Add a mass and make your life more complicated

Alternative title:
Add a mass and have fun with mathematics

Stefan Weinzierl

Uni Mainz

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Motivation

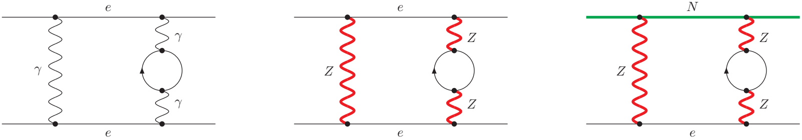
- The LHC and future colliders will provide many precision measurements.
- Of particular interest are processes involving **heavy particles** (Higgs, top-quark, Z/W-bosons)
- Precision calculations at high energies rely on perturbation theory and here in particular on **Feynman integrals**.

This talk: Feynman integrals with internal masses.

Loops and Legs

- **Rule of thumb** in massless theories: The complexity of a Feynman integral increases as the number of loops or the number of legs increases.
- More accurately: The complexity of a Feynman integral increases as the number of loops or the **number of kinematic variables** increases.
- **Adding a mass** increases the number of kinematic variables and hence increases the complexity of a Feynman integral.

Example



Physics is about numbers

What is the numerical precision we are aiming for?

- For a **physical observable** we usually only need a few digits for the highest term in perturbation theory.
- For **amplitudes** we may need quadruple precision in singular limits (soft/collinear).
- For **master integrals** / **special functions** we may want $\mathcal{O}(100) - \mathcal{O}(1000)$ digits to use PSLQ.

Hierarchies

There might be a hierarchy in the kinematic variables the Feynman integral depends on:

$$x_1 \quad \ll \quad x_2$$

We would like to have **stable numerical evaluations**.

Example: Møller scattering: $e^- e^- \to e^- e^-$

$$|t| \ll s \ll m_Z^2.$$

Options

From **many scales / automated** to **fewer scales / fast evaluations**:

- Purely numerical: Sector decomposition, numerical integration in loop momentum space.
- Semi-numerical: Unitarity methods, numerical integration of a differential equation, AMFlow
- Semi-analytical: Expansion in a small parameter
- Analytical: Reduction to standarised special functions

This talk: The method of differential equations

Section 1

Differential equations

Setting the stage

- Feynman integrals are regulated with the dimensional regularisation parameter ε.
- Feynman integrals depend on kinematic variables

$$x_1, x_2, \dots .$$

- Integration-by-parts allows us to reduce Feynman integrals to master integrals

$$I_1, I_2, \dots .$$

- Integration-by-parts allows us to derive a differential equation for the master integrals with differential one-forms

$$\omega_1, \omega_2, \dots .$$

Notation

$\mathbf{N_F} = N_{\mathrm{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

$\mathbf{N_B} = N_{\mathrm{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_{N_B})$.

$\mathbf{N_L} = N_{\mathrm{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, ..., \omega_{N_L})$.

The method of differential equations

We want to calculate $I(\varepsilon, x)$ **as a Laurent series in** ε.

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral (always possible).*

$$\left[d + A(\varepsilon, x) \right] I = 0.$$

(Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

2. *Transform the differential equation into an* ε*-factorised form (***bottle neck***).*

$$\left[d + \varepsilon A(x) \right] I = 0.$$

(Henn '13)

3. *Solve the latter differential equation with appropriate boundary conditions (always possible).*

Example for an ε-factorised form

$$A(x) = C_1 \omega_1 + C_2 \omega_2$$

with differential one-forms

$$\omega_1 = \frac{dx}{x}, \qquad \omega_2 = \frac{dx}{x-1},$$

and matrices

$$C_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Motivation

- If the differential equation is not in ε-form, we might have

$$\frac{dx}{x}, \quad \frac{dx}{x^2}, \quad \frac{dx}{x^3}, \quad \frac{dx}{x-1}, \quad \frac{dx}{(x-1)^2}, \quad \frac{dx}{(x-1)^3}.$$

- If the differential equation is in ε-form, this reduces to

$$\frac{dx}{x}, \quad \frac{dx}{x-1}.$$

Vector bundles

- Fibre spanned by the master integrals $I = (I_1, \ldots, I_{N_F})$.

  (The master integrals $I_1(x), \ldots, I_{N_F}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre.)
- Base space with coordinates $x = (x_1, \ldots, x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A with differential one-forms $\omega = (\omega_1, \ldots, \omega_{N_L})$.

Transformations on this vector bundle:

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Transformations

- **Change the basis of the master integrals**

$$I' \;=\; UI,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' \;=\; UAU^{-1} + UdU^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x'_i \;=\; f_i\left(x\right), \qquad 1 \le i \le N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \, \frac{\partial x_i}{\partial x'_j} \, dx'_j.$$

Remarks

- A change of the basis of the master integrals is like a **gauge transformation**:

$$A' = UAU^{-1} + UdU^{-1}.$$

- A coordinate transformation is like in **general relativity**:

$$A = \sum_{i=1}^{N_B} A_i dx_i \quad \Rightarrow \quad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x'_j} dx'_j.$$

Change the basis of the master integrals

- A change of the basis of master integrals is done to **transform** the system of differential equations **into an ε-factorised form**.
- **Conjecture**: Such a transformation always exists.
- Heuristic method for finding such a transformation: Analysing the **maximal cut**.

Coordinate transformation on the base manifold

- The transformation to an ε-factorised form may introduce algebraic or transcendental functions.
- A coordinate transformation may lead to a nicer form.
  Examples:
  - **Square roots**:

$$x = \frac{(1-x')^2}{x'}, \quad x' = \frac{1}{2}\left(2+x-\sqrt{x(4+x)}\right) \quad \Rightarrow \quad \frac{dx}{\sqrt{x(4+x)}} = -\frac{dx'}{x'}$$

  - **Elliptic case**:

$$x = -9\frac{\eta(\tau)^4\eta(6\tau)^8}{\eta(3\tau)^4\eta(2\tau)^8}, \quad \tau = \frac{\psi_2(x)}{\psi_1(x)} \quad \Rightarrow \quad \left(\frac{\pi}{\psi_1(x)}\right)^2 \frac{12dx}{x(x+1)(x+9)} = 2\pi i\, d\tau$$

Iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold B and $\gamma : [0,1] \rightarrow B$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \;=\; \gamma^* \omega_j.$$

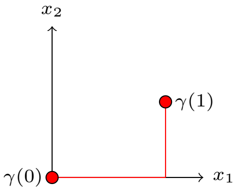
The **iterated integral** is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) \;=\; \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Chen '77

Translation to physics

- Manifold *B*: kinematic space, coordinates are the kinematic variables.
- $\gamma(0)$: Boundary point
- $\gamma(1)$: Point,where we would like to evaluate the integral.



Multiple polylogarithms

Consider differential one-forms on $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere) of the form

$$\omega^{\mathrm{mpl}}(z_j) = \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1, \dots, z_k; \lambda) = \int\limits_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int\limits_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \dots \int\limits_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \qquad z_k \neq 0$$

Shuffle product and trailing zeros

- Let U be a domain. Assume that by suitable coordinate transformation all ω_j are holomorphic in $U \backslash \{0\}$ and have at most a simple pole at $z = 0$.
- The iterated integrals have then at most logarithmic singularities $\ln^n(z)$ as $z \to 0$.
- Iterated integrals obey the shuffle product.
- Using the shuffle product, we may **make these logarithms explicit**. The remaining functions are then **regular** as $z \to 0$.

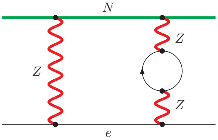
Strategy

- Find a transformation to an **ε-factorised form**.
- Choose as **boundary point** the **closest singular point**.
- Find a **coordinate transformation** such that all differential one-forms have at most a **simple pole** at the boundary point.
- Use the shuffle product to **make** the **logarithms explicit**.

Remark

- Consider $I_\gamma(\omega,\omega,\omega,\omega,\omega,\omega;\lambda)$, where $\omega = d \ln P_8(\lambda)$ and P_8 is a degree eight polynomial $P_8 \in \mathbb{R}[\lambda]$.
- This iterated integral can be expressed as a linear combination of $8^6 = 262144$ multiple polylogarithms with (in general) complex arguments (the roots of P_8).
  This is **highly inefficient** and gets worth at higher weight.
- **Better** to treat ω as a single integration kernel.
  (If all roots are roots of unity, this yields cyclotomic harmonic polylogarithms)

  Ablinger, Blümlein, Schneider, '11



Caveats of iterated integrals

- In general, an individual iterated integral is **not** homotopy invariant.
  The linear combination making up a Feynman integral is, since the connection A is flat (integrable).
- If the differential one-forms ω_k transform nicely under a group of coordinate transformations, this does in general not imply that iterated integrals transform nicely as well.
  However, the vector space spanned by the master integrals does again.
  Suggests to use different bases of master integrals in different kinematic regions.

Section 2

Examples

- Elliptic curves
- Curves of higher genus
- Manifolds of higher dimension

Subsection 1

Elliptic curves

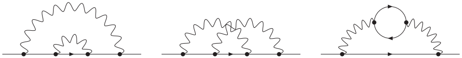
Beyond multiple polylogarithms

- Not every Feynman integral can be expressed in terms of multiple polylogarithms.
- Starting from two-loops, we encounter more complicated functions.
- The next-to-simplest Feynman integrals involve an **elliptic curve**.

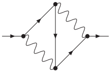
Elliptic curves

We do not have to go very far to encounter elliptic integrals in precision calculations: The simplest example is the two-loop electorn self-energy in QED:

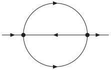
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the **kite graph**:



One sub-topology is the **sunrise graph** with three equal non-zero masses:



(Sabry, '62)

Where is the elliptic curve?

- *For the sunrise it's very simple: The second graph polynomial defines an elliptic curve in Feynman parameter space:*

$$-p^2 a_1 a_2 a_3 + \left(a_1 + a_2 + a_3\right)\left(a_1 a_2 + a_2 a_3 + a_3 a_1\right) m^2 \quad = \quad 0.$$

- *More general: If the maximal cut is of the form*

$$\int dz \, \frac{N(z)}{\sqrt{P(z)}},$$

where $P(z)$ is a polynomial of degree 3 or 4. This gives the elliptic curve

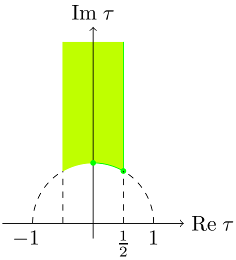
$$y^2 \quad = \quad P(z).$$

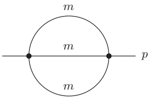
q-expansions

- Iterated integrals in the elliptic case are evaluated with the help of their q-expansions, $q = \exp(2\pi i \tau)$.
- The q-series converge for $|q| < 1$.
- By a modular transformation we may map τ to the fundamental domain, resulting in

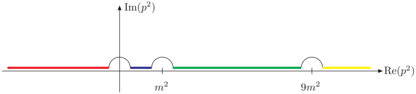
$$|q| \quad \leq \quad e^{-\pi\sqrt{3}} \approx 0.0043,$$

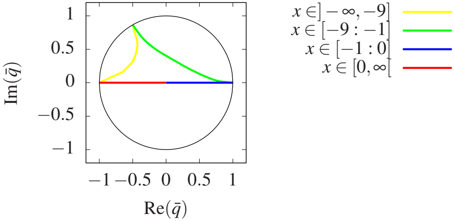
resulting in a fast converging series.





- Consider the equal mass sunrise integral with $x = -p^2/m^2$.
- Singularites at $x \in \{-9, -1, 0, \infty\}$.
- In the variable x we don't expect an expansion around one singular point to converge beyond the next singular point.
- In the variable q the expansion converges for all values $x \in \mathbb{R}$ except the three other singular points.





Physics is about numbers:

- Iterated integrals of modular forms and elliptic multiple polylogarithms can be evaluated numerically with arbitrary precision.
- Implemented in `GiNaC`.

Walden, S.W, '20

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.1)
  __,  _______  Copyright (C) 1999-2021 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> iterated_integral({Eisenstein_kernel(3,6,-3,1,1,2)},0.1);
0.23675657575197179243274817775862177623438999192840338805367
```

Subsection 2

Curves of higher genus

Hyperelliptic curves

Definition

A hyperelliptic curve is an algebraic curve of genus $g \geq 2$ whose defining equation takes the form

$$y^2 = P(z),$$

for some polynomial $P(z)$ of degree $(2g+1)$ or $(2g+2)$.

They generalise elliptic curves, whose defining equation takes the same form when $g = 1$.

We are interested in **Feynman integrals**, where the maximal cut takes the form

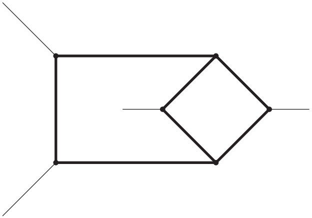
$$\int dz \, \frac{N(z)}{\sqrt{P(z)}}$$

Non-planar double boxes

Non-planar double boxes (with sufficient internal/external masses) provide examples of higher-genus Feynman integrals.

- In the loop momentum representation one obtains a genus 3 curve.

  Georgoudis, Zhang, ’15
- In the Baikov representation one obtains a genus 2 curve.



Can we understand this?

Yes we can!

R. Marzucca, A. McLeod, B. Page, S.Pögel, S.W., ’23

Extra involutions

- **The solution to this riddle**: The higher genus curve has an extra involution. In the simplest case, if $P(z)$ is of the form

$$P(z) = Q\left(z^2\right) = \left(z^2 - \alpha_1^2\right) \dots \left(z^2 - \alpha_{g+1}^2\right)$$

  the extra involution is given by $e_1 : z \to -z$.
- The substitution $w = z^2$ leads to a **genus drop**.

Why is there an extra involution?

For our example we can trace it back to discrete Lorentz transformations (parity, time reversal):

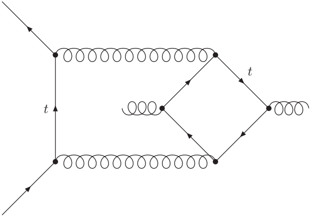
- In the Baikov representation everything is manifestly Lorentz invariant, the Baikov variables are Lorentz invariants:

$$z = k^2 - m^2.$$

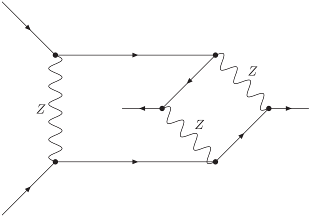
- In the loop momentum representation we choose a frame, we choose a parametrisation of the loop momenta, we choose an elimination order: The full Lorentz symmetry is not required to be trivially realised, but may manifest itself through extra symmetries of the curve.

Examples

- Top pair production at NNLO
  (genus drop from 3 to 2)



- Møller scattering at NNLO
  (genus drop from 3 to 2)



Subsection 3

Manifolds of higher dimension

Calabi-Yau manifolds

- Calabi-Yau manifold are studied in mathematics.
  - A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.
  - An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

  conjectured by Calabi, proven by Yau

- The **mirror map** relates a Calabi-Yau manifold A to another Calabi-Yau manifold B with Hodge numbers $h_B^{p,q} = h_A^{n-p,q}$.

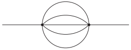
Candelas, De La Ossa, Green, Parkes '91

- **Calabi-Yau operators** have a special local normal form.

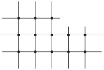
M. Bogner '13, D. van Straten '17

Fantastic Beasts and Where to Find Them

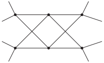
- Bananas



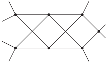
- Fishnets



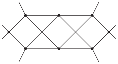
- Amoebas



- Tardigrades



- Paramecia



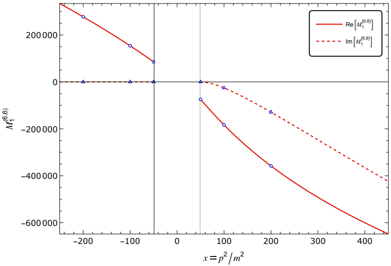
Aluffi, Marcolli, ’09, Bloch, Kerr, Vanhove, ’14
Bourjaily, McLeod, von Hippel, Wilhelm, ’18
Duhr, Klemm, Loebbert, Nega, Porkert, ’22

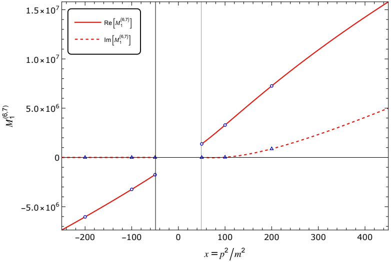
Bananas

- The l-loop banana integral with (equal) non-zero masses is related to a **Calabi-Yau $(l-1)$-fold**.
- An elliptic curve is a Calabi-Yau 1-fold, this is the geometry at two-loops.
- The system of differential equations for the equal mass l-loop banana integral can be transformed to an **ε-factorised form**.
  - Change of variables from $x = p^2/m^2$ to τ given by mirror map.
  - Transformation constructed from special local normal form of a Calabi-Yau operator.

  Pögel, Wang, S.W. '22
- Strong support for the conjecture that a transformation to an ε-factorised differential equation exists for all Feynman integrals.

Results: Six loops





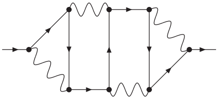
Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

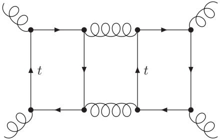
Pögel, Wang, S.W. '22

Examples

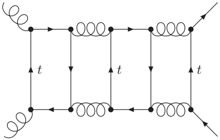
- Electron self-energy in QED
  (related to a Calabi-Yau 3-fold).



- Dijet production at N^3LO
  (related to a Calabi-Yau 2-fold).



- Top pair production at N^4LO
  (related to a Calabi-Yau 3-fold)



Conclusions

- Precision calculations with heavy particles lead to challenging Feynman integrals early on in the perturbative expansion.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- The differential one-forms are in a certain sense universal and geometric.
- A better understanding of the relation to algebraic geometry may lead to more efficient numerical evaluation algorithms and automatisation.