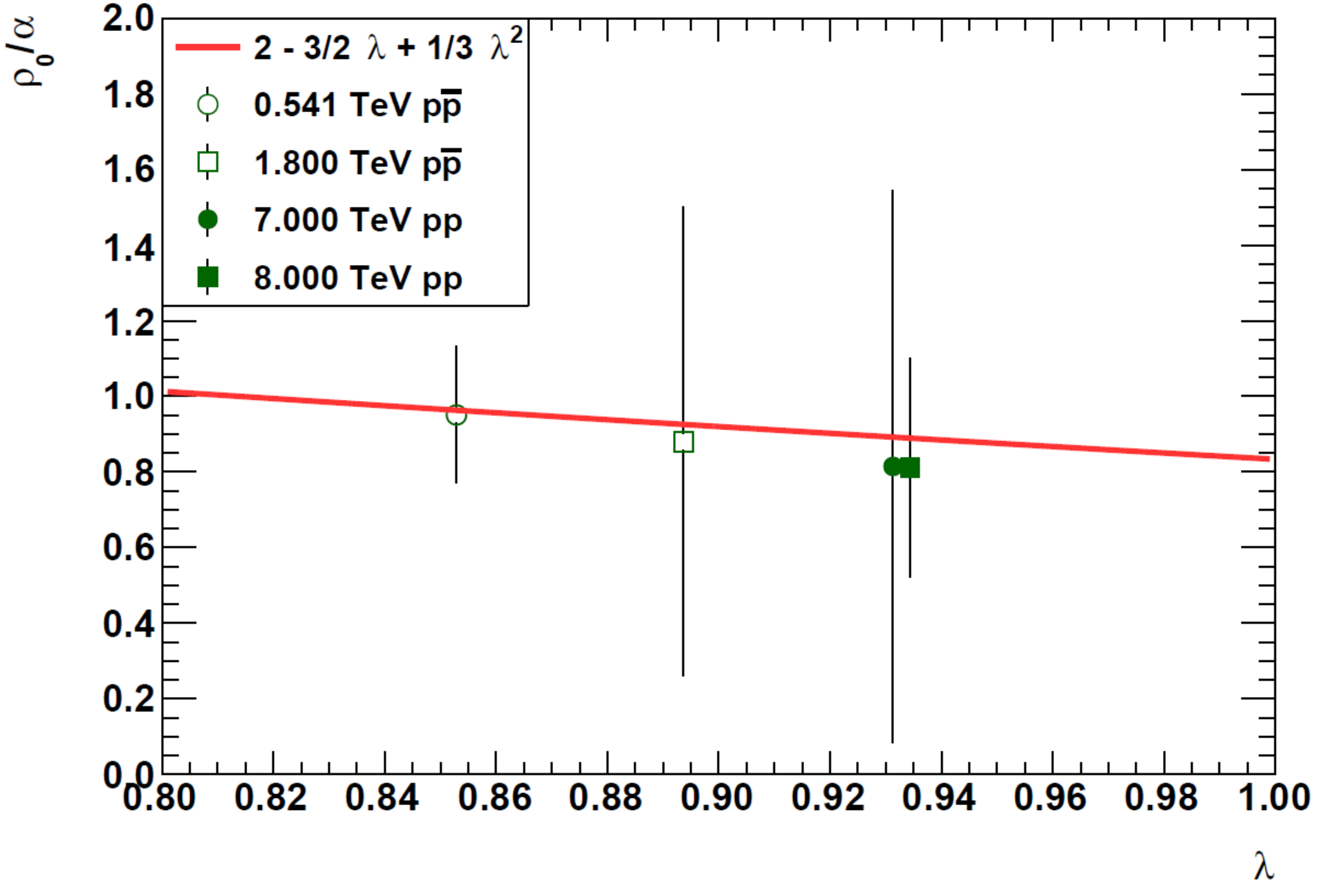
$$\mathrm{Im}\, t_{el}(s,b) \simeq \lambda(s)\exp\left(-\frac{b^2}{2R^2(s)}\right)$$

↓

$$\rho_0(s) = \alpha(s)\left(2 - \frac{3}{2}\lambda(s) + \frac{1}{3}\lambda^2(s)\right)$$

$$\lambda(s) = \mathrm{Im}\, t_{el}(s, b=0)$$

→ by rescaling one can get additional $\alpha$ parameter values at energies where $\rho_0$ is measured (and vice versa)




**The dependence of $\rho_0/\alpha$ on $\lambda = \mathrm{Im}\, t_{el}(s, b=0)$ in the TeV energy range. The data points are generated numerically by using the trends of the ReBB model scale parameters and the experimentally measured $\rho$-parameter values.**

# Measurable quantities

- **differential cross section:**

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{4\pi}|T(s,t)|^2$$

- **total, elastic and inelastic cross sections:**

$$\sigma_{tot}(s) = 2ImT(s,t=0)$$

$$\sigma_{el}(s) = \int_{-\infty}^{0} \frac{d\sigma(s,t)}{dt} dt$$

$$\sigma_{in}(s) = \sigma_{tot}(s) - \sigma_{el}(s)$$

- **ratio $\rho_0$:**

$$\rho_0(s) = \lim_{t\to 0} \rho(s,t) \equiv \frac{ReT(s,t\to 0)}{ImT(s,t\to 0)}$$

- **slope of dσ/dt:**

$$\mathrm{B}(\mathrm{s},\mathrm{t}) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\ln \frac{d\sigma}{\mathrm{dt}}(\mathrm{s},\mathrm{t})\right)$$

$$B_0(s) = \lim_{t\to 0} \mathrm{B}(\mathrm{s},\mathrm{t})$$