Resurgence

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motivation

"decode" the path integral

$$\int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S[\phi]}$$

$\uparrow$

- how to evaluate?
- how does this change as parameters (e.g. chemical potential, masses, couplings, ...) change?

- strong coupling $\Longleftrightarrow$ weak coupling
- Minkowski $\Longleftrightarrow$ Euclidean
- $\mu \Longleftrightarrow i\mu$
- $T \Longleftrightarrow 1/T$
- non-equilibrium
- intense background fields ...

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dual role of path integral

1. generator of perturbation theory
2. generator of saddle expansion

goal: we should understand how these fit together

$\Rightarrow$ connecting perturbative & non-perturbative physics: this is what "resurgence" is for!

e.g. propagator

$$\langle x_t | e^{-\frac{iHt}{\hbar}} | x_0 \rangle \nearrow \sum_n e^{-\frac{iE_n t}{\hbar}} \psi_n^*(x_t)\, \psi_n(x_0)$$

$$\searrow \sum_{\text{geodesics}} \frac{e^{\frac{i d^2(x_t, x_0)}{2\hbar t}}}{\sqrt{4\pi t}} (1 + \dots)$$

large $t \Leftrightarrow$ spectral information

small $t \Leftrightarrow$ geometric information

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cf. heat kernel

$$\mathrm{tr}(e^{-Ht}) \nearrow \sum_n e^{-\lambda_n t} \quad , \quad t \to \infty$$

$$\searrow \frac{1}{(4\pi t)^{d/2}} \sum_n c_n t^n \quad , \quad t \to 0$$

λ_n : spectral data

c_n : geometric data

hamiltonian $\Updownarrow$ Lagrangian

These come from the same place, so they must be related!

Q : is this useful?!?

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perturbation theory generically produces asymptotic series

– we need to learn efficient, practical ways to handle these

$\longrightarrow$ Borel-Écalle resurgent summation

- asymptotic series contain a wealth of information: we need to know how to extract this efficiently
- resurgence is a natural and efficient unification of perturbation theory and non-perturbative physics

"all-orders semiclassical physics"

"complex analysis with asymptotic series"

<u>trans-series</u>

asymptotic series $\left(f(\hbar) \sim \sum_n^\infty c_n \hbar^n \quad , \quad \hbar \to 0 \right)$

provides <u>some</u> information about the function $f(\hbar)$, but not <u>all</u>

a trans-series extension is designed to encode "<u>all</u>" information about the function

$$\sum_n c_n \hbar^n \longrightarrow \sum_{n=0}^\infty \sum_{k=1}^\infty \sum_{l=2}^k c_{nkl}\, \hbar^n e^{-\frac{kS}{\hbar}} (\ln \hbar)^l$$

fluctuation sum

"instanton sum"

(finite) sum over logs, associated with instanton - anti-instanton interactions

. this form is a "semiclassical" trans-series general trans-series is constructed out of all iterations of 3 monomials:

$$\hbar \quad , \quad e^{-1/\hbar} \quad , \quad \ln \hbar$$

$\Rightarrow$ in general include $\ln(\ln(\ln \ldots \ln(\hbar))\ldots)$
(e.g. familiar from Feynman integrals)

also $e^{e^{-1/\hbar}} \ldots$

- formal trans-series are "closed" under all operations of analysis
(contrast with series which are obviously not closed)

- this is a general structure, conjectured to be relevant for all "natural problems".

In practice, this means for classes of problems:

- coupled differential / difference / integral eq's
- finite dimensional exponential integrals

⋮

by now, many concrete examples in QM, matrix models, integrable QFT, lattice QFT, localizable SUSY QFT, Chern-Simons, Hopf algebra renormalization, ...

(note: integrability is <u>not</u> required for resurgence, even though many explicit examples are related to integrable structures)

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the story begins with Airy and Stokes (1830's — 1850's)

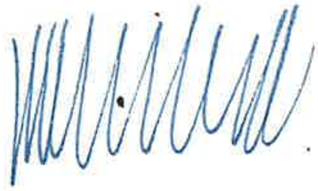
- "spurious"/"supernumerary" rainbows with ~30 bands were measured in laboratory conditions in 1830's
- Airy explained how rainbows work and in doing so derived an integral representation for what we now call the Airy function $Ai(x)$

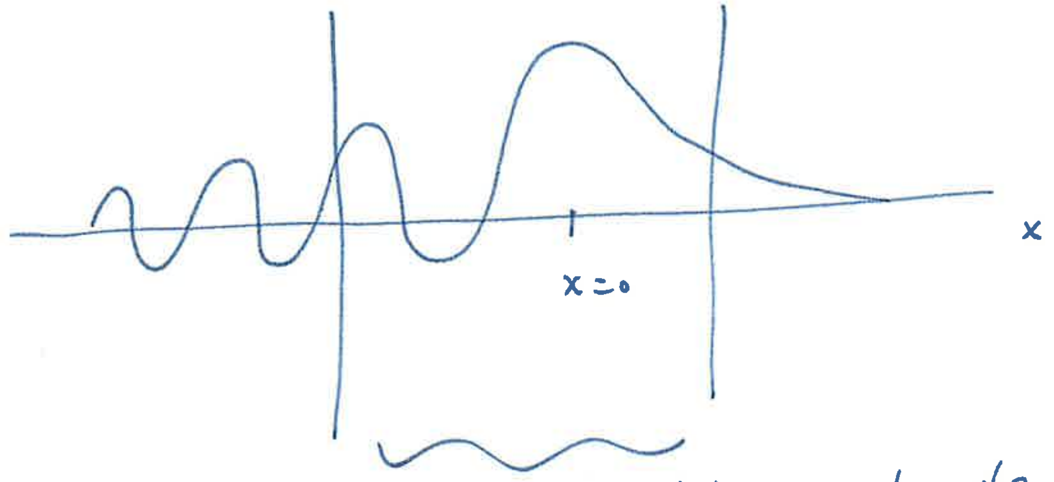
$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\left(\frac{t^3}{3} + x\,t\right)}$$

integration variable

argument of $Ai(x)$

this integral cannot be evaluated on the real line, especially at large x

integrand ~  → averages to a small value?



Airy could evaluate this much, but no more

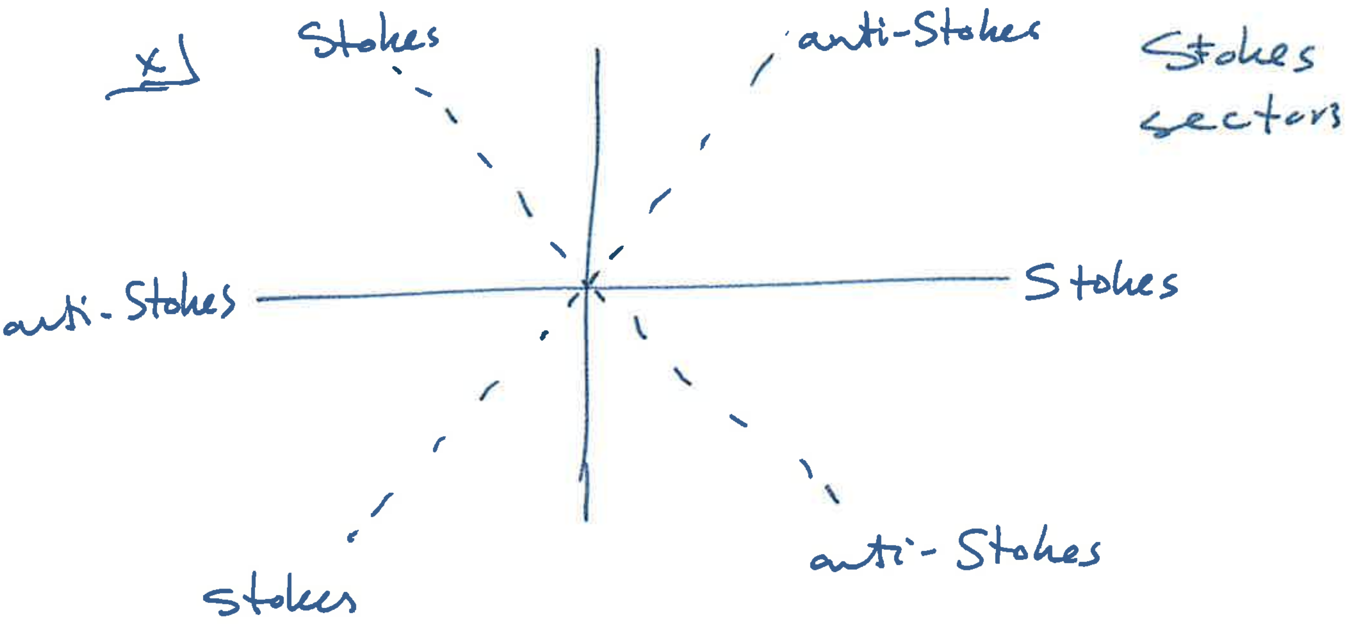
Stokes applied the (relatively new at the time) methods of Cauchy et al, to show that

$$\mathrm{Ai}(x) \sim \begin{cases} \dfrac{e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\pi}\; x^{1/4}}, & x \to +\infty \\[2ex] \dfrac{\cos\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}\;(-x)^{1/4}}, & x \to -\infty \end{cases}$$

$x \to +\infty$ governed by 1 saddle point

$x \to -\infty$ ——— 2 ———(s)

more importantly, he identified "Stokes sector" in the complex plane



Stokes phenomenon: there are changes in the saddle structure as the phase of the variable x crosses a Stokes line

- this is necessary in order to satisfy the non-perturbative "connection formulae" for $Ai(x)$, which is analytic in $\mathbb{C}$

Stokes: how is this analyticity compatible with asymptotic expansions

① we will study how the formal asymptotic expansions, at large x, can be upgraded to trans-series which are consistent with analyticity $\longrightarrow$ Borel summation

② next we show how a saddle expansion of the integral matches this structure

1 "perturbation theory" for Ai(x)

ODE : $y'' = xy$

Liouville-Green : write $y(x) = \dfrac{e^{-S(x)}}{\sqrt{S'(x)}}$
("WKB")

$$\Longrightarrow \quad S' \sim \pm\sqrt{x}$$
$$S \sim \pm \tfrac{2}{3} x^{3/2}$$
$$\sqrt{S'} \rightsquigarrow x^{1/4}$$

try ansatz for decaying solution as $x \to +\infty$

$$y \sim x^{\alpha} e^{-\frac{2}{3}x^{3/2}} (1 + \dots)$$

$$\rightarrow \quad y'' \sim \left(x - \frac{(2\alpha + 1/2)}{\sqrt{x}} + o\left(\tfrac{1}{x}\right)\right) y$$

$$\Downarrow \quad \Rightarrow \quad \alpha = -1/4 \quad \checkmark$$

extend ansatz

$$y \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \sum_{n=0}^{\infty} \frac{a_n}{x^{3n/2}}$$

$\rightarrow$ recurrence formula for a_n

$$\rightarrow \quad a_n = \frac{1}{2\pi} \frac{\Gamma(n+\frac{1}{6})\Gamma(n+5/6)}{n!\,\left(\frac{4}{3}\right)^n} = \left\{1, \frac{5}{48}, \frac{385}{4608}, \frac{85085}{663552}, \dots\right\}$$

remarkable (generic!) duality between large n behavior of a_n and large x behavior of $Ai(x)$

$$y \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{x^{1/4}} \sum_{n=0}^{\infty} \frac{a_n^{(\mp)}}{x^{3n/2}}, \qquad x \to +\infty$$

upper: $-$: Airy $Ai(x)$

lower: $+$: Airy $Bi(x)$

as $n \to +\infty$

$$a_n^{(+)} \sim \frac{1}{2\pi} \frac{(n-1)!}{\left(\frac{4}{3}\right)^n} \left(1 - \frac{5}{36}\frac{1}{n} + \frac{25}{2592}\frac{1}{n^2} + \dots\right)$$

$\downarrow$ convert to "diminishing factorial form"

$$= \frac{1}{2\pi} \frac{(n-1)!}{\left(\frac{4}{3}\right)^n} \left(1 - \left(\frac{4}{3}\right)\frac{\frac{5}{48}}{(n-1)} + \left(\frac{4}{3}\right)^2 \frac{\frac{385}{4608}}{(n-1)(n-2)} + \dots\right)$$

see exercise in Lecture 4 to see how to use Stirling numbers to do this conversion efficiently

"large order / low order resurgence relation"

- the large order in n <u>resurges</u> the low order coefficients in x - from the other saddle exercise 1.1

apply to $I_\nu(x)$, modified Bessel fn., where the coefficients a_n are not numbers, but polynomials in the index ν. Several interesting observations follow ...

result of "perturbation theory" $x \to +\infty$

normalization convention

$$\left.\begin{matrix} 2\,Ai(x) \\ Bi(x) \end{matrix}\right\} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{\sqrt{\pi}\; x^{\frac{1}{4}}} \sum_{n=0}^{\infty} (\mp)^n \frac{\Gamma(n+\frac{1}{6})\,\Gamma(n+\frac{5}{6})}{(2\pi)(n!)\,\left(\frac{4}{3}x^{3/2}\right)^n}$$

$$x \to e^{\mp\frac{2\pi i}{3}} x \quad \Rightarrow \quad x^{3/2} \to -x^{3/2}$$

so it looks like $Ai \Longleftrightarrow Bi$ interchange

(up to a factor

$$Ai\left(e^{\mp\frac{2\pi i}{3}}x\right) \overset{?}{\longrightarrow} \frac{1}{2}\, e^{\pm\frac{i\pi}{6}}\, Bi(x)$$

(from $\frac{1}{x^{1/4}}$)

growing as $x \to +\infty$

but this is not correct - it is missing an exponentially suppressed term, which needs to be included ($\to$ "2-term trans-series") in order to satisfy the monodromy properties of $Ai(x)$

the correct "connection formula" is

$$Ai\left(e^{\mp\frac{2\pi i}{3}}x\right) = \frac{1}{2}e^{\pm i\pi/6}Bi(x) + \underbrace{\frac{1}{2}e^{\mp\frac{i\pi}{3}}Ai(x)}_{\text{suppressed by an } e^{-\frac{4}{3}x^{3/2}} \text{ factor}}$$

Q: how do we recover this from the perturbative series ?

"non-perturbative completion"

(ie. restoring a symmetry [global property] of the functions $Ai(x)$, $Bi(x)$ which is broken by the formal $x \to +\infty$ asymptotic series)

→ Borel summation

introduce the basic idea via the classic example studied by Euler ("De Seriebus Divergentibus", 1760)
arxiv translation: 1808.02841

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} = ?$$

write $n! = \int_0^\infty dt\, e^{-t} t^n$

$$\rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} \overset{?}{=} \int_0^\infty dt\, e^{-t} \frac{1}{x} \sum_{n=0}^{\infty} \left(-\frac{t}{x}\right)^n$$

$$= \int_0^\infty dt\, e^{-xt} \frac{1}{1+t}$$

"Borel transform" $\mathcal{B}(t)$

- so the series is the $x \to +\infty$ asymptotic expansion of the (convergent!) Laplace integral $\int_0^\infty dt\, e^{-xt} \frac{1}{1+t}$. So $B(t) = \frac{1}{1+t}$ is the inverse Laplace transform.

- Note also that

$$\int_0^\infty dt\, e^{-xt} \frac{1}{1+t} = e^x \Gamma(0,x)$$

an incomplete Γ function

- see slides for plots of $e^x \Gamma(0,x)$ and truncated expansions

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more interesting question: what about $\sum_{n=0}^\infty \frac{n!}{x^{n+1}}$ non-alternating in sign!

formally $\sum_{n=0}^\infty \frac{n!}{x^{n+1}} \longrightarrow \int_0^\infty dt\, e^{-xt} \cdot \frac{1}{1-t}$

but there is a pole at $t=1$ on the contour of integration

counter-rotate t as we rotate $x \to -x$; but t can be rotated both ways $t \to e^{\mp i\pi} t$ as $x \to e^{\pm i\pi} x$

- The difference is given by the pole term,

$\pm i\pi\, e^{-x}$, which is:

① imaginary even though every term in the series is real

② unclear which sign?

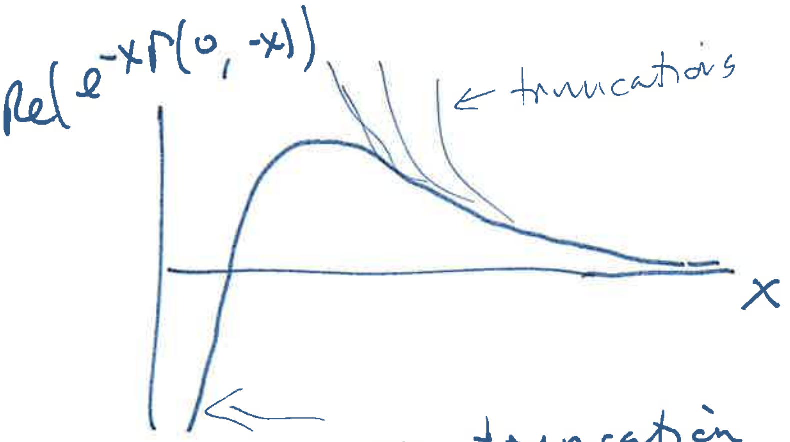
compare with the (non-perturbative) connection formula for $e^x \Gamma(0,x)$:

$$e^{-x}\Gamma(0, e^{i\pi}x) - e^{x}\Gamma(0, e^{-i\pi}x) = -2\pi i\, e^{-x}$$

so $$\mathrm{Im}\left(e^{-x}\Gamma(0, e^{\pm i\pi}x)\right) = \mp \pi e^{-x} \qquad (x>0)$$

$$\mathrm{Re}\left(e^{-x}\Gamma(0,-x)\right) = P\int_0^\infty dt\, \frac{e^{-xt}}{t-1}$$

principal part integral



no truncation order sees this turnover behavior of the real part!

- in the end all we need to make everything consistent is one direction in which the integral makes sense, and knowledge of the singularity structure of the Borel transform function $B(t)$
- more generally (see exercise 1.2)

$$x^{-a} e^{x}\Gamma(a,x) = \int_0^\infty dt\, e^{-xt} \frac{1}{(1+t)^{1-a}}$$

$$a \notin \mathbb{Z} \Rightarrow \text{a branch point at } t=-1$$

- exercise 1.3 : an infinite number of simple poles : note the interpretation of non-perturbative completion of the gamma function, to make it consistent with the reflection formula, which is the key to analytic continuation of the gamma function

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Borel summation

$$f(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^{n+1}} \quad \text{with} \quad c_n \sim n!\,(1+\dots)$$

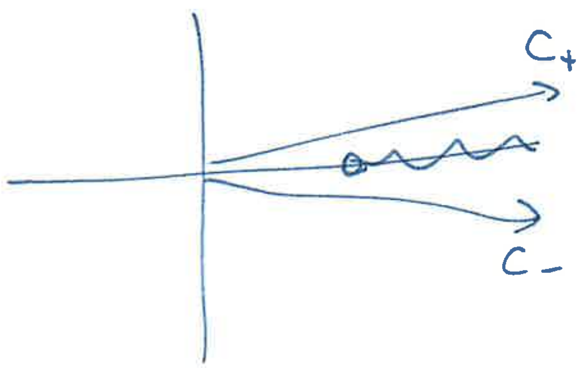
$$\xrightarrow[\text{transform}]{\text{Borel}} \quad \mathcal{B}[f](t) := \sum_{n=0}^{\infty} \frac{c_n}{n!} \frac{1}{x^{n+1}}$$

then The "Borel sum of $f(x)$" is given by the Laplace integral

$$\mathcal{S} f(x) = \int_0^\infty dt\, e^{-xt}\, \mathcal{B}[f](t)$$

"directional Borel sum"

$$\mathcal{S}_\theta f(x) = \int_0^{\infty e^{i\theta}} dt\, e^{-xt}\, \mathcal{B}[f](t)$$



rotating phase of x

→ counter-rotate phase of t

⇒ deform contour$^\wedge$ and collect the (non-perturbative in $\frac{1}{x}$) ~~pole~~ pole and branch cut contributions

Borel summation = "regularization" of the formal asymptotic series

$$\boxed{\text{Borel singularities} \equiv \text{non-perturbative physics}}$$

- so The problem of understanding $f(x)$ becomes one of understanding the singularity structure of the Borel transform $\mathcal{B}[f](t)$
- if we divided out the correct leading factorial growth then $\mathcal{B}[f](t)$ will have a finite radius of convergence

  $\Rightarrow$ it must have at least 1 singularity in $\mathbb{C}$

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illustrate with $Ai(x)$

recall that perturbation theory produced the expansions

$$\left\{ \begin{matrix} 2Ai(x) \\ Bi(x) \end{matrix} \right\} \sim \frac{e^{\mp \frac{2}{3} x^{3/2}}}{\sqrt{\pi}\, x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{2\pi\, n!\, (\frac{4}{3} x^{3/2})^n}$$

factorially divergent!

Borel transform for $Ai(x)$

$$\rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{1}{6})\Gamma(n+5/6)}{(2\pi)\, n!\, n!} t^n = {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; -t\right)$$

Borel sum

$$\mathrm{Ai}(x) = \frac{e^{-2/3\, x^{3/2}}}{\sqrt{4\pi}\, x^{1/4}} \left(\tfrac{4}{3} x^{3/2}\right) \int_0^\infty dt\, e^{-\frac{4}{3} x^{3/2} t}\, {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; -t\right)$$

$$x \to e^{\mp 2\pi i/3} \Rightarrow x^{3/2} \to e^{\mp i\pi} x^{3/2}$$

$\Rightarrow$ counter-rotate $t \to e^{\pm i\pi}$

$\Rightarrow$ we hit the branch cut from above and below

$\Rightarrow$ we need to know the jump across the cut for ${}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; -t\right)$:

see DLMF

$${}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; t+i\varepsilon\right) - {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; t-i\varepsilon\right) = i\, {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}, 1; 1-t\right)$$

$(t > 0$

<u>exercise</u>: check that this generates the correct non-perturbative connection formula

$$\mathrm{Ai}\left(e^{\mp \frac{2\pi i}{3}} x\right) = e^{\pm i\frac{\pi}{6}} \frac{\mathrm{Bi}(x)}{2} + \frac{1}{2} e^{\mp i\frac{\pi}{3}} \mathrm{Ai}(x)$$

from formal series

from the cut discontinuity

<u>moral</u>: we showed that if we know the singularity structure of the Borel transform function, then we can compute the non-perturbative connection formula

see ex. 1.4 for another example