Weinberg's Theorem:

"(...) QFT has no content beyond analyticity, unitarity, cluster decomposition and symmetry."

Weinberg (1979).

One example: consider a theory of two kinds of scalars ϕ and ψ defined by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\phi\psi^*\psi$$

We consider the 2-point function of ϕ:

$\langle 0|T\phi(x)\phi(y)|0\rangle$ and its Fourier transform:

$$G^{(2)}(p,p') \equiv (2\pi)^4\,\delta^4(p+p')\,\tilde{D}_r(p^2)$$

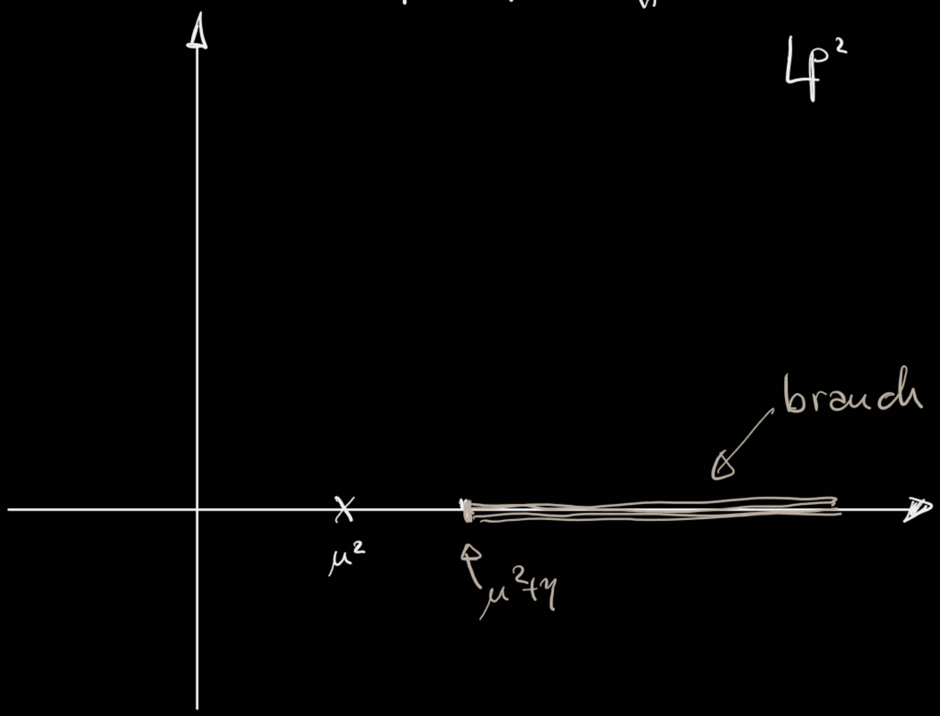
$\tilde{D}_r(p^2)$ satisfies the Källen-Lehmann repres.:

$$\tilde{D}_r(p^2) = \frac{i}{p^2-\mu^2+i\varepsilon} + \int_{\mu^2+\eta}^{\infty} ds \, \frac{i\,\sigma(s)}{p^2-s+i\varepsilon}$$

where

$$\sigma(q^2)\,\theta(q^0) = \sum_n{}' \, (2\pi)^3 \delta^4(q-p_n) \, |\langle n|\phi(0)|0\rangle|^2$$

$$\llcorner p^2$$



$-i\tilde{D}_r(p^2)$ is analytic everywhere other than for the pole at μ^2 and the branch cut;

it is real on the real axis for $p^2<\mu^2+\eta$;

$$\Rightarrow \quad \left[-i\tilde{D}_r(p^2)\right]^* = -i\,\tilde{D}_r(p^{2*})$$

$$\Rightarrow \quad \mathrm{Disc}\left[-i\,\tilde{D}_r(p^2)\right] = -i\,2\pi\,\sigma(s)$$

$$\left[\lim_{\varepsilon\to 0} \frac{1}{x+i\varepsilon} = -i\pi\,\delta(x) + P\,\frac{1}{x}\right]$$

Self-energy:

$$\equiv -i\,\tilde{\Pi}_r(p^2)$$

$\tilde{D}_r(p^2) \equiv$ 

after resumming the geometric series:

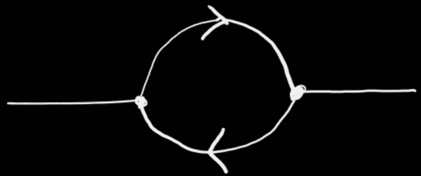
$$\tilde{D}_r(p^2) = \frac{i}{p^2-\mu^2-\tilde{\Pi}_r(p^2)+i\varepsilon}$$

Renormalization conditions:

1. $\tilde{D}_r(p^2)$ has a pole at $p^2=\mu^2$ $\Rightarrow$ $\tilde{\Pi}_r(\mu^2)=0$

2. residue $= i$ $\Rightarrow$ $\left.\dfrac{\partial\tilde{\Pi}_r(p^2)}{\partial p^2}\right|_{p^2=\mu^2}=0$

Calculation of $\tilde{\Pi}_r(p^2)$ in perturbation theory at order g^2:



$$= g^2\int\frac{d^4q}{(2\pi)^4}\int_0^1 dx\,\frac{1}{\left[q^2+p^2x(1-x)-m^2+i\varepsilon\right]^2}$$

After imposing the two renorm. conditions:

$$\tilde{\Pi}_r(p^2) = \tilde{\Pi}_f(p^2) - \tilde{\Pi}_f(\mu^2) - (p^2-\mu^2)\frac{\partial}{\partial p^2}\tilde{\Pi}_f(p^2)\Big|_{\mu^2}$$

we end up with

$$\tilde{\Pi}_r(p^2) = \frac{g^2}{16\pi^2}\int_0^1 dx\left[\ln\left(\frac{m^2-p^2x(1-x)-i\varepsilon}{m^2-\mu^2x(1-x)}\right) + \frac{(p^2-\mu^2)x(1-x)}{m^2-x(1-x)\mu^2}\right]$$

The calculation of the imaginary part is straightforward and comes solely from the log. Remembering: $\log(-x \pm i\varepsilon) = \pm i\pi$

$$\mathrm{Im}\,\tilde{\Pi}_r(p^2) = \frac{-g^2}{16\pi^2}\int_{-1/2\beta}^{1/2\beta} dy\,\pi = -\frac{g^2}{16\pi}\beta(p^2)$$

with $\beta(p^2) = \sqrt{1-\frac{4m^2}{p^2}}$

Clearly I should be able to get the same result from the Källen-Lehmann representation.

Indeed, considering that at $O(g^2)$ the only contribution is due to the 2N state (with N the scalar φ), then we have

$$|\langle 2N|\phi(0)|0\rangle|^2 = \frac{g^2}{(p^2-\mu^2)^2}$$

which gives us g^2 for the self-energy.

Integral over the phase space:

$$\int\frac{d^3p_1}{(2\pi)^3 2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}(2\pi)^4\,\delta^4(q-(p_1+p_2))$$

$= \frac{1}{8\pi}\beta(q^2)$ which we need to multiply by g^2 and divide by 2π if we want to get $\sigma(q^2)$

$$\Rightarrow \boxed{\sigma(q^2) = -\frac{g^2}{16\pi^2}\beta(q^2)}$$

which confirms the result of the explicit calculation.

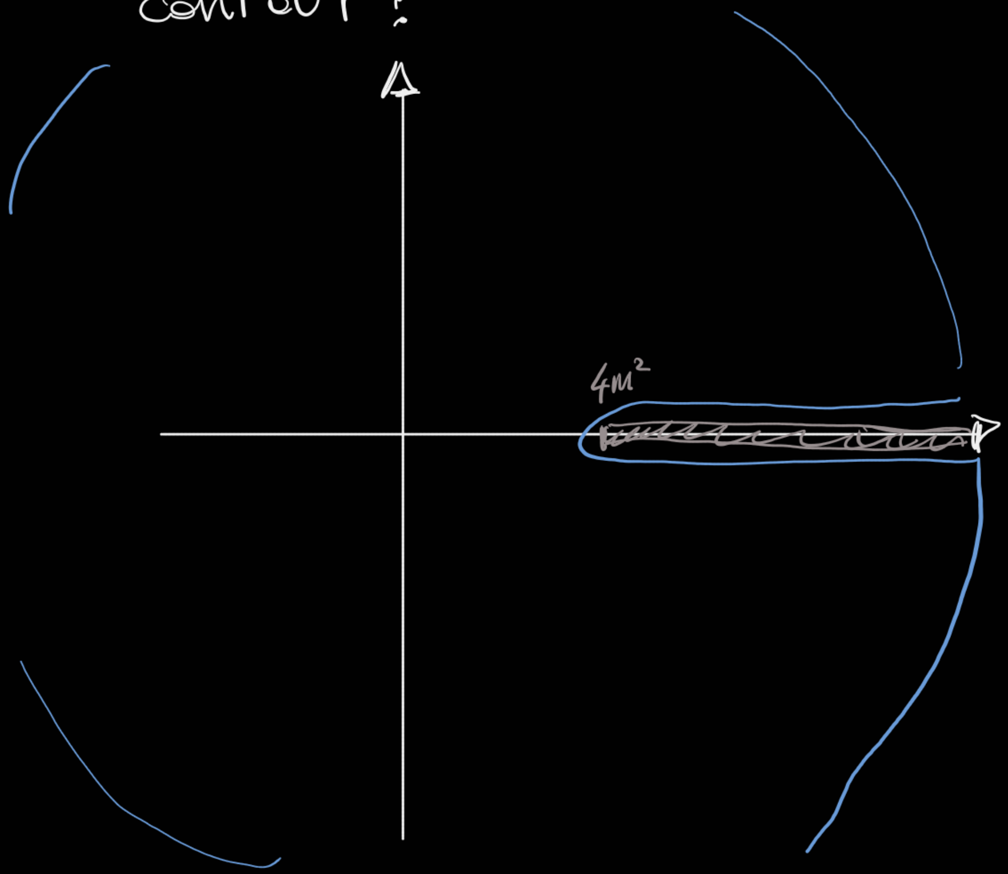
So, unitarity (or Källen-Lehmann) allows us to get the correct imaginary part without having to do an explicit calculation. What about the real part?

If $f(z)$ is an analytic function, then I can write:

$$f(s) = \frac{1}{2\pi i} \oint dz \, \frac{f(z)}{z-s}$$

for a closed contour in the domain of analyticity.

The self-energy has no pole at μ^2, only the branch cut, so that we can use the following contour:



Pushing the radius of the contour to infinity we get:

$$\tilde{\Pi}_f(p^2) = \frac{1}{2\pi i}\int_{4m^2}^{\infty} ds\, \frac{\mathrm{Disc}\,\tilde{\Pi}_f(s)}{s-p^2} = \frac{1}{\pi}\int_{4m^2}^{\infty} ds\, \frac{\mathrm{Im}\,\tilde{\Pi}_f(s)}{s-p^2}$$

After subtracting $\tilde{\Pi}_f(\mu^2)$ and $(p^2-\mu^2)\frac{\partial}{\partial p^2}\tilde{\Pi}_f(p^2)\Big|_{\mu^2}$

we obtain:

$$\tilde{\Pi}_r(p^2) = -\frac{(p^2-\mu^2)^2}{16\pi^2} g^2 \int_{4m^2}^{\infty} ds\, \frac{\beta(s)}{(s-\mu^2)^2(s-p^2)}$$

which provides a different representation of the result of the explicit calculation, with which it coincides.

Summary:

Unitarity and analyticity allow for an alternative derivation of the $O(g^2)$ perturbative result. However, a simultaneous application of perturbation theory is not necessary, and the unitarity based result is more general.

Indeed we can consider the following theory, one in which we add a $\lambda(\varphi^*\varphi)^2$ term to the Lagrangian with λ large. In such a theory we wouldn't be able to calculate the N-meson mass in perturbation theory, nor would g be a good approximation to the matrix element

$$\langle NN|\phi(0)|0\rangle = \frac{g}{p^2-\mu^2}\cdot F_{\phi NN}(p^2)$$

which would have to be described by a form factor $F_{\phi NN}(s)$ which is a non-perturbative, unknown function of λ and s. However, since we did not rely on perturbation theory for the derivation of the formulae, we can adapt it to the present case by replacing $g \to g\, F_{\phi NN}(q^2)$

$$\tilde{\Pi}_r^{NN}(p^2) = -\frac{(p^2-\mu^2)}{16\pi^2}\, g^2 \int_{4M^2}^{\infty} ds\, \frac{\beta(s)\, |F_{\phi NN}(s)|^2}{(s-\mu^2)^2(s-p^2)}$$

This is the contribution of the 2N intermediate state, but we could certainly have more, even to leading order in g, because of the non-perturbative self-interaction of the N mesons.

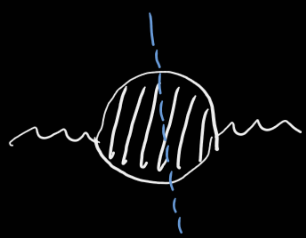
This is analogous to the situation we have for the hadronic contribution to the self-energy of the photon, usually called: hadronic vacuum polarization (HVP).

$$i e^2 \int d^4x \, e^{iqx} \langle 0 | T \, j_\mu(x) j_\nu(0) | 0 \rangle = \Pi_{\mu\nu}(q)$$
$$= \left(q^2 g_{\mu\nu} - q_\mu q_\nu \right) \Pi(q^2)$$

Renormalization condition: $\Pi(0) = 0$

Dispersive representation:

$$\Pi(s) = \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \, \frac{\mathrm{Im}\,\Pi(s')}{s'(s'-s)}$$

$\mathrm{Im}\,\Pi(s) =$  $\propto \Big|$  $\Big|^2$

$$= \frac{s}{4\pi\alpha} \, \sigma(e^+e^- \to \text{hadrons})$$

which is then rewritten in terms of the R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

In deriving this expression we have done a perturbative expansion in α_{em}, so that the two-point function of the em current is to be evaluated in pure QCD, with QED switched off.

The HVP is relevant for two important applications:
- the running of $\alpha_{em}(q^2)$;
- the contribution to the muon $g-2$.

The <u>running of α_{em}</u> is directly given by $\Pi(s)$ (modulo simple factors), so that:

$$\Delta\alpha^{(5)}_{had}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi}\int_{4M_\pi^2}^{\infty} ds \, \frac{R(s)}{s(M_Z^2-s)}$$

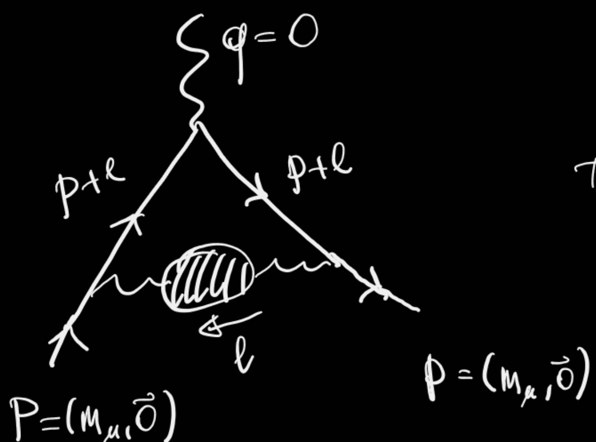
The running of α_{em} due to leptons can be calculated directly, and is (to leading order in α) given by the same expr. above with $R(s) \to 1$, as well as $4M_\pi^2 \to 4m_e^2$. The integral can then be calculated analytically and is given by

$$\frac{\alpha M_Z^2}{3\pi}\int_{4m_e^2}^{\infty} ds\, \frac{1}{s(M_Z^2-s)} = \frac{\alpha}{3\pi}\ln\left(\frac{M_Z^2}{4m_e^2}\right)$$

This is the kind of large log which needs to be resummed by RGE. For the hadronic contribution the solution of the RGE is not known. Moreover the effective log is more $\ln\left(\frac{M_Z^2}{M_\rho^2}\right)$, so much smaller than what one gets from the electron.

a_μ^{HVP}.

The HVP contribution to the muon $g-2$: a_μ^{HVP} can be evaluated as follows:



$$\Pi(l^2) = \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\, \frac{\mathrm{Im}\,\Pi(s')}{s'(s'-l^2)}$$

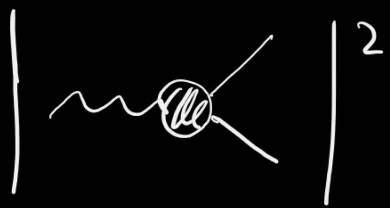
With the dispersive representation above, the only dependence on ℓ^2 in $\Pi(\ell^2)$ is propagator-like, with a variable mass s', over which one needs to integrate at the end $\Rightarrow$ the 1-loop integral can be carried out explicitly:

$$a_\mu^{HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\hat{K}(s)}{s^2} R(s)$$

$\hat{K}(s)$ is a smooth function which starts at 0 for $s=0$ and grows monotonically to 1 for $s \to \infty$. $\Rightarrow$ Thanks to the $\frac{1}{s^2}$ in the integrand, the low-energy region receives much weight in the integral.

Form factor.

2π contribution to the HVP:

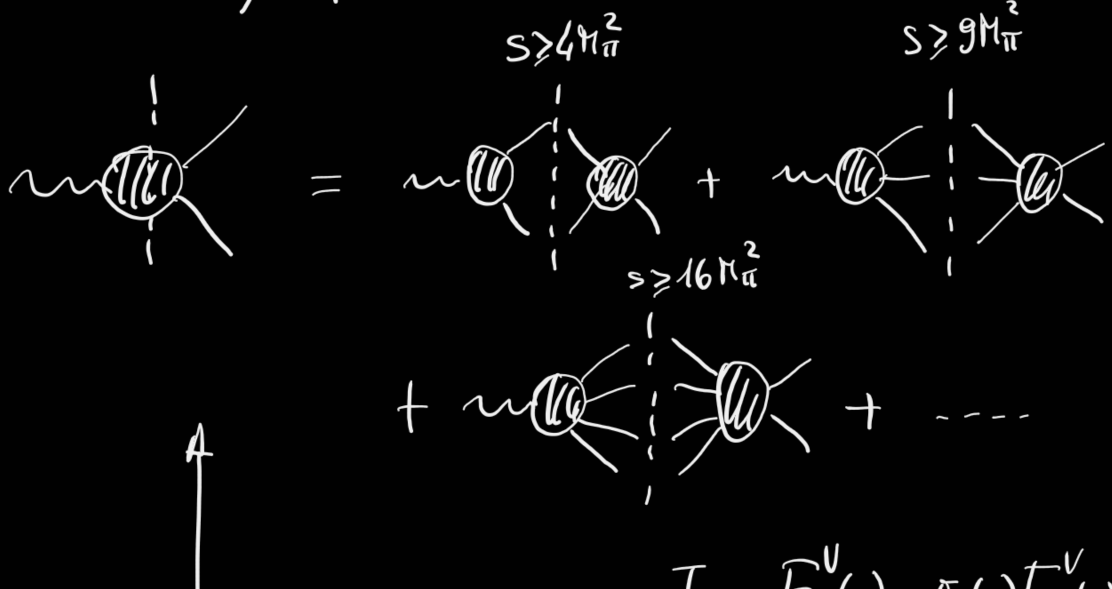


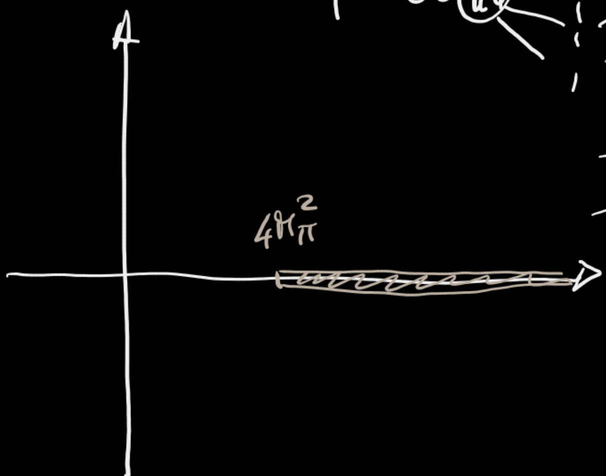
with  $= F_\pi^V(s)$

$$\langle \pi^+(p') | j^\mu_{em}(0) | \pi^+(p) \rangle = (p+p')^\mu F_\pi^V((p-p')^2)$$

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\sigma_\pi^3(s)\,|F_\pi^V(s)|^2$$

Unitarity for F_π^V:





$$\mathrm{Im}_{2\pi} F_\pi^V(s) = \sigma_\pi(s) F_\pi^V(s)\, t_1^{1*}(s)$$

$t_\ell^I(s)$ — I: isospin, ℓ: angular momentum

$$t_1^1(s) = \sin\delta_1^1(s) e^{i\delta_1^1(s)}$$

$$\Rightarrow \quad F_\pi^V(s) \sin\delta_1^1(s) e^{-i\delta_1^1(s)} \in \mathbb{R}$$

$$\Rightarrow \quad F_\pi^V(s) = |F_\pi^V(s)| e^{i\delta_1^1(s)}$$ in the elastic region.

Beyond the elastic region, ($s \geq 9M_\pi^2$, if IB, $s \gtrsim 16M_\pi^2$ in the IL), the phase of the form factor is not related to the $\pi\pi$ phase shift any more, but the fact that it has a cut, and that it satisfies the Schwartz reflection principle means that the discontinuity can be described by a phase. Let us define:

$$F_\pi^V(s) = |F_\pi^V(s)| e^{i\delta(s)}$$

with $\delta(s) = \delta_1^1(s)$ for $s \leq s_{in}$

$$e^{-2i\delta} F_\pi^V(s+i\varepsilon) = F_\pi^V(s-i\varepsilon)$$

Take the logarithm:

$$\ln(F_\pi^V(s+i\varepsilon)) - \ln(F_\pi^V(s-i\varepsilon)) = 2i\delta(s)$$

$$\Rightarrow \ln F_\pi^V(s) = \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}$$

the subtraction being necessary if $\lim_{s\to\infty}\delta(s) = \text{const.}$

The condition $F_\pi^V(0) = 1$ fixes the subtraction constant for the log to be equal to zero.

$$\Rightarrow F_\pi^V(s) = \Omega(s) \equiv \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right]$$

In principle any polynomial multiplying $\Omega(s)$ does not change the analytic properties, but only implies the presence of zeros $\Rightarrow$ changes the behaviour at infinity. We will not consider this possibility.

How useful is this representation?

If we are interested in the low-energy region, say below 1 GeV, we can rely on three additional informations:

1. The $\pi\pi$ phase shifts, and in particular the $I=1$ P-wave, are quite well known, in particular thanks to Roy eqs. and the numerical solutions thereof;

2. The 3π inelasticity is strongly suppressed, as it is a isospin-violating effect. It is only significant because of a resonance, the ω;

3. Inelasticity due to other multiparticle states, in particular 4π states and higher are phenomenologically constrained by the Eidelman-Lukaszuk bound, which shows that it starts to be visibly different from zero at $s_{in} = (M_\pi + M_\omega)^2$. Moreover, it starts slowly and remains quite small until above 1 GeV.

$$\Rightarrow \quad F_\pi^V(s) = \Omega_1^1(s) \cdot G_\omega(s) \cdot \Omega_{in}(s)$$

The factorization follows directly from the Omnès solution:

$$\delta(s) = \delta_1^1(s) + \delta_\omega(s) + \delta_{in}(s)$$

Information about $\delta_{in}(s)$ and $\Omega_{in}(s)$ is scarce. We therefore rely on a polynomial representation in a conformal variable, which is designed to start developing an imaginary part for $s \geq s_{in}$.

$$z = \frac{\sqrt{s_{in} - s_1} - \sqrt{s_{in} - s}}{\sqrt{s_{in} - s_1} + \sqrt{s_{in} - s}}$$

s_1 is an arbitrary energy parameter which is mapped to $z = 0$.

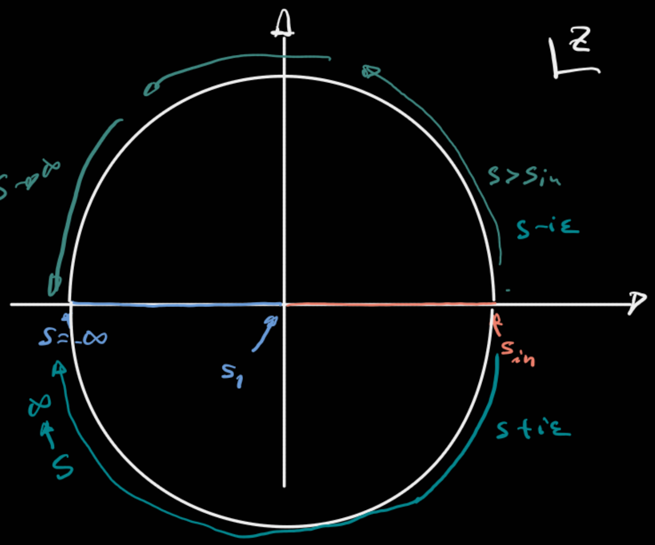
For $\quad s_1 \leq s \leq s_{in} \Rightarrow 0 \leq z \leq 1$; for $s > s_{in}$ $\mathrm{Im}(z) \neq 0$;

moreover $z = \dfrac{a-ib}{a+ib} \Rightarrow |z| = 1.$ $\quad z = \dfrac{(a-ib)^2}{a^2+b^2} = \dfrac{a^2-b^2-2iab}{a^2+b^2}$

$$\mathrm{Im}\, z = \frac{-ab}{a^2+b^2} \begin{array}{l} <0 \text{ for } s+i\varepsilon \\ >0 \text{ for } s-i\varepsilon \end{array}$$

For $s \to +\infty$ $\quad z \to -1$.

The real axis below s_1 is mapped to the segment $-1 \le z \le 0$.



$$\Omega_{in} = 1 + \sum_{k=1}^{N} c_k \left(z^k(s) - z^k(0) \right)$$

to ensure $\Omega_{in}(s=0) = 1$.

Moreover we impose a threshold behaviour like $(s-s_{in})^{3/2}$ as required for a P-wave, which is achieved by imposing

$$c_1 = -\sum_{k=2}^{N} k\, c_2 .$$

Equipped with this parametrization we can fit data sets for $e^+e^- \to \pi^+\pi^-$ for $s \le 1\,\mathrm{GeV}^2$.