A novel method for calculating Bose-Einstein correlation functions with Coulomb final-state interaction

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Based on: arXiv:2308.10745; just about to be published in EPJ C.

Day of Femtoscopy 2023, Gyöngyös
October 30, 2023
Outline

- **Introduction**
  - HBT correlations, Coulomb effect, basic formulas
  - Need for refinement: non-Gaussian sources, precision measurements
  - Numerical & methodological motivation

- **New method for treatment of Coulomb interaction**
  - Calculation of the Coulomb integral kernel
  - Rigorous mathematics needed
  - Spherically symmetric case: limiting functional expressed
  - Implementation; esp. for Lévy-type sources

- **Outlook**
  - Ready to use in experimental analyses
  - Generalizations: non-spherically symmetric case, strong interaction
Bose-Einstein-correlations of like particles ($\pi^+\pi^+$, $\pi^-\pi^-$, $K^+K^+$...): measure fm-scale space-time extent of particle emitting source

Some definitions:

- **source function:** $S(x, p)$
- **single part. distr.:** $N_1(p) = \int dx \, S(x, p)$
- **pair wave function:** $\psi^{(2)}(x_1, x_2)$
- **pair mom. distr.:** $N_2(p_1, p_2) = \int dx_1 dx_2 \, S(x_1, p_1) S(x_2, p_2) |\psi^{(2)}(x_1, x_2)|^2$
- **corr. function:** $C(p_1, p_2) = \frac{N_2(p_1, p_2)}{N_1(p_1) N_1(p_2)}$
- **pair source:** $D(r, K) = \int d^4 \rho \, S(\rho + \frac{r}{2}, K) S(\rho - \frac{r}{2}, K)$

Approximately thus

$$C(k, K) = \frac{\int D(r, K) |\psi_k(r)|^2 dr}{\int D(r, K) dr}, \quad K := \frac{p_1 + p_2}{2}, \quad k := \frac{p_1 - p_2}{2}.$$
Core-halo model intercept parameter $\lambda$: $S = \sqrt{\lambda}S_c + (1 - \sqrt{\lambda})S_h$

$S_h$,”large”: $\Rightarrow$ observable $C(k, K) = 1 - \lambda + \lambda \frac{\int D_c(r, K) |\psi_k(r)|^2 dr}{\int D_c(r, K) dr}$.

No final state interactions: $C(k) \equiv C^{(0)}(k)$, Fourier transform of source

$|\psi^{(0)}_k(r)|^2 = 1 + \cos(2kr) \Rightarrow C^{(0)}(k) = 1 + \lambda \frac{\int D_c(r, K) \cos(2kr) dr}{\int D_c(r, K) dr}$.

Final state Coulomb interaction: $\psi^{(0)}$ replaced by solution of two-body Coulomb Schr. eq.; NR case: well known formulas (see below)

$C^{(0)}(k) = \frac{C(k)}{K(k)}$, $K(k) \equiv \frac{\int D_c(r) |\psi_k(r)|^2 dr}{\int D_c(r) |\psi^{(0)}_k(r)|^2 dr}$ Coulomb correction

Final state strong interaction: small (?) for $\pi\pi, KK$

Usual treatment: only s-wave (1 parameter: strong scattering length $f_0$)
Source types

- **Gaussian**: usual choice; \( D_{cc}(r) \propto \exp(-r_{kl}R_{kl}^{-1}) \).
  - Fit parameters: \( R_{kl}(K) \) and \( \lambda(K) \)
  - A generalization: Edgeworth expansion of \( C(k) \); in this source: FT of \( C^{(0)}(k) \)

- **Lévy-type sources**: generalized Gaussian; new parameter \( \alpha \in \mathbb{R}^+ \): stability index; \( \alpha \leq 2 \).
  - Expressed as a Fourier transform:
    \[
    D_{cc}(r) = \frac{1}{(2\pi)^3} \int d^3 q \, e^{iqr} \exp(-|qR|^\alpha).
    \]
  - Arises in natural processes: stability property (just as for Gaussian)
  - Generalization: Levy polynomials (same as Edgeworth for Gaussians)

- **Cauchy sources**: \(\Leftrightarrow\) exponential \( C(k) \): special case of Levy (\( \alpha = 1 \))
  - employed at CMS for HBT in p+p collisions...
Illustration of Lévy sources

- $\alpha=2$: Gaussian, $\alpha=1$: Cauchy distribution
- For $\alpha \neq 2$, power law like $r \to \infty$ decrease ($\sim r^{-3-\alpha}$); no finite variance

For such sources, $C_2^{(0)}(Q) = 1 + \lambda \exp(-|QR|^\alpha)$ easy, $D_{cc}(r)$ source itself calculable only numerically (for $\alpha \neq 1, 2$)
Lévy sources in heavy ion collisions

- **Non-Gaussian behavior:**
  - Model independent source extraction („imaging“)
  - PHENIX, PRL 98 (2007) 132301

- **PHENIX measurement with Lévy assumption**
  - $\alpha \neq 2$ confirmed $m_t$-independently

- **Coulomb effect:** an essential ingredient
Introduction

Coulomb interaction

- Non-relativistic treatment: valid in Pair Co-Moving System (PCMS).

- \( p = \hbar k \): relative momentum, \( E = \frac{p^2}{2m} \), \( m \): reduced mass

- Sommerfeld parameter (Coulomb parameter) \( \eta \): ratio of classical closest distance \( r_0 \equiv \frac{q_e^2}{4\pi\varepsilon_0} \frac{1}{E} \) to wavelength \( \lambda \equiv \frac{2\pi\hbar}{p} \):

\[
\eta \equiv \alpha_{em} \frac{mc}{\hbar k} = \frac{\pi r_0}{\lambda}, \quad \text{with} \quad \alpha_{em} \equiv \frac{q_e^2}{4\pi\varepsilon_0} \frac{1}{\hbar c} \approx \frac{1}{137}.
\]

- Two-particle wave function: symmetrized scattering ,,out” state
  - ,,out” states asymptotically plane wave + incoming spherical wave
  - alternate ,,in” state (plane wave + outgoing spherical wave) yields same results

\[
\psi^{(C)} = e^{iKR} \times \frac{\mathcal{N}^*}{\sqrt{2}} e^{-ikr} \left\{ M(1-i\eta, 1, i(kr+k)) + (k \leftrightarrow -k) \right\}.
\]

Making use of the \( M(a, b, z) \) confluent hypergeometric function

- Normalization (\( \mathcal{N} \)) and Gamow factor (\( |\mathcal{N}|^2 \)):

\[
\mathcal{N} = e^{-\pi\eta/2} \Gamma(1+i\eta), \quad |\mathcal{N}|^2 = e^{-\pi\eta} |\Gamma(1+i\eta)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}. \quad (1)
\]
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections

\[ |\psi^{(0)}|^2, k = 50 \text{ MeV/c} \]

\[ |\psi^{\text{(Coulomb)}}|^2, k = 50 \text{ MeV/c} \]
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- Gamow correction captures only the value at the origin

Calculational methods:
- Direct integrating
- Pre-calculate a ,,Coulomb correction" with fix parameters (say, $R = 5$ fm Gaussian): fast but inconsistent
- Use iterative method, use memory lookup table...
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- Calculational methods:
  - Direct integrating $D(r)|\psi_k^{(2)}(r)|^2$ during fit: time-consuming, even nowadays
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New method for calculating the Coulomb effect

New method needed

- Many (if not all) interesting source types defined as Fourier transforms
  \[
  D(r) := \int \frac{d^3 q}{(2\pi)^3} f(q) e^{iqr} \Leftrightarrow f(q) = \int d^3 r \, D(r) e^{-irq}
  \]

- In many cases (eg. Lévy sources), even this is possible only numerically

- Direct numerical calculation of \( C_2(Q) \) thus (although used) **very** problematic
  - Slow decrease of \( D(r) \), oscillating asymptotic \( \psi^{(2)}_k(r) \)
  - Awkward: Fourier transform, then ,,almost inverse” Fourier transform, *numerically*

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Not working in this form: Fourier transform \( \neq \) integral (Lebesgue)

Workaround: regularization, \( \lambda \in \mathbb{R}^+ \), then \( \lambda \to 0 \).

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\[ C_2(Q) \]
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= \lim_{\lambda \to 0} \int d^3 r \ e^{-\lambda r} \left| \psi_k^{(2)}(r) \right|^2 D(r)
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Details of derivation

- Interchanging our integrals in a careful way:

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- "Ingredients":

- Conditions of (Lebesgue) integrability
- Lebesgue theorem (for interchanging integrals and limits)
- Fubini's theorem (for interchanging repeated integrals)

In last step, cannot interchange $\int d^3 q$ and $\lim_{\lambda \to 0}$. As of now, continuing only in the spherically symmetric case: $f(q) \equiv f_s(q)$, $D_{cc}(r) = 2\pi R_\infty \int_0 d^3 q q^2 \sin(qr) f_s(q)$.
Details of derivation

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- „Ingredients“:
- Conditions of (Lebesgue) integrability
New method for calculating the Coulomb effect

Details of derivation

- Interchanging our integrals in a careful way:

\[ C_2(Q) = \int d^3r \ |\psi_k^{(2)}(r)|^2 D(r) = \int d^3r \ \lim_{\lambda \to 0} e^{-\lambda r} |\psi_k^{(2)}(r)|^2 D(r) = \]

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Details of derivation

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Details of derivation

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Details of derivation

Interchanging our integrals in a careful way:

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,,Ingredients”:

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As of now, continuing only in the spherically symmetric case:

\[ f(q) \equiv f_s(q), \ D_{cc}(r) = 2\pi \int_0^{\infty} \text{d} q \, q^2 \sin(qr)f_s(q). \]
Details of derivation (cont’d)

- After substituting $\psi_k^{(2)}(r)$, „master” formula thus reads as

$$C_2(Q) = \frac{|N|^2}{2\pi^2} \lim_{\lambda \to 0} \int_0^\infty dq \, q^2 f_s(q) \left[ D_{1s}(q) + D_{2s}(q) \right],$$

where

$$D_{1s}(q) = \int d^3r \frac{\sin(qr)}{qr} e^{-\lambda r} M(1+i\eta, 1, -i(kr+kr)) M(1-i\eta, 1, i(kr+kr)),
$$

$$D_{2s}(q) = \int d^3r \frac{\sin(qr)}{qr} e^{-\lambda r} M(1+i\eta, 1, -i(kr-kr)) M(1-i\eta, 1, i(kr+kr)).$$

- These can be calculated (using complex analysis; method pioneered by Nordsieck in the theory of bremsstrahlung & pair creation)


$$D_{1s}(q) = \frac{4\pi}{q} \text{Im} \left[ \frac{1}{(\lambda-iq)^2} \left( 1 + \frac{2k}{q+i\lambda} \right)^{2i\eta} \mathcal{F}_+ \left( \frac{4k^2}{(q+i\lambda)^2} \right) \right],$$

$$D_{2s}(q) = \frac{4\pi}{q} \text{Im} \left[ \frac{(\lambda-iq-2ik)^i\eta (\lambda-iq+2ik)^{-i\eta}}{(\lambda-iq)^2+4k^2} \right].$$

Here $\mathcal{F}_+(x) \equiv {}_2F_1(i\eta, 1+i\eta, 1, x)$ is the hypergeometric function.
The main result

- For $\lim_{\lambda \to 0}$, function forms of $D_{1\lambda s}, D_{2\lambda s}$ become ,,ill-behaved''
  
  $\textit{(Remark: a simple well known similar case is the approximation of} \, \delta(x) \, \text{Dirac delta with smooth peaked functions)}$

- Need to calculate & simplify $\lambda \to 0$ limit (numerical limit-taking . . . $\Rightarrow$)
  $\Rightarrow$ result: $\textit{functional}$, not a proper integral transform of $f_s(q)$

- Result of the calculation:

\[
C_2(Q)=|N|^2 \left(1+f_s(2k)+\frac{\eta}{\pi} \left[ A_{1s} + A_{2s} \right] \right), \text{ where }
\]

\[
A_{1s} = -\frac{2}{\eta} \int_0^\infty dq \frac{f_s(q) - f_s(0)}{q} \text{Im} \left[ \left(1+\frac{2k}{q}\right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0\right) \right],
\]

\[
A_{2s} = -\frac{2}{\eta} \int_0^\infty dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \text{Im} \left(\frac{q+2k}{q-2k+i0}\right)^{i\eta}.
\]

- $\eta \to 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)

- $|N|^2$ factor only: Gamow correction
  $\Rightarrow A_{1s}, A_{2s}$ ,,correct the Gamow correction''

- $A_{1s}$ and $A_{2s}$: well-defined functionals of $f_s(q)$

- Care needed about branch cuts ($\pm i0$ terms) of $\mathcal{F}_+(x)$ and complex powers
Numerical implementation

- Final numerical integrals needed: in $A_{1\lambda s}$ and $A_{2\lambda s}$
- Transform integral to $x \in [0, 1] \Rightarrow$ smooth, „beautiful” integrands
- Gauss-Krohnrod quadrature (from C++ boost library) used:
  - Main parameters: # of max iterations & tolerance
  - Investigated; optimal value found: few hundred integrand evaluations (instead of many 10000-s)

Real-time calculation (during fit procedure) possible!

Codes archived at: github.com/csanadm/CoulCorrLevyIntegral
Example calculations: illustrations

- For Lévy sources, for pion \((\pi^+ \pi^+, \pi^- \pi^-)\) pairs:

\[
\begin{align*}
\alpha \in (0.6, 2.0) \\
\textcolor{red}{R = 3 \text{ fm}} & \quad \textcolor{magenta}{R = 6 \text{ fm}} \\
\textcolor{blue}{R = 9 \text{ fm}} & \quad \textcolor{cyan}{R = 12 \text{ fm}} \\
\textcolor{red}{\alpha = 2} \\
\textcolor{blue}{\alpha = 0.6} \\
\pi^+ \pi^+
\end{align*}
\]

- most frequent target of HBT measurements
- Shaded region „swept” over by \(C_2(Q)\) as \(\alpha\) changes
- Apparent „nodes” disappear with increased zooming in
Example calculations: illustrations

- For Lévy sources, for kaon \((K^+K^+, K^-K^-)\) pairs:

  \[
  C_2(Q) = 2 \alpha \\
  \alpha \in (0.6, 2.0) \\
  R = 3 \text{ fm} \quad R = 6 \text{ fm} \\
  R = 9 \text{ fm} \quad R = 12 \text{ fm} \\
  \alpha = 2 \\
  \alpha = 0.6
  \]

- Similarly to the case of pions; „nodes” are only apparent
- Coulomb effect stronger \((m_K > m_\pi; \eta \text{ increases})\)
- Considerable interplay of experimentally measurable \(\lambda, R, \alpha\)
Summary and outlook

- Efficient new method Coulomb interacting HBT correlation function calculation
  - Calculations directly in momentum (Fourier) space
  - Careful mathematical methods invoked, distribution theory motivated
  - Cross-checked with previous direct calculations
  - Numerical implementation done, ready for use in data analysis

- As of now, implementation only for spherically symmetric sources
  - Prospective generalization I: go beyond spherical symmetry
    This is where efficiency becomes crucial...
  - Prospective generalization II: short-range strong interactions
  - Prospective generalization (in fact, simplification) for non-identical particle correlations: only $D_{1\lambda s}$ (ie. $A_{1s}$) term needed

***

New exact analytic formulas for QM Coulomb problem! 😊

Thank you for your attention!