Exercises for Introductory Lectures on Resurgence

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Continuum Foundations of Lattice Gauge Theories

Exercises: Introductory Lectures on Resurgence

1. Lecture 1: Airy, Stokes, Borel and Trans-series
2. Lecture 2: Nonlinear Stokes, Painlevé and Gross-Witten-Wadia Model
3. Lecture 3: QFT example: Euler-Heisenberg Effective Action
4. Lecture 4: Resurgent Extrapolation, Padé & Analytic Continuation

Exercise 1.1: the modified Bessel function $I_\nu(x)$ has the large x asymptotic expansion (dlmf.10.40.E5):

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n(\nu)}{x^n} \quad , \quad x \to +\infty$$

the coefficients depend on the Bessel index parameter ν

$$\alpha_n(\nu) = (-1)^n \frac{\cos(\pi\nu)}{\pi} \frac{\Gamma\left(n+\frac{1}{2}-\nu\right)\Gamma\left(n+\frac{1}{2}+\nu\right)}{2^n\,\Gamma(n+1)}$$

1. Show that the large-order growth ($n \to \infty$) is

$$\alpha_n(\nu) \sim \frac{1}{\pi} \frac{(-1)^n (n-1)!}{2^n} \left(\alpha_0(\nu) - \frac{2\,\alpha_1(\nu)}{(n-1)} + \frac{2^2\,\alpha_2(\nu)}{(n-1)(n-2)} - \dots\right)$$

2. What is the significance of the $\cos(\pi\nu)$ prefactor?

Exercise 1.2: Consider the Borel transform $B(t) = \frac{1}{(t+1)^\beta}$, which has a *branch point* at $t = -1$, with exponent $0 < \beta < 1$, and a branch cut along the negative axis: $t \in (-\infty, -1]$:

$$x^{\beta-1}\, e^x\, \Gamma\left(1-\beta, x\right) = \int_0^\infty dt\, e^{-x\,t}\, \frac{1}{(t+1)^\beta}$$

1. Generate an expression for the $x \to +\infty$ asymptotic expansion of the function $x^{\beta-1}\, e^x\, \Gamma\left(1-\beta, x\right)$.
2. Using the discontinuity of the Borel transform function $B(t) = \frac{1}{(t+1)^\beta}$ across the cut, derive the general connection formula dlmf.8.2.E10 for the incomplete gamma function:

$$e^{\pi\, i\, \beta}\Gamma\left(1-\beta, z\, e^{\pi i}\right) - e^{-\pi\, i\, \beta}\Gamma\left(1-\beta, z\, e^{-\pi i}\right) = \frac{2\pi i}{\Gamma\left(\beta\right)}$$

Exercise 1.3: Consider the asymptotic expansion of the trigamma function ($B_n(q)$ is the Bernoulli polynomial)

$$\psi^{(1)}\left(\frac{1+x}{2}\right) \sim \sum_{n=0}^{\infty} \frac{2^{n+1} B_n\left(\frac{1}{2}\right)}{x^{n+1}} \qquad , \quad x \to +\infty$$

1. Use Borel summation to show that

$$\psi^{(1)}\left(\frac{1+x}{2}\right) = 2\int_0^\infty dt\, e^{-x\,t} \frac{t}{\sinh(t)}$$

   Hint: see dlmf.24.4.E27 & dlmf.24.4.E2.

   Note that there is an *infinite number* of Borel poles.

2. Show that the real part of this function, along the imaginary axis, has an infinite series of exponential terms

$$\text{Re}\left[\psi^{(1)}\left(\frac{1+i\,x}{2}\right)\right] \sim 0 \; -2\pi^2 \sum_{k=1}^{\infty} (-1)^k k\, e^{-k\,\pi\,x} \qquad , \quad x \to +\infty$$

   and show that these terms are required for consistency with dlmf.5.15.E6, the non-perturbative reflection formula

Exercise 1.4: The modified Bessel function $K_\nu(x)$ has a Borel representation (Airy is associated with $\nu = \frac{1}{3}$) for $x > 0$

$$K_\nu(x) = \sqrt{2\pi x}\, e^{-x} \int_0^\infty dt\, e^{-2\,x\,t}\ {}_2F_1\left(\frac{1}{2} - \nu, \frac{1}{2} + \nu, 1; -t\right)$$

1. Derive the asymptotic expansion:

$$K_\nu\left(x\right) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{x^k} \quad , \quad x \to +\infty$$

$$a_k(\nu) = \frac{\cos(\pi\,\nu)}{\pi} \left(-\frac{1}{2}\right)^k \frac{\Gamma\left(k + \frac{1}{2} - \nu\right)\Gamma\left(k + \frac{1}{2} + \nu\right)}{\Gamma(k+1)}$$

2. Use the discontinuity of the hypergeometric function (dlmf.15.2.E3) to derive the non-perturbative connection formula of $K_\nu(x)$ (dlmf.10.34.E2):

$$K_\nu\left(ze^{m\pi i}\right) = e^{-m\nu\pi i} K_\nu\left(z\right) - \pi i \sin\left(m\nu\pi\right)\csc\left(\nu\pi\right) I_\nu\left(z\right)$$

Exercise 1.5:

1. Translate standard notation for the Mathieu equation $w'' + (a - 2q\cos(2z))w = 0$, from https://dlmf.nist.gov/28.2.E1, into Schrödinger form $-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi(x) + \cos(x)\psi(x) = E\psi(x)$.
2. Hence convert the perturbative small $\hbar$ expansion from https://dlmf.nist.gov/28.8.E1 into an expression for the first 8 terms of the perturbative energy expansion $E_{\text{pert}}(\hbar, N)$, where N is the band label.
3. Compute the small $\hbar$ expansion of

$$\frac{\partial E_{\text{pert}}}{\partial N}\exp\left[8\int_0^{\hbar}\frac{1}{\hbar^3}\left(\frac{\partial E_{\text{pert}}}{\partial N} - \hbar + \frac{\hbar^2}{8}\left(N + \frac{1}{2}\right)\right)\right]$$

and compare with the leading non-perturbative Mathieu splitting in https://dlmf.nist.gov/28.8.E2

Resurgence in Nonlinear ODEs: Painlevé II = "nonlinear Airy"

$$y''(x) = x\, y(x) + 2\, y^3(x)$$

Exercise 2.1:

1. Show that the general Painlevé II solution has a meromorphic expansion with only poles for moveable singularities (those associated with boundary conditions):

$$y(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0 (x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + \ldots$$

2. Show that all the coefficients of this expansion are expressed in terms of the pole location x_0 and the coefficient h_0 of the cubic term.

3. Change the nonlinearity of the equation from $y^3(x)$ to $y^4(x)$ and show that this Painlevé integrability condition fails (comment: nevertheless, despite being nonintegrable, all the subsequent resurgent trans-series analysis still holds)

Exercise 2.2:

1. Generate many terms, and verify the large order behavior of the coefficients of the formal $x \to -\infty$ series for the Painlevé II Hastings-McLeod solution
2. Numerically identify the Stokes constant to high precision

$$0.1466323... = \frac{1}{\pi}\sqrt{\frac{2}{3\pi}}$$

3. Confirm the large-order/low-order resurgence relation

Exercise 2.3: Consider the Painlevé III (Okamoto form) for the GWW model:

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

1. Show that in the $t > 1$ region this equation linearizes to

$$t^2\Delta'' + t\Delta' + \frac{N^2}{t^2}\left(1-t^2\right)\Delta \approx 0$$

   which is solved by the Bessel functions $J_N\left(\frac{N}{t}\right)$, $Y_N\left(\frac{N}{t}\right)$

2. Show that for $t < 1$ the dominant large N solution is algebraic, $\Delta(t) \sim \sqrt{1-t}$, from which the formal large N series solution can be generated.

Exercise 2.4:

1. Generate many terms of the formal large N solution for $\Delta(t, N)$ in the small t regime

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(\lambda)}{N^{2n}}$$

2. Derive the form of the first non-perturbative correction to this formal large N expansion, including the first few fluctuation corrections:

$$\Delta_{NP}(t, N) \sim f(t)\, e^{-N\, S_{\rm weak}(t)} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(\lambda)}{N^n} + \dots$$

3. Show that the subleading corrections to the large n growth of the coefficient functions $d_n^{(0)}(\lambda)$ are associated with the expansion terms $d_n^{(1)}(\lambda)$ of the first non-perturbative correction to the formal large N expansion.

Exercise 3.1: Analytically continue the integral representation of the Hurwitz zeta function (for $Re(s) > 1$, $Re(z) > 0$)

$$\zeta_H(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, e^{-z\,t} \frac{t^{s-1}}{1-e^{-t}} \qquad (1)$$

into the region $Re(s) > -2$ to obtain

$$\begin{aligned}\zeta_H(s,z) &= \frac{z^{1-s}}{s-1} + \frac{z^{-s}}{2} + \frac{s\,z^{-1-s}}{12} \\ &+ \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1-s}}\, e^{-2z\,t} \left(\coth(t) - \frac{1}{t} - \frac{t}{3}\right) \qquad (2)\end{aligned}$$

Hence show that $\zeta_H(-1,z) = -\frac{1}{12} + \frac{z}{2} - \frac{z^2}{2}$, and

$$\begin{aligned}\zeta'_H(-1,z) &= \frac{1}{12} - \frac{z^2}{4} - \zeta_H(-1,z)\ln z \\ &- \frac{1}{4} \int_0^\infty \frac{dt}{t^2}\, e^{-2z\,t} \left(\coth(t) - \frac{1}{t} - \frac{t}{3}\right) \qquad (3)\end{aligned}$$

Exercise 3.2: In the zeta function method we define

$$\ln \det(\text{operator}) := -\zeta'(0) \qquad \text{where} \qquad \zeta(s) := \sum_{\text{spectrum}\,\lambda} \frac{1}{\lambda^s}$$

Given that the eigenvalues of the Dirac operator in a constant magnetic field B are given by the Landau level result

$$\lambda_n^{\pm} = m^2 + p_\perp^2 + eB(2n + 1 \pm 1) \qquad , \qquad n = 0, 1, 2, \ldots$$

with the Landau degeneracy factor $\frac{eB}{2\pi}$, derive the Euler-Heisenberg effective action $\mathcal{L}$ by showing that

$$\mathcal{L} = \frac{e^2 B^2}{2\pi^2} \left\{ \zeta_H' \left(-1, \frac{m^2}{2eB} \right) + \zeta_H \left(-1, \frac{m^2}{2eB} \right) \ln \left(\frac{m^2}{2eB} \right) - \frac{1}{12} + \frac{1}{4} \left(\frac{m^2}{2eB} \right)^2 \right\} \tag{4}$$

Exercise 3.3:

1. Show that the Euler-Heisenberg effective action can be expressed in terms of the log of the Barnes gamma function (https://dlmf.nist.gov/5.17)

$$\mathcal{L}(b) = \frac{b^2}{2\pi^2}\left[-\log(A) + \frac{1}{16b^2} + \left(-\frac{1}{8b^2} + \frac{1}{4b} - \frac{1}{12}\right)\log\left(\frac{1}{2b}\right) - \log\left(G\left(\frac{1}{2b}\right)\right) - \left(1 - \frac{1}{2b}\right)\log\left(\Gamma\left(\frac{1}{2b}\right)\right)\right]$$

where $b \equiv \frac{eB}{m^2}$.

2. Hence study the small b and large b expansions, showing that the strong field expansion has a finite radius of convergence.

Exercise 3.4: For scalar QED the spectrum of the Klein-Gordon operator in a constant B field has no spin projection term, so it is given by

$$\lambda_n = m^2 + p_\perp^2 + eB(2n+1) \qquad , \qquad n = 0, 1, 2, \ldots$$

1. Hence use the zeta function method to show that the EH effective action for scalar QED has the integral representation
$$\mathcal{L}_{\rm scalar} = \frac{e^2 B^2}{16\pi^2} \int_0^\infty ds \, \frac{1}{s^2} \left(\frac{1}{\sinh(s)} - \frac{1}{s} + \frac{s}{6} \right)$$
2. Generate the asymptotic weak field expansion of the scalar QED effective action and comparee the large-order behavior of the expansion coefficients with the spinor QED case.
3. Compute the leading strong-field behavior by inspection of the Borel integral representation, and relate this to the scalar QED beta function.

Exercise 3.5: Consider an electric field, directed in the z direction, with a one-dimensional cosine inhomogeneity: $E(t) = E\cos(\omega t)$.

1. Compute the LCFA effective action by integrating the Euler-Heisenberg effective action over one period of the field, with the constant field replaced by its time-dependent form.

2. Show that the coefficients of the weak field expansion grow factorially with perturbative order, and with subleading corrections of both power-law and exponential form. Compute the first few power-law correction terms.

3. Demonstrate the resurgence relation by showing that the power-law corrections in the previous part are related to the fluctuations about the instanton factors for the imaginary part of the effective action.

Exercise 3.6: Consider the classical Euclidean equations of motion for scalar QED in a (generally inhomogeneous) background electromagnetic field :

$$\ddot{x}_\mu = 2ieF_{\mu\nu}(x)\,\dot{x}_\nu$$

where the dots refers to derivatives wrt the proper-time and x_μ is the 4 dim spacetime coordinate.

1. Show that for any solution $\dot{x}_{\rm cl}^2$ is a constant of motion.
2. Show that the closed trajectory (with period T) for a constant E field is a circle, and evaluate the classical action.
3. Show that for a time dependent (but spatially constant) linearly polarized electric field, with Euclidean vector potential $A_3 = -i\frac{E}{\omega}\,f(\omega\,x_4)$, where ω is a frequency scale parameter, the classical action can be expressed as

$$S[x_{\rm cl}](T) = -\frac{\dot{x}_{\rm cl}^2}{4}\,T + \frac{1}{2}\int_0^T d\tau\,(\dot{x}_4^{\rm cl})^2$$

Exercise 4.1: In resurgence it is convenient to re-write a factorial large order growth expression, with power-law corrections, as an expansion in "diminishing" factorials:

$$b_n \sim \Gamma(n+a) \sum_{m=0}^{\infty} \frac{d_m}{n^m} \qquad , \quad n \to +\infty$$

can be written as

$$b_n \sim \sum_{k=0}^{\infty} c_k \, \Gamma(n+a-k) \qquad , \quad n \to +\infty$$

where the coefficients c_k are expressed in terms of the d_m via the Stirling numbers of the first kind (hint: dlmf.26.8.ii):

$$c_k = \sum_{l=0}^{k} S^{(1)}(k,l) \sum_{j=0}^{l} (-a)^l \binom{j-l}{j} d_{l-j}$$

Verify this with some examples.

Exercise 4.2: The perturbative expansion $C(x) \sim \sum_{n=1}^{\infty} c_n\, x^n$ determines the anomalous dimension in the Hopf algebraic renormalization of 4 dimensional Yukawa theory. The coefficients c_n are positive integers, enumerating combinatorial objects known as "connected chord diagrams". This sequence is listed on the OEIS as https://oeis.org/A000699.

1. Generate 100 terms using the recursion formula listed on the OEIS and then analyze them using Richardson acceleration to show that

$$c_n \sim \frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{e\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}+O\left(\frac{1}{n^3}\right)\right)$$

2. $C(x)$ satisfies a nonlinear ODE: $C(x)\left(1-2x\frac{d}{dx}\right)C(x)=x-C(x)$. Show that the first non-perturbative correction term $C_{\rm np}(x)$ satisfies a linear ODE $\frac{d}{dx}\ln\left(C(x)C_{\rm np}(x)/x\right)=\frac{1}{2xC(x)}$. Therefore $C_{\rm np}(x)$ is immediately solved in terms of $C(x)$.

3. Hence expand $C_{\rm np}(x)$ at small x and compare with part 1.

Exercise 4.3:

1. investigate Darboux's theorem numerically for the hypergeometric function, which has a branch point at $t = 1$

$$
{}_2F_1\left(a, b, c; t\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)n!} t^n
$$

2. Compare with the exact expansion of the hypergeometric function about $t = 1$ (see dlmf.15.8)
3. What happens if $a + b - c =$ integer?

Exercise 4.4: Explore the Padé pole structures for various functions with interesting singularities.

Exercise 4.5: Use the BenderWu package of Sulejmanpasic and Ünsal (https://arxiv.org/abs/1608.08256) to compute many terms of the perturbative expansion of the ground state energy of the anharmonic oscillator, and of the symmetric double-well potential, and use various extrapolation methods to explore the Borel plane structure of the (truncated) Borel transform functions. Comment on the physical meaning of what you find.

Exercise 4.6: Generate a finite amount (e.g. 50 orders) of perturbative data by expanding the function $x^{-\frac{1}{3}}\, e^{x}\, \Gamma\left(\frac{1}{3}, x\right)$ as $x \to +\infty$, and use this as input for resummation using the following methods:
(i) optimal truncation; (ii) Padé in the x plane; (iii) Padé in the Borel plane; (iv) Padé in the Borel plane after a conformal map into the unit disc; (v) Padé in the Borel plane after a uniformizing map.
Explore how things change as you change the amount of perturbative data.
Comment on the similarities and differences between the resulting reconstructions of the function.

Exercise 4.7: Generate a finite amount (e.g. 20 orders) of perturbative data by expanding the Borel transform $B(p)$ of the Airy function: $B(p) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -p\right)$, and use this as input for extrapolation using the following methods. Probe specifically the vicinity of the branch point at $p = -1$ and cut $p \in (-\infty, -1]$

1. Padé
2. Padé after a conformal map into the unit disc
$$p = \frac{4z}{(1-z)^2} \quad \longleftrightarrow \quad z = \frac{\sqrt{1+p}-1}{\sqrt{1+p}+1}$$
3. Padé after a uniformizing map via the elliptic nome function
$$p = -\varphi(z) = -16z + 128z^2 - 704z^3 + \ldots \longleftrightarrow z = \exp\left[-\pi \frac{\mathbb{K}(1+p)}{\mathbb{K}(-p)}\right]$$
where $\varphi(z) =$ InverseEllipticNomeQ[z] in Mathematica.

Exercise 4.8: Painlevé I equation: $y''(x) = 6\,y^2(x) - x$

1. Show that the Écalle critical variable is $\frac{(24x)^{5/4}}{30}$
2. With the *tritronquée* ansatz

$$y(x) \sim -\sqrt{\frac{x}{6}} \left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{30}{(24\,x)^{5/4}} \right)^{2n} \right) \qquad , \qquad x \to +\infty$$

show that the coefficients c_n satisfy the recursion formula:

$$c_n = -4(n-1)^2 c_{n-1} - \frac{1}{2} \sum_{m=2}^{n-2} c_m\, c_{n-m} \quad , \quad n \geq 3$$

with $c_1 = \frac{4}{25}$ and $c_2 = -\frac{392}{625}$.

3. Show that the large order growth of the coefficients is

$$c_n \sim \frac{1}{\pi} \sqrt{\frac{6}{5\pi}} (-1)^{n+1} \Gamma\left(2n - \frac{1}{2}\right) \left(1 - \frac{\frac{1}{8}}{\left(2n - \frac{3}{2}\right)} + \frac{\frac{9}{128}}{\left(2n - \frac{3}{2}\right)\left(2n - \frac{5}{2}\right)} + \ldots \right)$$

Exercise 4.9: Painlevé I equation: $y''(x) = 6\,y^2(x) - x$

1. Find the form of the first non-perturbative correction to the perturbative *tritronquée* solution of the previous problem, and compute the first few fluctuation terms.
2. Show how these fluctuation terms relate to the subleading corrections of the large order growth in part 3 of the previous question.