A novel method for calculating Bose-Einstein correlation functions with Coulomb final-state interaction

Márton Nagy, Máté Csanád, Aletta Purzsa, Dániel Kincses
(Eötvös University, Budapest)

Based on: arXiv:2308.10745; just about to be published in EPJ C.

Day of Femtoscopy 2023, Gyöngyös
October 30, 2023

Outline

- Introduction
  - HBT correlations, Coulomb effect, basic formulas
  - Need for refinement: non-Gaussian sources, precision measurements
  - Numerical & methodological motivation
- New method for treatment of Coulomb interaction
  - Calculation of the Coulomb integral kernel
  - Rigorous mathematics needed
  - Spherically symmetric case: limiting functional expressed
  - Implementation; esp. for Lévy-type sources
- Outlook
  - Ready to use in experimental analyses
  - Generalizations: non-spherically symmetric case, strong interaction

Introduction

- Bose-Einstein-correlations of like particles ($\pi^+\pi^+$, $\pi^-\pi^-$, $K^+K^+\dots$): measure fm-scale space-time extent of particle emitting source
- Some definitions:

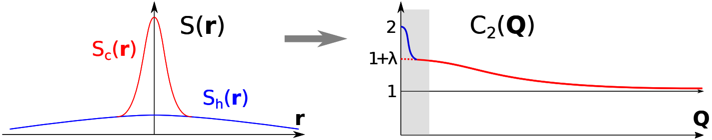
source function:	$S(x, \mathbf{p})$
single part. distr.:	$N_1(\mathbf{p}) = \int \mathrm{d}x\, S(x, p)$
pair wave function:	$\psi^{(2)}(x_1, x_2)$
pair mom. distr.:	$N_2(\mathbf{p}_1, \mathbf{p}_2) = \int \mathrm{d}x_1 dx_2\, S(x_1, p_1) S(x_2, p_2) \left\vert\psi^{(2)}(x_1, x_2)\right\vert^2$
corr. function:	$C(\mathbf{p}_1, \mathbf{p}_2) = \frac{N_2(\mathbf{p}_1, \mathbf{p}_2)}{N_1(\mathbf{p}_1) N_1(\mathbf{p}_2)}$
pair source:	$D(\mathbf{r}, \mathbf{K}) = \int \mathrm{d}^4\rho\, S\left(\rho + \frac{r}{2}, \mathbf{K}\right) S\left(\rho - \frac{r}{2}, \mathbf{K}\right)$

- Approximately thus

$$C(\mathbf{k}, \mathbf{K}) = \frac{\int D(\mathbf{r}, \mathbf{K}) \vert\psi_{\mathbf{k}}(\mathbf{r})\vert^2 \mathrm{d}\mathbf{r}}{\int D(\mathbf{r}, \mathbf{K}) \mathrm{d}\mathbf{r}}, \qquad \mathbf{K} := \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, \quad \mathbf{k} := \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}.$$

Introduction

- Core-halo model intercept parameter λ: $S = \sqrt{\lambda} S_c + (1 - \sqrt{\lambda}) S_h$



$$S_h \text{ „large"}: \quad \Rightarrow \quad \text{observable } C(\mathbf{k},\mathbf{K}) = 1 - \lambda + \lambda \frac{\int D_c(\mathbf{r},\mathbf{K}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 \mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r},\mathbf{K}) \mathrm{d}\mathbf{r}}.$$

- No final state interactions: $C(\mathbf{k}) \equiv C^{(0)}(\mathbf{k})$, Fourier transform of source

$$\left|\psi^{(0)}_{\mathbf{k}}(\mathbf{r})\right|^2 = 1 + \cos(2\mathbf{kr}) \quad \Rightarrow \quad C^{(0)}(\mathbf{k}) = 1 + \lambda \frac{\int D_c(\mathbf{r},\mathbf{K}) \cos(2\mathbf{kr}) \mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r},\mathbf{K}) \mathrm{d}\mathbf{r}}.$$

- Final state Coulomb interaction: $\psi^{(0)}$ replaced by solution of two-body Coulomb Schr. eq.; NR case: well known formulas (see below)

$$C^{(0)}(\mathbf{k}) = \frac{C(\mathbf{k})}{K(\mathbf{k})}, \quad K(\mathbf{k}) \equiv \frac{\int D_c(\mathbf{r}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 \mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r}) |\psi^{(0)}_{\mathbf{k}}(\mathbf{r})|^2 \mathrm{d}\mathbf{r}} \quad \text{Coulomb correction}$$

- Final state strong interaction: small (?) for $\pi\pi$, KK
  Usual treatment: only s-wave (1 parameter: strong scattering length f_0)

Source types

- Gaussian: usual choice; $D_{cc}(\mathbf{r})\propto\exp\left(-r_k r_l R^{-1}_{kl}\right)$.
  - Fit parameters: $R_{kl}(\mathbf{K})$ and $\lambda(\mathbf{K})$
  - A generalization: Edgeworth expansion of $C(k)$; in thisource: FT of $C^{(0)}(k)$

    see eg. T. Csörgő, S. Hegyi, Phys. Lett. B **489**, 15 (2000)
- Lévy-type sources: generalized Gaussian; new parameter $\alpha\in\mathbb{R}^+$: stability index; $\alpha\leq 2$. Expressed as a Fourier transform:

$$D_{cc}(\mathbf{r})=\frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{q}\, e^{i\mathbf{qr}}\exp\left(-|\mathbf{q}R|^{\alpha}\right).$$

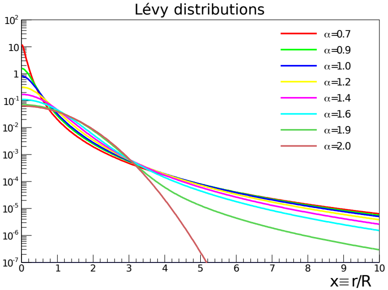
  - Arises in natural processes: stability property (just as for Gaussian)
  - Generalization: Levy polynomials (same as Edgeworth for Gaussians)

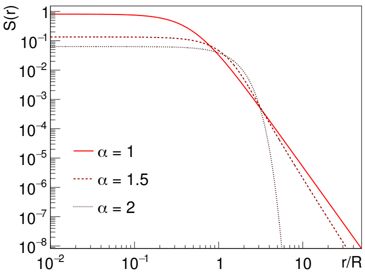
    T. Novák, *et al.*, Acta Phys. Polon. Supp. **9**, 289 (2016)
- Cauchy sources $\Leftrightarrow$ exponential $C(k)$: special case of Levy ($\alpha=1$)

employed at CMS for HBT in p+p collisions...

Illustration of Lévy sources

- α=2: Gaussian, α=1: Cauchy distribution
- For $\alpha \neq 2$, power law like $r \to \infty$ decrease ($\sim r^{-3-\alpha}$); no finite variance

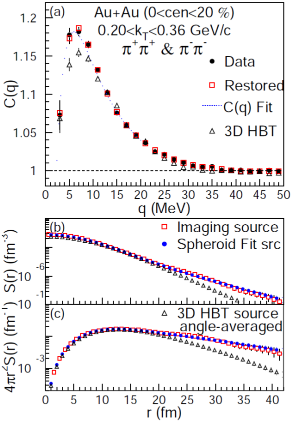




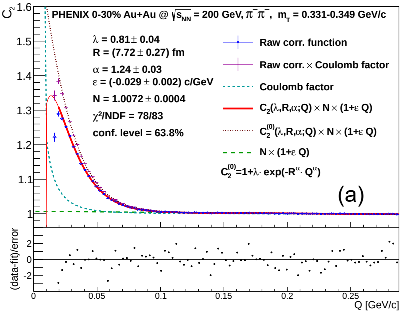
- For such sources, $C_2^{(0)}(\mathbf{Q}) = 1 + \lambda \exp\left(-|QR|^\alpha\right)$ easy, $D_{cc}(\mathbf{r})$ source itself calculable only numerically (for $\alpha \neq 1, 2$)

Lévy sources in heavy ion collisions

- Non-Gaussian behavior:
  - Model independent source extraction (,,imaging")
  - PHENIX, PRL 98 (2007) 132301



- PHENIX measurement with Lévy assumption
  - $\alpha \neq 2$ confirmed m_t-independently



- Coulomb effect: an essential ingredient

Coulomb interaction

- Non-relativistic treatment: valid in Pair Co-Moving System (PCMS).
- $\mathbf{p}=\hbar\mathbf{k}$: relative momentum, $E=\frac{p^2}{2m}$, m: reduced mass
- Sommerfeld parameter (Coulomb parameter) η: ratio of classical closest distance $r_0\equiv\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{E}$ to wavelength $\lambda\equiv\frac{2\pi\hbar}{p}$:

$$\eta\equiv\alpha_{\rm em}\frac{mc}{\hbar k}=\frac{\pi r_0}{\lambda},\quad \text{with}\quad \alpha_{\rm em}\equiv\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c}\approx\frac{1}{137}.$$

- Two-particle wave function: symmetrized scattering „out" state
  - „out" states asymptotically plane wave + *incoming* spherical wave
  - alternate „in" state (plane wave + outgoing spherical wave) yields same results

$$\psi^{(C)}=e^{i\mathbf{KR}}\times\frac{\mathcal{N}^*}{\sqrt{2}}e^{-ikr}\Big\{M\big(1-i\eta,1,i(kr+\mathbf{kr})\big)+(\mathbf{k}\leftrightarrow-\mathbf{k})\Big\}.$$

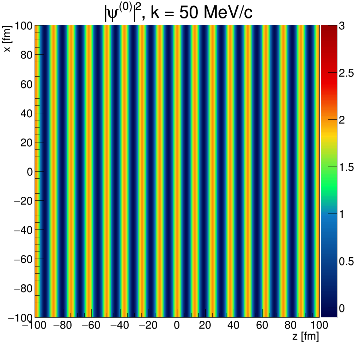
Making use of the $M(a,b,z)$ confluent hypergeometric function

- Normalization ($\mathcal{N}$) and Gamow factor ($|\mathcal{N}|^2$):

$$\mathcal{N}=e^{-\pi\eta/2}\Gamma(1+i\eta),\qquad |\mathcal{N}|^2=e^{-\pi\eta}|\Gamma(1+i\eta)|^2=\frac{2\pi\eta}{e^{2\pi\eta}-1}. \tag{1}$$

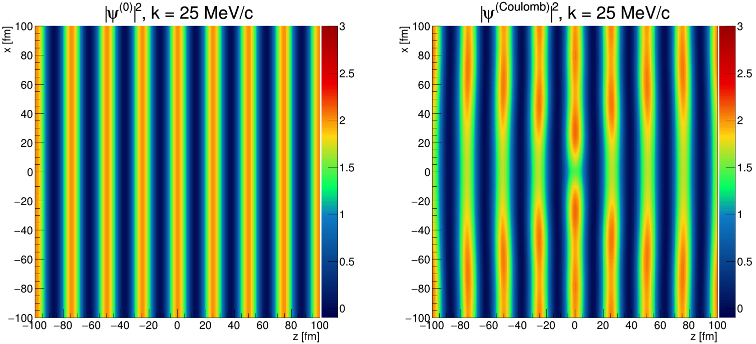
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



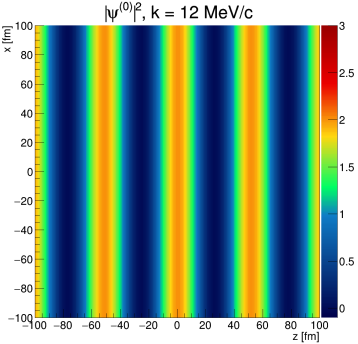
Coulomb interaction

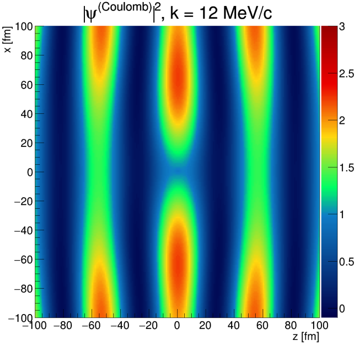
- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



Coulomb interaction

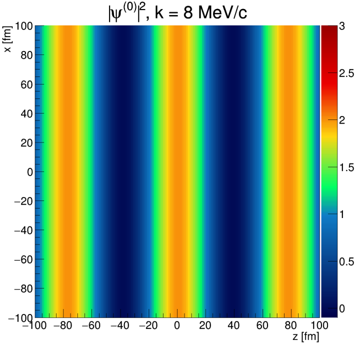
- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections

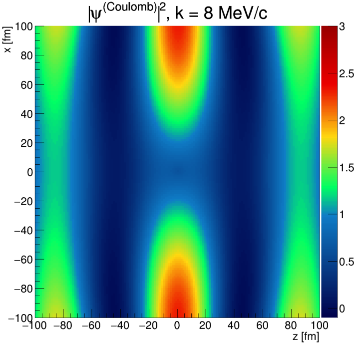




Coulomb interaction

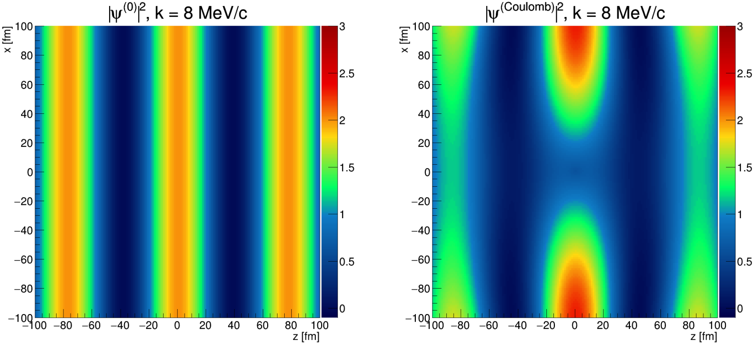
- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections





Coulomb interaction

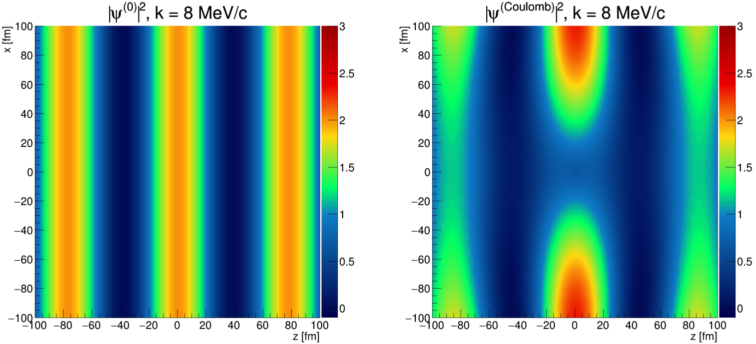
- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



- Gamow correction captures only the value at the origin

Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



- Gamow correction captures only the value at the origin
- Calculational methods:
  - Direct integrating $D(\mathbf{r})|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$ during fit: time-consuming, even nowadays
  - Pre-calculate a ,,Coulomb correction" with fix parameters (say, $R = 5\,\mathrm{fm}$ Gaussian): fast but inconsistent
  - Use iterative method, use memory lookup table. . .

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})\ldots$
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})\dots$
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \overset{??}{=}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \stackrel{??}{=}$$

$$\stackrel{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \overset{??}{=}$$

$$\overset{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 =$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}):=\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,f(\mathbf{q})e^{i\mathbf{qr}}\quad\Leftrightarrow\quad f(\mathbf{q})=\int\mathrm{d}^3\mathbf{r}\,D(\mathbf{r})e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q})=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})e^{i\mathbf{qr}}=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{r}\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\overset{??}{=}$$

$$\overset{??}{=}\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{q}\int\mathrm{d}^3\mathbf{r}\,f(\mathbf{q})e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2=$$

$$=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})\int\mathrm{d}^3\mathbf{r}\,e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \overset{??}{=}$$

$$\overset{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 =$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{qr}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \lightning\lightning\lightning$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \overset{??}{=}$$

$$\overset{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 =$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad ϟϟϟ$$

- Not working in this form

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \overset{??}{=}$$

$$\overset{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 =$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{qr}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \quad ϟϟϟ$$

- Not working in this form: Fourier transform $\neq$ integral (Lebesgue)

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \overset{??}{=}$$

$$\overset{??}{=} \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q} \int \mathrm{d}^3\mathbf{r}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 =$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 \quad ϟϟϟ$$

- Not working in this form: Fourier transform $\neq$ integral (Lebesgue)
  - Workaround: regularization, $\lambda \in \mathbb{R}^+$, then $\lambda \to 0$.

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}):=\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,f(\mathbf{q})e^{i\mathbf{qr}}\quad\Leftrightarrow\quad f(\mathbf{q})=\int\mathrm{d}^3\mathbf{r}\,D(\mathbf{r})e^{-i\mathbf{qr}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse" Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals"

$$C_2(\mathbf{Q})=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})e^{i\mathbf{qr}}=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{r}\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\overset{??}{=}$$

$$\overset{??}{=}\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{q}\int\mathrm{d}^3\mathbf{r}\,f(\mathbf{q})e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2=$$

$$=\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{q}\,f(\mathbf{q})\int\mathrm{d}^3\mathbf{r}\,e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\quad ⚡⚡⚡$$

- Not working in this form: Fourier transform $\neq$ integral (Lebesgue)
  - Workaround: regularization, $\lambda\in\mathbb{R}^+$, then $\lambda\to0$.
  - Careful math needed (once in a physicist's lifetime...)

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int\mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int\mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r} \lim_{\lambda\to 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,e^{i\mathbf{q}\mathbf{r}}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r}\, \lim_{\lambda\to 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}=$$

$$=\lim_{\lambda\to 0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2.$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,e^{i\mathbf{q}\mathbf{r}}=$$

$$=\lim_{\lambda\to 0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}e^{i\mathbf{q}\mathbf{r}}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2}}.$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2.}}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}} = \int \mathrm{d}^3\mathbf{r}\, |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r} \lim_{\lambda\to 0} e^{-\lambda r} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r}) =$$

$$= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} =$$

$$= \underline{\underline{\lim_{\lambda\to 0} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 .}}$$

- „Ingredients”:

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int\mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2D(\mathbf{r})=\int\mathrm{d}^3\mathbf{r}\lim_{\lambda\to0}e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to0}\int\mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to0}\int\mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\int\mathrm{d}^3\mathbf{r}\,e^{-\lambda r}e^{i\mathbf{q}\mathbf{r}}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.$$

- ,,Ingredients":
  - Conditions of (Lebesgue) integrability

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\,e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\,e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}e^{i\mathbf{q}\mathbf{r}}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.}}$$

- ,,Ingredients”:
  - Conditions of (Lebesgue) integrability
  - *Lebesgue theorem* (for interchanging integrals and limits)

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}}|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.}}$$

- ,,Ingredients":
  - Conditions of (Lebesgue) integrability
  - *Lebesgue theorem* (for interchanging integrals and limits)
  - *Fubini's theorem* (for interchanging repeated integrals)

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\,e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\,e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\,e^{-\lambda r}e^{i\mathbf{q}\mathbf{r}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2}}.$$

- „Ingredients”:
  - Conditions of (Lebesgue) integrability
  - *Lebesgue theorem* (for interchanging integrals and limits)
  - *Fubini's theorem* (for interchanging repeated integrals)
- In last step, *cannot* interchange $\int \mathrm{d}^3\mathbf{q}$ and $\lim_{\lambda\to 0}$.

Details of derivation

- Interchanging our integrals in a careful way:

$$\underline{\underline{C_2(\mathbf{Q})}}=\int \mathrm{d}^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2 D(\mathbf{r})=\int \mathrm{d}^3\mathbf{r}\,\lim_{\lambda\to 0} e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, D(\mathbf{r})=$$

$$=\lim_{\lambda\to 0}\int \mathrm{d}^3\mathbf{r}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}}=$$

$$=\underline{\underline{\lim_{\lambda\to 0}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\int \mathrm{d}^3\mathbf{r}\, e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2.}}$$

- ,,Ingredients":
  - Conditions of (Lebesgue) integrability
  - *Lebesgue theorem* (for interchanging integrals and limits)
  - *Fubini's theorem* (for interchanging repeated integrals)
- In last step, *cannot* interchange $\int \mathrm{d}^3\mathbf{q}$ and $\lim_{\lambda\to 0}$.
- As of now, continuing only in the spherically symmetric case: $f(\mathbf{q})\equiv f_s(q)$, $D_{cc}(r)=2\pi\int_0^\infty \mathrm{d}q\, q^2 \sin(qr) f_s(q)$.

Details of derivation (cont'd)

- After substituting $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$, ,,master" formula thus reads as

$$C_2(Q)=\frac{|\mathcal{N}|^2}{2\pi^2}\lim_{\lambda\to 0}\int_0^\infty \mathrm{d}q\, q^2 f_s(q)\Big[\mathcal{D}_{1\lambda s}(q)+\mathcal{D}_{2\lambda s}(q)\Big], \qquad \text{where}$$

$$\mathcal{D}_{1\lambda s}(q)=\int \mathrm{d}^3\mathbf{r}\,\frac{\sin(qr)}{qr}e^{-\lambda r}M\big(1+i\eta,1,-i(kr+\mathbf{kr})\big)M\big(1-i\eta,1,i(kr+\mathbf{kr})\big),$$

$$\mathcal{D}_{2\lambda s}(q)=\int \mathrm{d}^3\mathbf{r}\,\frac{\sin(qr)}{qr}e^{-\lambda r}M\big(1+i\eta,1,-i(kr-\mathbf{kr})\big)M\big(1-i\eta,1,i(kr+\mathbf{kr})\big).$$

- These can be calculated (using complex analysis; method pioneered by Nordsieck in the theory of bremsstrahlung & pair creation)

A. Nordsieck, Phys. Rev. **93**, 785 (1954).

$$\mathcal{D}_{1\lambda s}(q)=\frac{4\pi}{q}\mathrm{Im}\left[\frac{1}{(\lambda-iq)^2}\left(1+\tfrac{2k}{q+i\lambda}\right)^{2i\eta}\mathcal{F}_+\left(\tfrac{4k^2}{(q+i\lambda)^2}\right)\right],$$

$$\mathcal{D}_{2\lambda s}(q)=\frac{4\pi}{q}\mathrm{Im}\left[\frac{(\lambda-iq-2ik)^{i\eta}(\lambda-iq+2ik)^{-i\eta}}{(\lambda-iq)^2+4k^2}\right],$$

Here $\mathcal{F}_+(x)\equiv {}_2F_1(i\eta,1+i\eta,1,x)$ is the hypergeometric function

The main result

- For $\lim_{\lambda\to 0}$, function forms of $\mathcal{D}_{1\lambda s}$, $\mathcal{D}_{2\lambda s}$ become ,,ill-behaved" (*Remark:* a simple well known similar case is the approximation of $\delta(x)$ Dirac delta with smooth peaked functions)
- Need to calculate & simplify $\lambda\to 0$ limit (numerical limit-taking. . . $\Rightarrow$ ↯)
  $\Rightarrow$ result: *functional*, not a proper integral transform of $f_s(q)$
- Result of the calculation:

$$C_2(Q)=|\mathcal{N}|^2\left(1+f_s(2k)+\frac{\eta}{\pi}\left[\mathcal{A}_{1s}+\mathcal{A}_{2s}\right]\right),\quad \text{where}$$

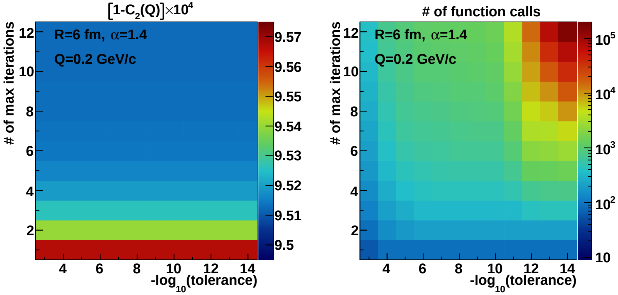
$$\mathcal{A}_{1s}=-\frac{2}{\eta}\int_0^\infty \mathrm{d}q\,\frac{f_s(q)-f_s(0)}{q}\,\mathrm{Im}\left[\left(1+\frac{2k}{q}\right)^{2i\eta}\mathcal{F}_+\left(\frac{4k^2}{q^2}-i0\right)\right],$$

$$\mathcal{A}_{2s}=-\frac{2}{\eta}\int_0^\infty \mathrm{d}q\,\frac{f_s(q)-f_s(2k)}{q-2k}\,\frac{q}{q+2k}\,\mathrm{Im}\,\frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

- $\eta\to 0$: free $C_2^{(0)}(Q)=1+f_s(Q)$ recovered (NB: $Q=2k$)
- $|\mathcal{N}|^2$ factor only: Gamow correction
  $\Rightarrow$ $\mathcal{A}_{1s}$, $\mathcal{A}_{2s}$,,correct the Gamow correction"
- $\mathcal{A}_{1s}$ and $\mathcal{A}_{2s}$: well-defined *functionals* of $f_s)(q)$
- Care needed about branch cuts ($\pm i0$ terms) of $\mathcal{F}_+(x)$ and complex powers

Numerical implementation

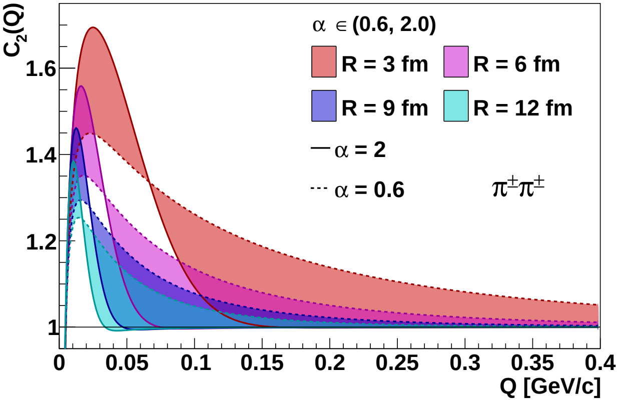
- Final numerical integrals needed: in $\mathcal{A}_{1\lambda_s}$ and $\mathcal{A}_{2\lambda_s}$
- Transform integral to $x \in [0,1] \Rightarrow$ smooth, „beautiful" integrands
- Gauss-Krohnrod quadrature (from `C++ boost` library) used:
  - Main parameters: # of max iterations & tolerance
  - Investigated; optimal value found:
    few hundred integrand evaluations (instead of many 10000-s)



- Real-time calculation (during fit procedure) possible!
- Codes archived at: `github.com/csanadm/CoulCorrLevyIntegral`

Example calculations: illustrations

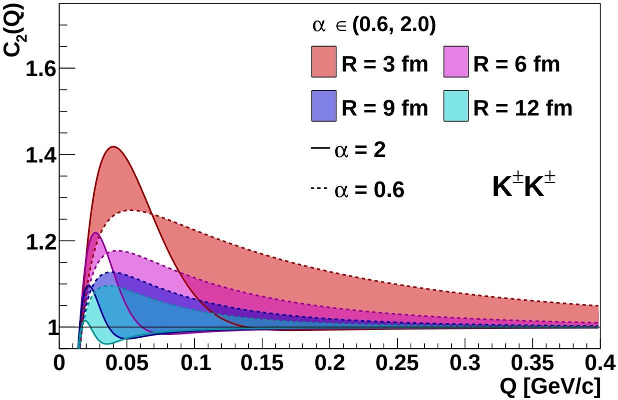
- For Lévy sources, for pion ($\pi^+\pi^+$, $\pi^-\pi^-$) pairs:



- most frequent target of HBT measurements
- Shaded region ,,swept" over by $C_2(Q)$ as α changes
- Apparent ,,nodes" disappear with increased zooming in

Example calculations: illustrations

- For Lévy sources, for kaon (K^+K^+, K^-K^-) pairs:



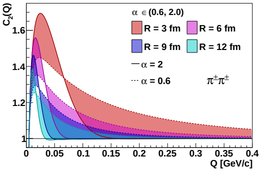
- Similarly to the case of pions; ,,nodes" are only apparent
- Coulomb effect stronger ($m_K > m_\pi$; η increases)
- Considerable interplay of experimentally measurable λ, R, α

Summary and outlook

- Efficient new method Coulomb interacting HBT correlation function calculation
  - Calculations directly in momentum (Fourier) space
  - Careful mathematical methods invoked, distribution theory motivated
  - Cross-checked with previous direct calculations
  - Numerical implementation done, ready for use in data analysis
- As of now, implementation only for spherically symmetric sources
  - Prospective generalization I: go beyond spherical symmetry
    This is where efficiency becomes crucial...
  - Prospective generalization II: short-range strong interactions
  - Prospective generalization (in fact, simplification) for non-identical particle correlations: only $\mathcal{D}_{1\lambda s}$ (ie. $\mathcal{A}_{1s}$) term needed

***

New exact analytic formulas for QM Coulomb problem! ☺



Thank you for your attention!