Introductory Lectures on Resurgence

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Continuum Foundations of Lattice Gauge Theories

Basic Introduction to Resurgence

1. - Stokes Phenomenon and Trans-series
   - Borel Summation basics
   - Recovering Non-perturbative Connection Formulas
2. - Nonlinear Stokes Phenomenon
   - Parametric Resurgence & Phase Transitions
   - 2d U(N) lattice = Gross-Witten-Wadia matrix model
3. - QFT: Euler-Heisenberg and Effective Field Theory
   - Resurgence analysis
   - Inhomogeneous fields
4. - Resurgent Extrapolation
   - The Physics of Padé Approximation
   - Probing the Borel Plane Numerically

Resurgence

the remarkable message from Écalle's theory of resurgence: expansions at different critical/singular/saddle/... points are related in subtle and potentially powerful ways

"The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions."

G. H. Hardy, *Orders of Infinity*, 1910

- trans-series generated by iterations of "trans-monomials"

$$\hbar,\ e^{-1/\hbar},\ \ln\hbar \quad \text{or} \quad \frac{1}{x},\ e^{-x},\ \ln x \quad \text{or} \quad \frac{1}{n},\ e^{-n},\ \ln n$$

- theorem: closure of formal trans-series under all operations of analysis (Écalle, ...)

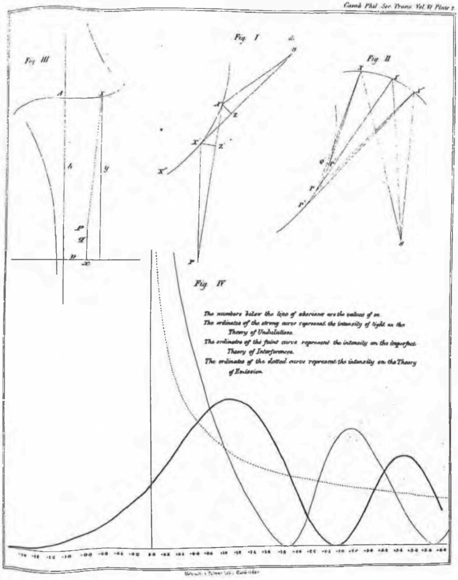
- conjecture: trans-series are practically sufficient "for all natural problems"

Airy, "Spurious Rainbows", and the Airy Function

(Mika-Pekka Markkanen, via Wikimedia Commons)

Airy and Rainbows: The Original "Sign Problem"

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\phi\, e^{i\left(\frac{1}{3}\phi^3 + x\,\phi\right)}$$



Stokes: Solution of The Original "Sign Problem"

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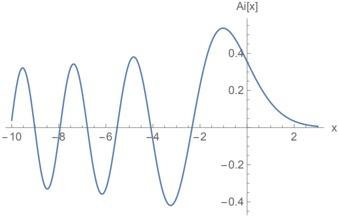
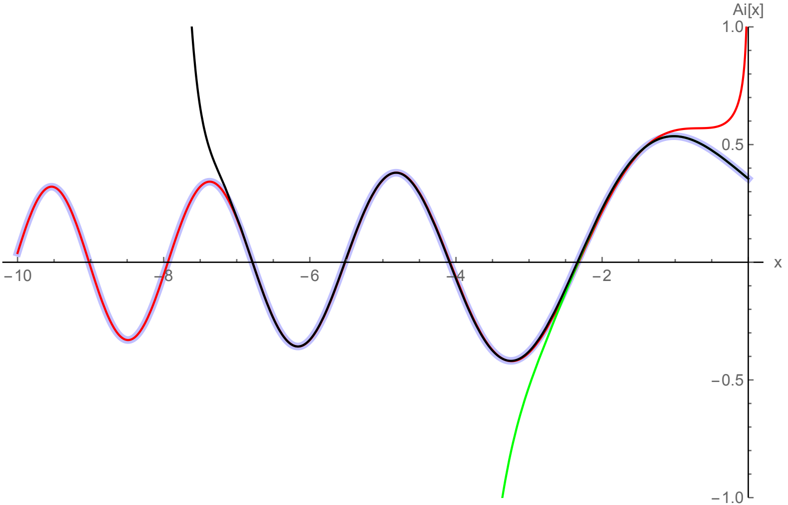


$$\mathrm{Ai}\,(x) \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

"Stokes, by mathematical supersubtlety, transformed Airy's integral into a form by which the light at any point of any of those thirty bands, and any desired greater number of them, could be calculated with but little labour" Lord Kelvin in Stokes's Obituary, 1903

Asymptotic Series are often "better than" Convergent Series



Resurgence of the Airy function

- formal large x solution = "perturbation theory"

$$y(x) \sim \# \frac{e^{\mp \frac{2}{3} x^{\frac{3}{2}}}}{x^{\frac{1}{4}}} \sum_{n=0}^{\infty} \frac{c_n^{(\mp)}}{\left(x^{3/2}\right)^n} \quad , \quad x \to +\infty$$

- recursion relation determines c_n: factorially divergent

$$c_n^{(\mp)} = (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{2\pi\, n!\, \left(\frac{4}{3}\right)^n} = (\mp 1)^n \left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \ldots\right\}$$

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- factorial large order behavior as $n \to \infty$

$$c_n^{+} \sim \frac{1}{2\pi} \frac{(n-1)!}{\left(\frac{4}{3}\right)^n} \left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} - \dots\right)$$

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- generic large order/low order resurgence relation

$$c_n^+ \sim \frac{(n-1)!}{(2\pi)\left(\frac{4}{3}\right)^n} \left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(n-1)} + \left(\frac{4}{3}\right)^2 \frac{385}{4608}\frac{1}{(n-1)(n-2)} - \dots\right)$$

Exercise 1.1: the modified Bessel function $I_\nu(x)$ has the large x asymptotic expansion (dlmf.10.40.E5):

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(\nu)}{x^n} \pm i e^{i\nu\pi} \frac{e^{-x}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\alpha_n(\nu)}{x^n}, \quad \left|\arg(x) - \frac{\pi}{2}\right| < \pi$$

the coefficients depend on the Bessel index parameter ν

$$\alpha_n(\nu) = (-1)^n \frac{\cos(\pi\nu)}{\pi} \frac{\Gamma\left(n + \frac{1}{2} - \nu\right) \Gamma\left(n + \frac{1}{2} + \nu\right)}{2^n \, \Gamma(n+1)}$$

1. Show that the large-order growth ($n \to \infty$) is

$$\alpha_n(\nu) \sim \frac{1}{\pi} \frac{(-1)^n (n-1)!}{2^n} \left(\alpha_0(\nu) - \frac{2\,\alpha_1(\nu)}{(n-1)} + \frac{2^2\,\alpha_2(\nu)}{(n-1)(n-2)} - \dots \right)$$

2. What is the significance of the $\cos(\pi\nu)$ prefactor?

Airy function: the need for non-perturbative completion

- formal large x solution to ODE: "perturbation theory"

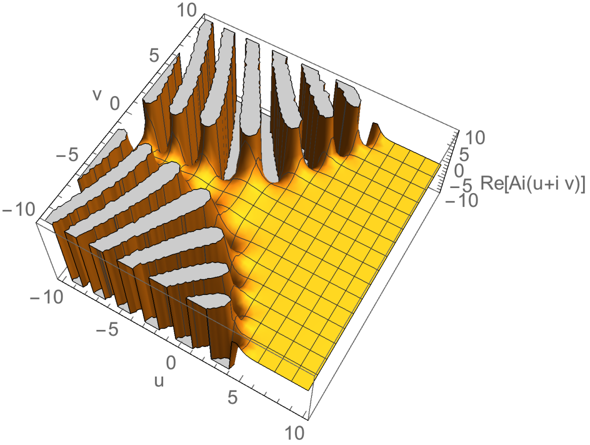
$$y'' = x\,y \;\Rightarrow\; \left\{ \begin{matrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \right\} \sim \frac{e^{\mp\frac{2}{3}x^{\frac{3}{2}}}}{\pi^{\frac{1}{2}}\,x^{\frac{1}{4}}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\,n!\,\left(\frac{4}{3}\,x^{3/2}\right)^n}$$

- these formal asymptotic series satisfy the ODE but do not satisfy the non-perturbative connection formula:

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\,x\right) = \frac{1}{2}e^{\pm\frac{\pi i}{6}}\,\mathrm{Bi}\,(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\,\mathrm{Ai}\,(x)$$

- how do we recover this from the perturbative series?
- "non-perturbative completion"

Stokes Sectors and the Airy Function



- non-perturbative connection formulas connect sectors

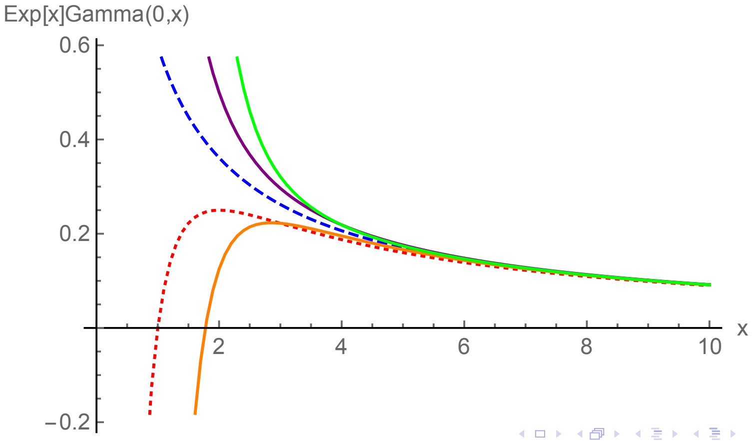
$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}x\right) = \frac{1}{2}e^{\pm\frac{\pi i}{6}}\,\mathrm{Bi}\,(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\,\mathrm{Ai}\,(x)$$

Borel Summation

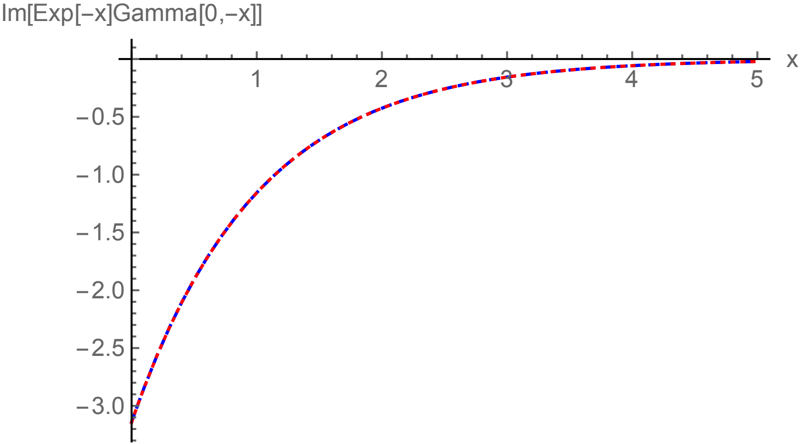
... enter Borel summation ...

Borel Summation: The Basic Idea

$$e^x\,\Gamma(0,x) = \int_0^\infty dt\,\frac{e^{-x\,t}}{t+1} \quad \sim \quad \sum_{n=0}^{\infty}(-1)^n \frac{n!}{x^{n+1}} \quad , \quad x \to +\infty$$

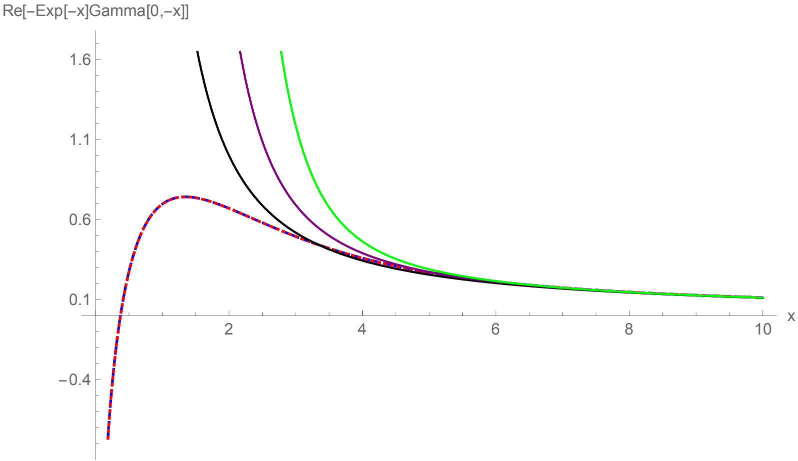


$$\mathrm{Im}\left[e^{-x}\,\Gamma\left(0, e^{\pm i\pi} x\right)\right] \quad = \quad \mp\,\pi\,e^{-x} \qquad , \quad x \to +\infty$$



Borel Summation: The Basic Idea

$$\mathrm{Re}\left[-e^{-x}\,\Gamma\left(0,-x\right)\right] = \mathcal{P}\int_0^\infty dt\, e^{-x\,t}\,\frac{1}{1-t} \sim \sum_{n=0}^{\infty}\frac{n!}{x^{n+1}}$$



Exercise 1.2: Consider the Borel transform $B(t) = \frac{1}{(t+1)^\beta}$, which has a *branch point* at $t = -1$, with exponent $0 < \beta < 1$, and a branch cut along the negative axis: $t \in (-\infty, -1]$. See dlmf.8.6.E5:

$$x^{\beta-1}\, e^x\, \Gamma\left(1-\beta, x\right) = \int_0^\infty dt\, e^{-x\,t}\, \frac{1}{(t+1)^\beta}$$

1. Generate an expression for the $x \to +\infty$ asymptotic expansion of the function $x^{\beta-1}\, e^x\, \Gamma\left(1-\beta, x\right)$.
2. Using the discontinuity of the Borel transform function $B(t) = \frac{1}{(t+1)^\beta}$ across the cut, derive the general connection formula dlmf.8.2.E10 for the incomplete gamma function:

$$e^{\pi\, i\, \beta}\Gamma\left(1-\beta, x\, e^{\pi i}\right) - e^{-\pi\, i\, \beta}\Gamma\left(1-\beta, x\, e^{-\pi i}\right) = \frac{2\pi i}{\Gamma\left(\beta\right)}$$

Exercise 1.3: Consider the asymptotic expansion of the "trigamma" function ($B_n(q)$ is the Bernoulli polynomial)

$$\psi^{(1)}\left(\frac{1+x}{2}\right) \sim \sum_{n=0}^{\infty} \frac{2^{n+1} B_n\left(\frac{1}{2}\right)}{x^{n+1}} \quad , \quad x \to +\infty$$

1. Use Borel summation to show that

$$\psi^{(1)}\left(\frac{1+x}{2}\right) = 2\int_0^{\infty} dt\, e^{-x\,t} \frac{t}{\sinh(t)}$$

   Hint: see dlmf.24.4.E27 & dlmf.24.4.E2.
   Note that there is an *infinite number* of Borel poles.

2. Show that the real part of this function, along the imaginary axis, has an infinite series of exponential terms

$$\mathrm{Re}\left[\psi^{(1)}\left(\frac{1+i\,x}{2}\right)\right] \sim 0 - 2\pi^2 \sum_{k=1}^{\infty} (-1)^k k\, e^{-k\,\pi\,x} \quad , \quad x \to +\infty$$

   and show that these terms are required for consistency with dlmf.5.15.E6, the non-perturbative reflection formula

Borel Summation and the Airy function

- formal large x solution to ODE: "perturbation theory"

$$y'' = x\,y \;\Rightarrow\; \left\{ \begin{matrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \right\} \sim \frac{e^{\mp\frac{2}{3}x^{\frac{3}{2}}}}{\pi^{\frac{1}{2}}\,x^{\frac{1}{4}}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\,n!\,\left(\frac{4}{3}\,x^{3/2}\right)^n}$$

- Borel transform for the $\mathrm{Ai}(x)$ series factor:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{(2\pi)\,n!} \frac{t^n}{n!} = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

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- Laplace transform recovers the $\mathrm{Ai}(x)$ formal series:

$$\mathrm{Ai}(x) = \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{4\pi}\,x^{\frac{1}{4}}} \left(\frac{4}{3}x^{\frac{3}{2}}\right) \int_0^{\infty} dt\, e^{-\frac{4}{3}x^{\frac{3}{2}}t}\; {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -t\right)$$

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- cut for $t \in (-\infty, -1]$: rotate t contour as x rotates

$${}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t+i\,\epsilon\right) - {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t-i\,\epsilon\right) = i\;{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t\right)$$

- discontinuity across cut $\Rightarrow$ non-pert. connection formula:

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}x\right) = \frac{1}{2}e^{\pm\frac{\pi i}{6}}\,\mathrm{Bi}\,(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\,\mathrm{Ai}\,(x)$$

Borel Summation and the Airy function

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- lesson: Borel transform singularity encodes the connection formula

Exercise 1.4: The modified Bessel function $K_\nu(x)$ has a Borel representation (Airy is associated with $\nu = \frac{1}{3}$) for $x > 0$

$$K_\nu(x) = \sqrt{2\pi x}\, e^{-x} \int_0^\infty dt\, e^{-2\,x\,t}\ {}_2F_1\left(\frac{1}{2} - \nu, \frac{1}{2} + \nu, 1; -t\right)$$

1. Derive the asymptotic expansion:

$$K_\nu\left(x\right) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{x^k} \quad , \quad x \to \infty, \quad |\arg(x)| \leq \frac{3\pi}{2} - \delta$$

$$a_k(\nu) = \frac{\cos(\pi\,\nu)}{\pi} \left(-\frac{1}{2}\right)^k \frac{\Gamma\left(k + \frac{1}{2} - \nu\right)\Gamma\left(n + \frac{1}{2} + \nu\right)}{\Gamma(k+1)}$$

2. Use the discontinuity of the hypergeometric function (dlmf.15.2.E3) to derive the non-perturbative connection formula of $K_\nu(x)$ (dlmf.10.34.E2):

$$K_\nu\left(ze^{m\pi i}\right) = e^{-m\nu\pi i} K_\nu\left(z\right) - \pi i \sin\left(m\nu\pi\right) \csc\left(\nu\pi\right) I_\nu\left(z\right)$$

. . . now use a "semiclassical" saddle approach, and compare

. . .

- "parametric resurgence"
- for a large class of QM spectral problems

$$E(\hbar, N) = E_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{\#}{\hbar} \right)^{N+\frac{1}{2}} e^{-S/\hbar} \mathcal{P}_{\text{inst}}(\hbar, N) + \ldots$$

- one-instanton fluctuation factor:

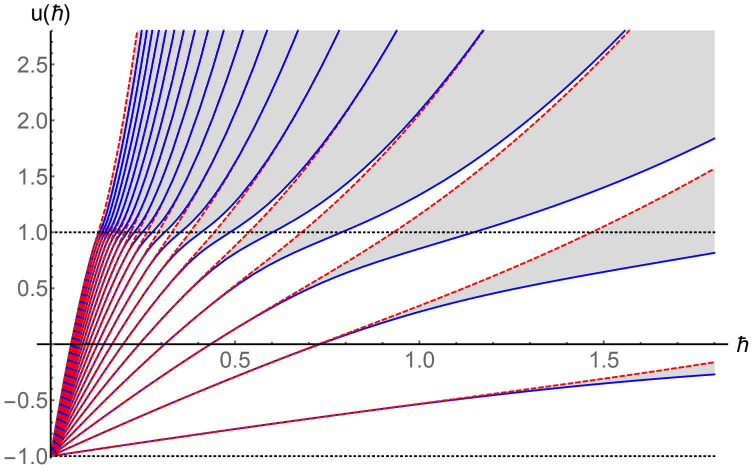
$$\mathcal{P}_{\text{inst}}(\hbar, N) = \frac{\partial E_{\text{pert}}}{\partial N} \exp \left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N + \frac{1}{2}\right) \hbar^2}{S} \right) \right]$$

- the entire trans-series can be decoded in terms of the perturbative series

Resurgence in Infinite Dimensions: the QM Path Integral

the remarkable message from Écalle's theory of resurgence: expansions at different critical/singular/saddle/... points are related in subtle and potentially powerful ways

... even with an infinite number of saddles !



- the trans-series transmutes across phase transitions

Exercise 1.5

1. Translate the standard notation for the Mathieu equation $w'' + (a - 2q\cos(2z))w = 0$, from https://dlmf.nist.gov/28.2.E1, into Schrödinger form $-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi(x) + \cos(x)\psi(x) = E\psi(x)$.
2. Hence convert the large q expansion from https://dlmf.nist.gov/28.8.E1 into an expression for the first 8 terms of the perturbative small $\hbar$ expansion of the energy, $E_{\text{pert}}(\hbar, N)$, where N is the band label.
3. With this information, compute the small $\hbar$ expansion of

$$\frac{\partial E_{\text{pert}}}{\partial N}\exp\left[8\int_0^{\hbar}\frac{1}{\hbar^3}\left(\frac{\partial E_{\text{pert}}}{\partial N} - \hbar + \frac{\hbar^2}{8}\left(N + \frac{1}{2}\right)\right)\right]$$

and compare with the leading non-perturbative Mathieu band splitting in https://dlmf.nist.gov/28.8.E2