FOUNDATIONS OF QUANTUM COMPUTING FOR LATTICE GAUGE THEORIES

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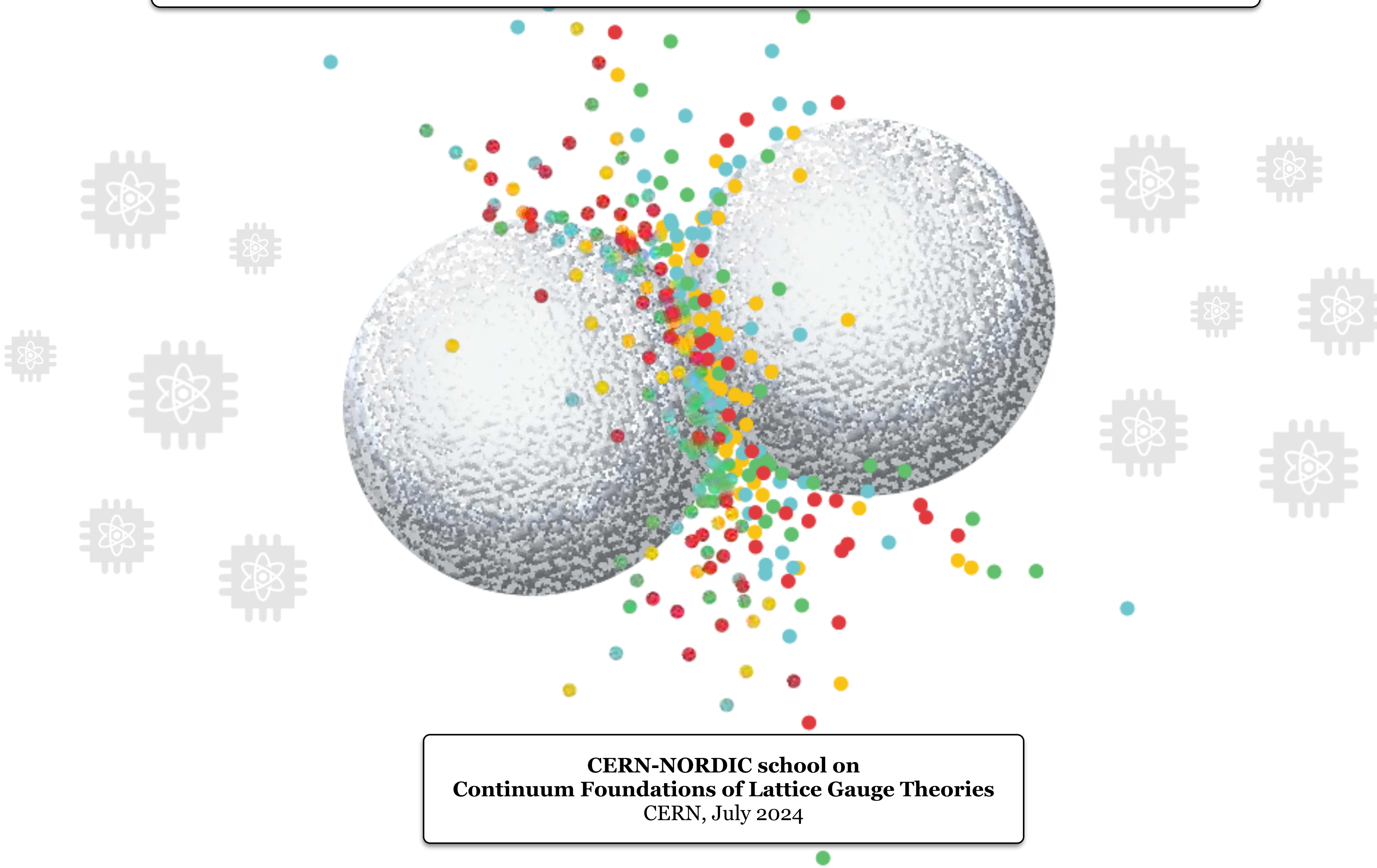
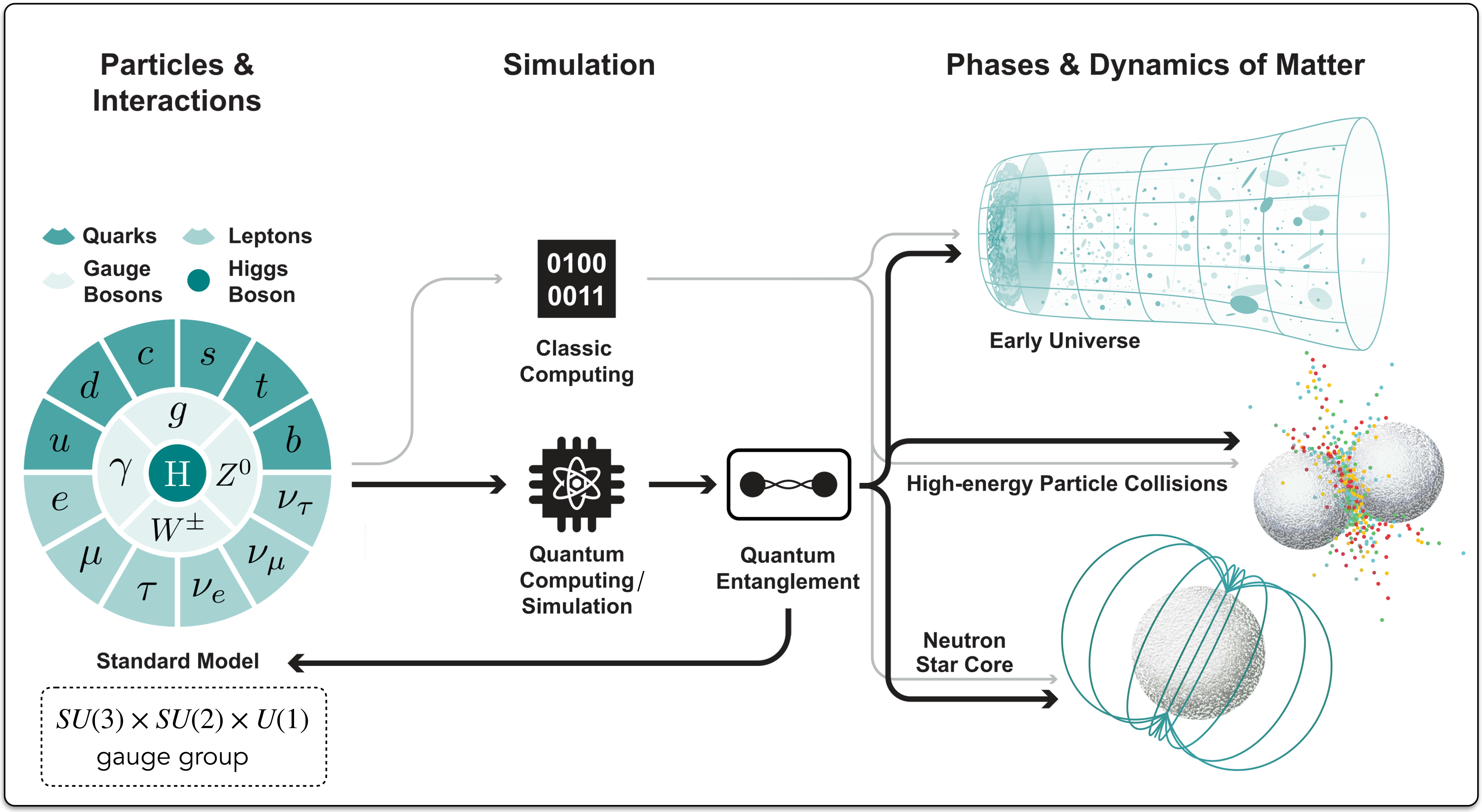


CERN-NORDIC school on
Continuum Foundations of Lattice Gauge Theories
CERN, July 2024

PRE-LECTURE [TO SET THE CONTEXT]
QUANTUM SIMULATION OF FUNDAMENTAL PARTICLES AND FORCES, WHY AND HOW?

SIMULATION ROADMAP FOR NUCLEAR AND HIGH-ENERGY PHYSICISTS



Bauer, ZD, Klco, and Savage, Quantum simulation of fundamental particles and forces, *Nature Rev. Phys.* 5 (2023) 7, 420-432.

See also: Quantum Simulation for High Energy Physics, Bauer, ZD et al, PRX Quantum 4 (2023) 2, 027001, and Quantum Information Science and Technology for Nuclear Physics, Beck, Carlson, ZD, Formaggio, Quaglioni, Savage, et al, arXiv:2303.00113 [nucl-ex].

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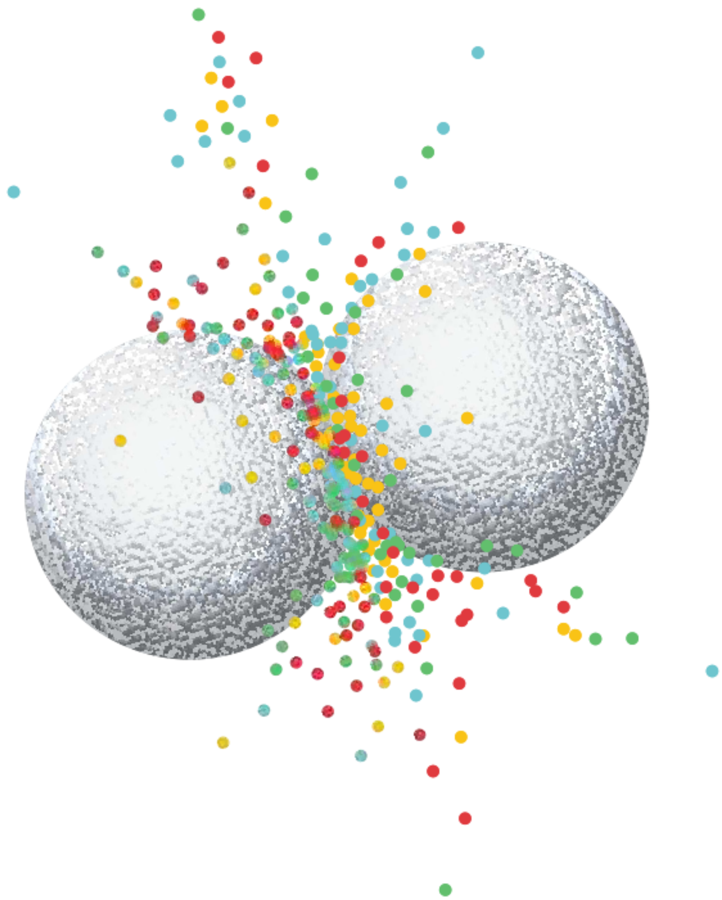
Classic Computing

Can we classically simulate scattering of composite particles from the Standard Model?

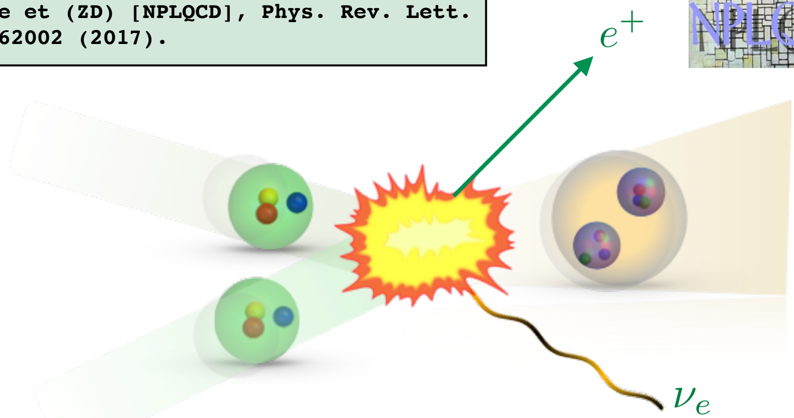


Lattice gauge theory methods based on Monte-Carlo sampling in Euclidean (imaginary) time have enabled this…but only at low energies so far…

TWO EXAMPLES: REACTIONS OF NUCLEONS

proton-proton fusion

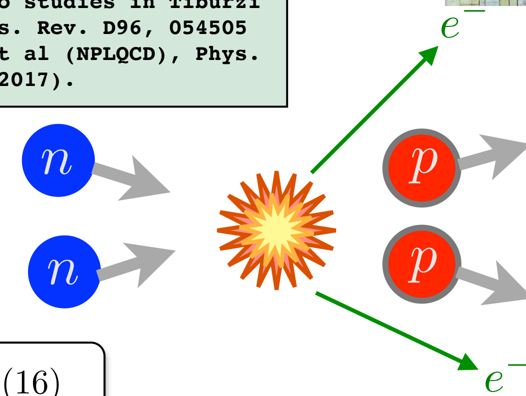
Savage et (ZD) [NPLQCD], Phys. Rev. Lett. 119,062002 (2017).



$$L_{1,A} = 3.9(0.1)(1.0)(0.3)(0.9)\ \mathrm{fm}^3 \quad @\ \mu = m_\pi^{\mathrm{phys.}} = 140\ \mathrm{MeV}$$

Neutrinoless double-β decay

ZD, Detmold, Fu, Grebe, Jay, Murphy, Oare, Shanahan, Wagman, arXiv:2402.09362 [hep-lat]. See also our two-neutrino studies in Tiburzi et al (ZD) (NPLQCD), Phys. Rev. D96, 054505 (2017)and Shanahan, ZD et al (NPLQCD), Phys. Rev. Lett. 119, 062003 (2017).



$$a^2 \mathcal{A}^{nn \to pp} = 0.078(16)$$
$$@\ m_\pi = 806\ \mathrm{MeV}$$

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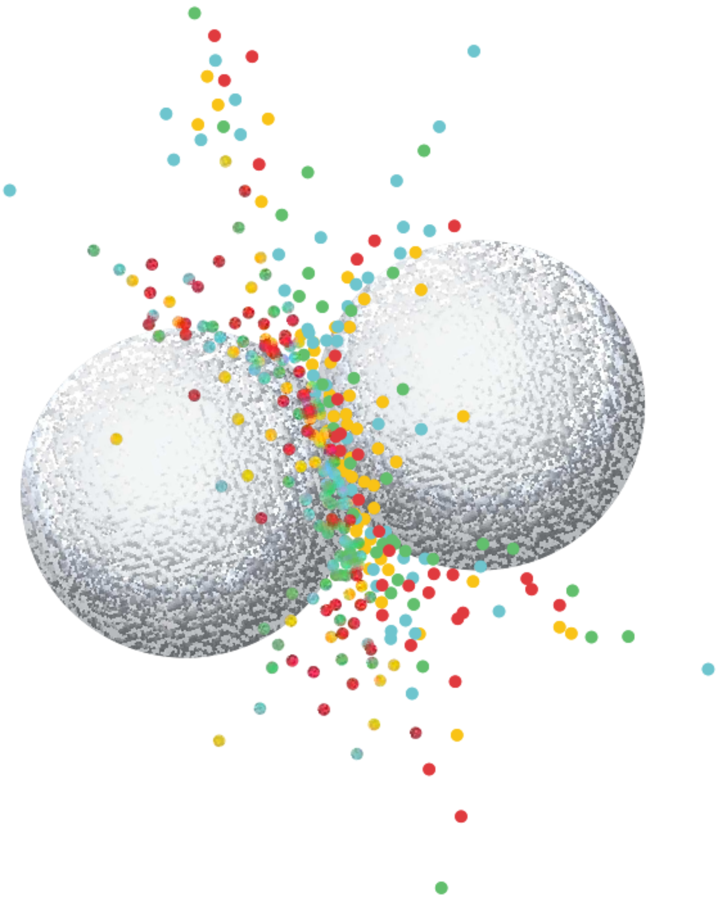
Classic Computing

What about high energies, like events at the Large Hadron Collider or the Relativistic Heavy-Ion Collider?



There are mainly two issues…

i) making complicated states, i.e., high-energy protons, or heavy ions, etc.,
ii) imaginary time nature of the classical Monte-Carlo calculations…no access to states as a function of Minkowski time elapsed after the collision!

CLASSICAL COMPUTATIONS OF NUCLEI BASED ON LATTICE GAUGE THEORY IS HARD.

i) The complexity of systems grows factorially with the number of quarks in the nucleus.

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Detmold and Orginos (2013)
Detmold and Savage (2010)
Doi and Endres (2013)
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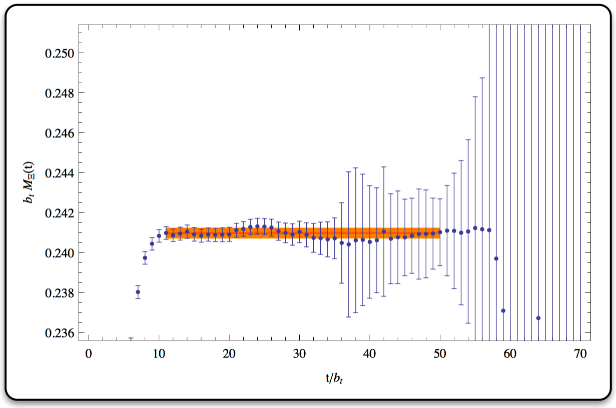

ii) There is a severe (exponentially bad in atomic number) signal-to-noise degradation.

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Paris (1984) and Lepage (1989)
Wagman and Savage (2017, 2018)
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iii) Excitation energies of nuclei are orders of magnitudes smaller than their masses.

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Beane at al (NPLQCD) (2009)
Beane, Detmold, Orginos, Savage (2011)
ZD (2018)
Briceno, Dudek and Young (2018)
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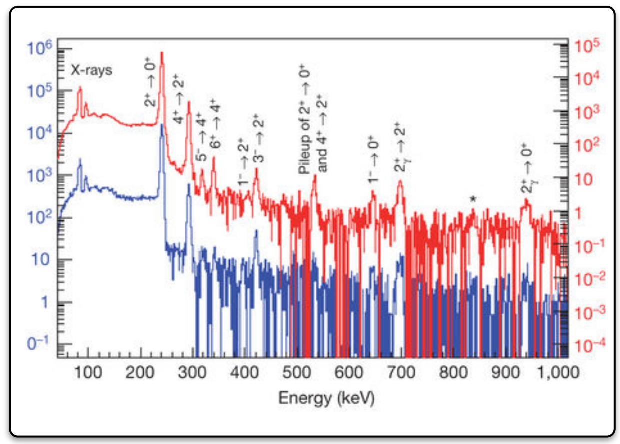
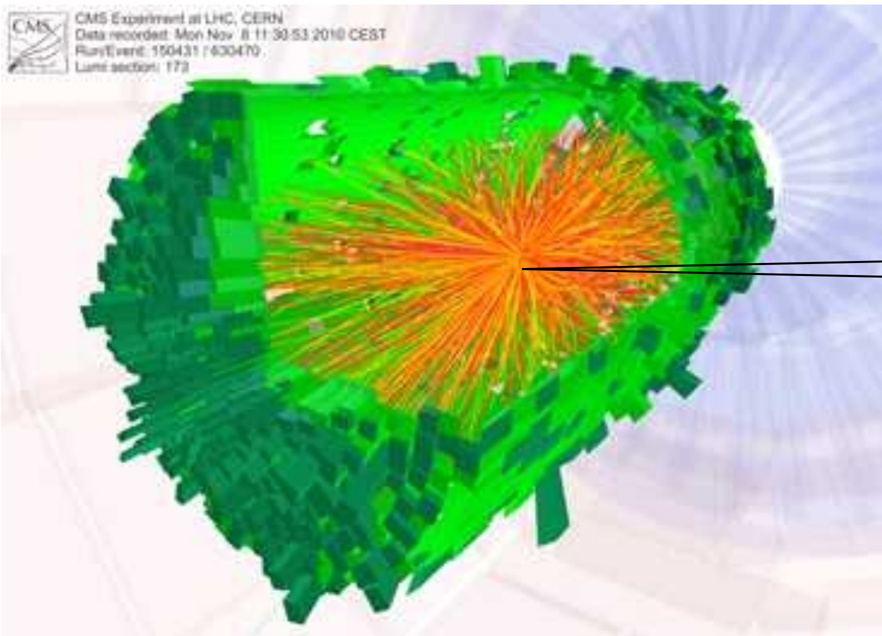
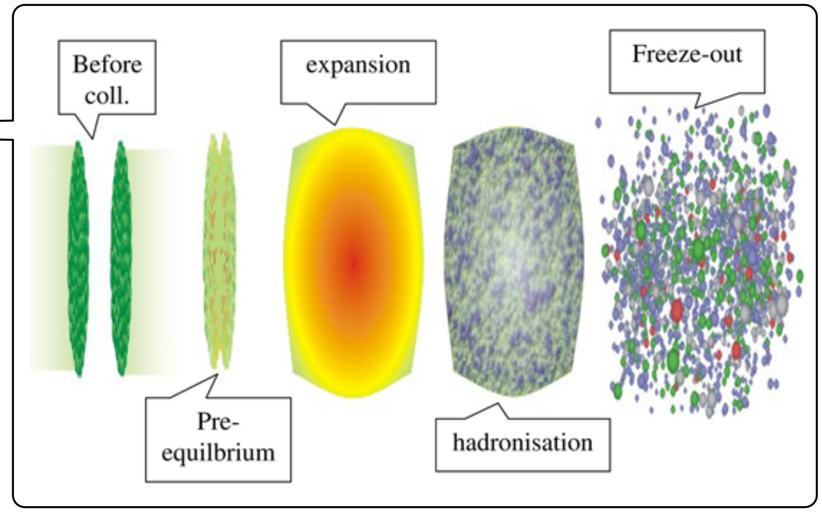


SIGN PROBLEM MAKES CONVENTIONAL LATTICE-GAUGE-THEORY METHODS INTRACTABLE.

No access to real-time non-equilibrium dynamics of matter in heavy-ion collisions or after the Big Bang…





…and to a wealth of dynamical response functions, transport properties, structure functions, etc.

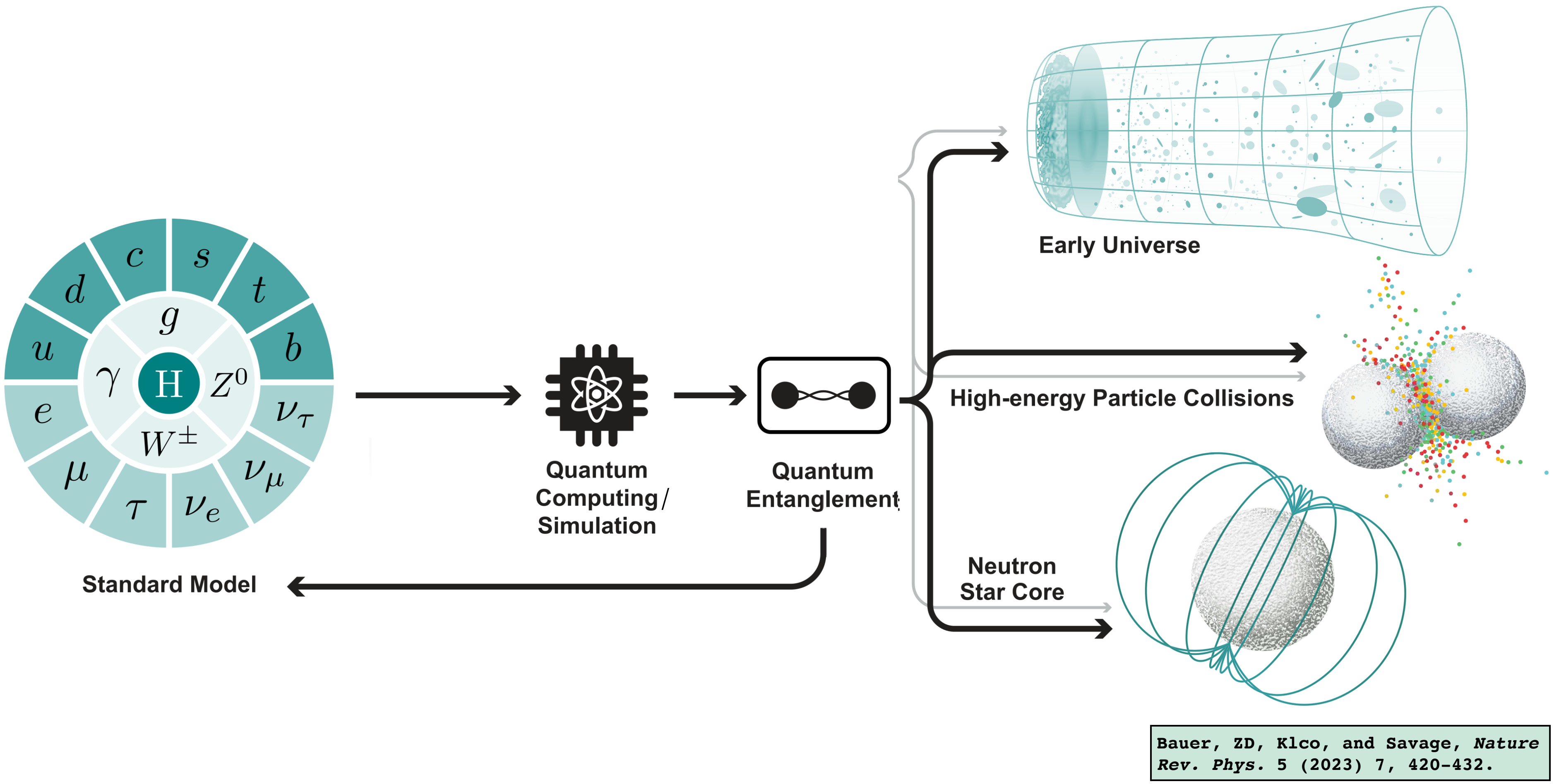
Path integral formulation:

$$e^{iS[U,q\bar{q}]}$$

Hamiltonian evolution:

$$U(t) = e^{-iHt}$$

STUDY HIGH-ENERGY, HIGH-DENSITY PHENOMENA VIA QUANTUM SIMULATION?



OUTLINE OF PART I: HAMILTONIAN FORMULATION OF LATTICE GAUGE THEORIES

i) Hamiltonian vs. Lagrangian formulation of LGTs

ii) Kogut-Susskind formulation: Basis states, Hilbert space, and constraints

- An Abelian case: U(1) LGT
- A non-Abelian case: SU(2) LGT

iii) Kogut-Susskind formulation: Hamiltonian

iv) A variety of formulations: a brief overview

v) Classical Hamiltonian-simulation methods: a brief discussion

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	Path integral (Lagrangian)	Hamiltonian
Degrees of freedom	Fields and their derivatives	Fields and their conjugate variables
Spacetime signature	Often Euclidean	Minkowski
Starting point	$\mathcal{L}[\varphi, \partial\varphi]$	$\hat{H}[\hat{\varphi}, \hat{\pi}]$
Hilbert space	Not explicitly constructed/relevant	Built out of $O^\dagger \lvert\text{vac.}\rangle^*$ $^*\lvert\text{vac.}\rangle = \lvert\text{empty state}\rangle$
Expectation values	$\frac{1}{\mathcal{Z}}\int \mathcal{D}\varphi\; e^{-S} O$	$\langle\psi\lvert\hat{O}\rvert\psi\rangle$
Dynamical quantities	Sometimes accessible with indirect methods, e.g., Luescher method.	In principle accessible: $\langle\psi\lvert e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}\rvert\psi\rangle$
Computational methods	Monte Carlo, etc.	Classical Hamiltonian methods like exact diag., tensor networks/ quantum simulation
Computational challenge	Sign and signal-to-noise problem for real-time quantities and finite-density systems.	Exponential scaling of the Hilbert space with the number of DOF.

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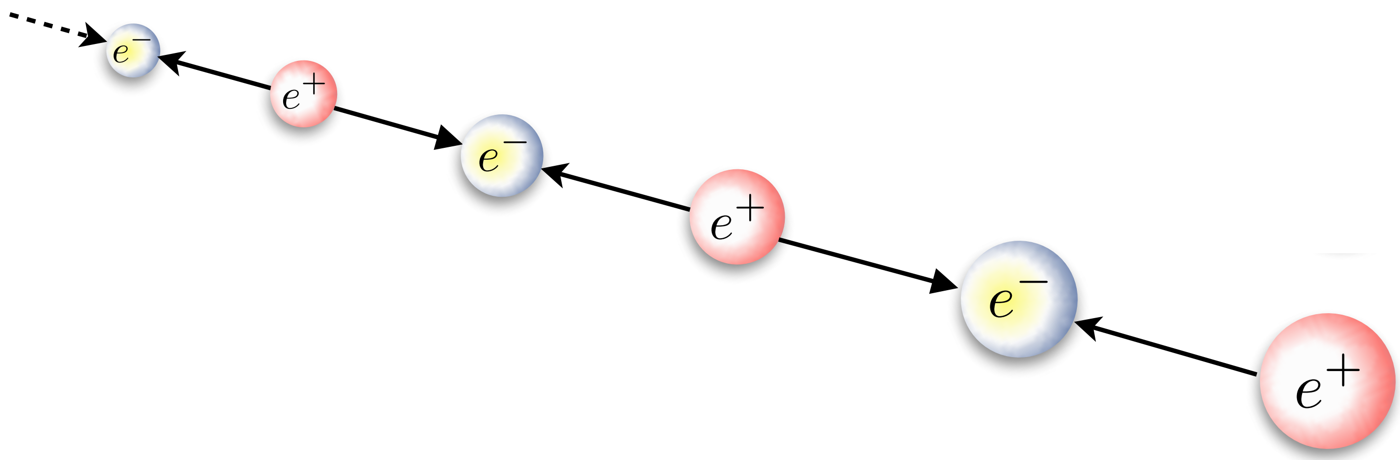
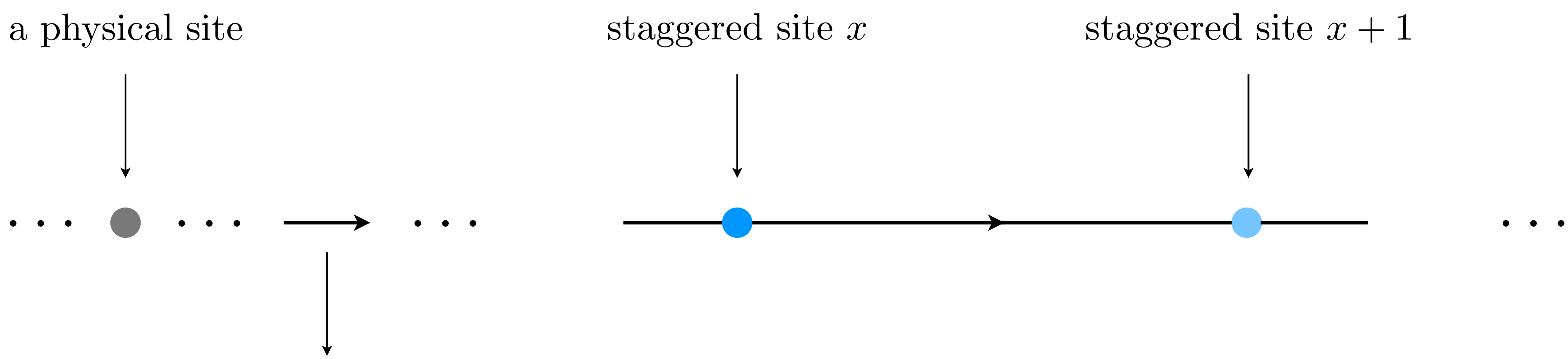
FOCUSING ON A SIMPLE EXAMPLE: THE 1+1 DIMENSIONAL QUANTUM ELECTRODYNAMICS COUPLED TO MATTER (SCHWINGER MODEL)

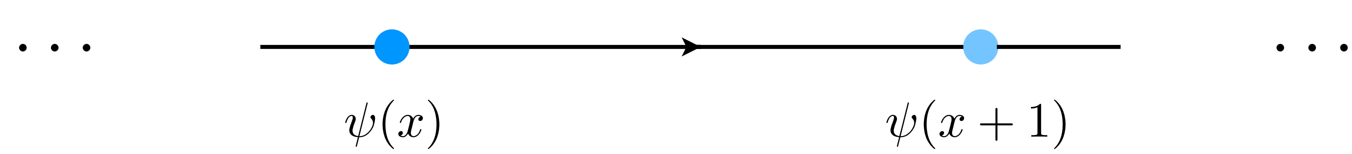


THE BASICS

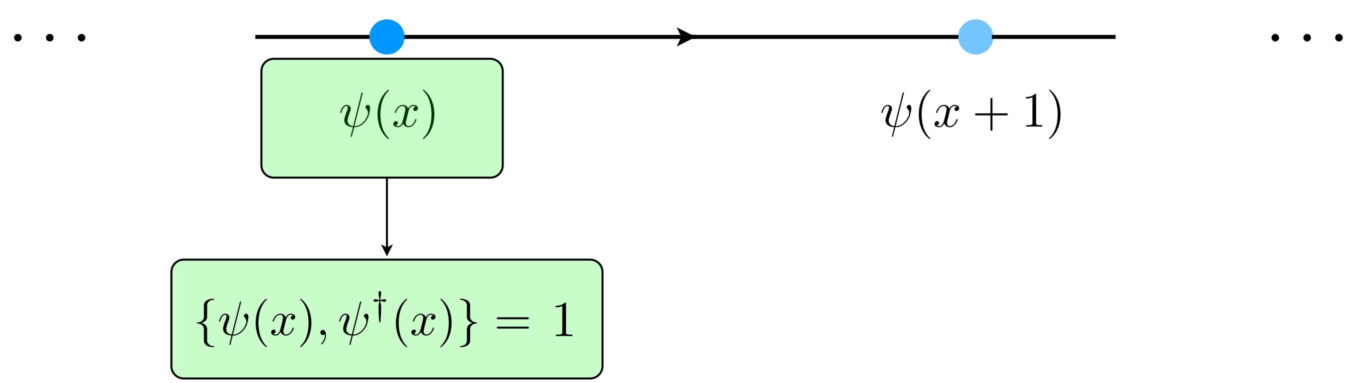


Staggering: Multi-component Dirac fermion field is split to single-component fermion fields each occupying one site.

THE BASICS

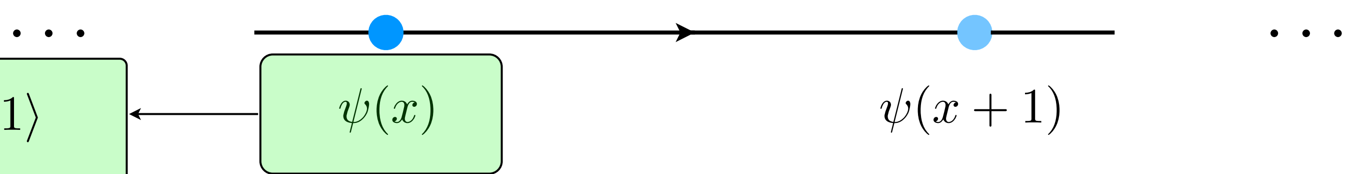


THE BASICS



All other anticommutations are zero.

THE BASICS



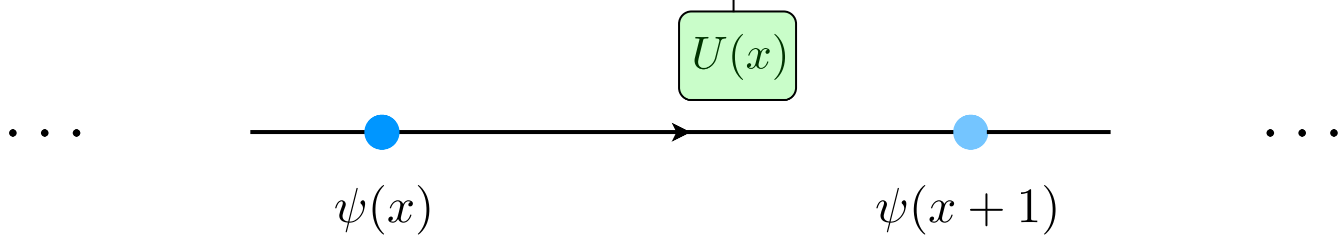
$$\psi|f\rangle = (1-\delta_{f,0})|f-1\rangle$$
$$\psi^\dagger|f\rangle = (1-\delta_{f,1})|f+1\rangle$$

Fermions can have occupation number zero or one.

THE BASICS

Picking the temporal gauge, and introducing the link variable *U*

$$U(x) = e^{iagA(x)} \quad (A_0 = 0,\ A_1 = A)$$

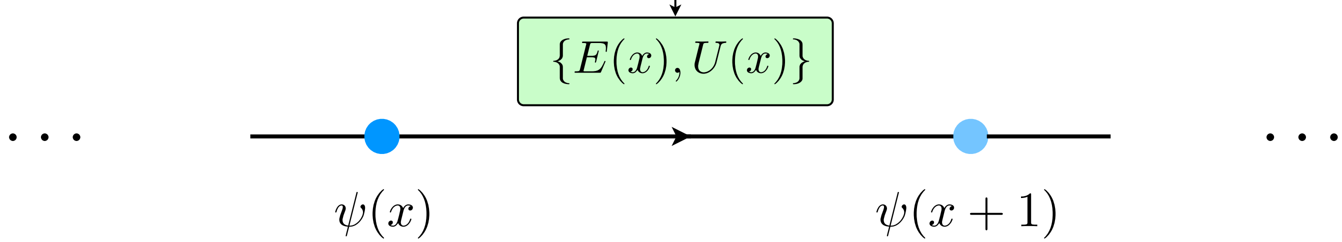


THE BASICS

E and *U* are conjugate variable pairs.
Not simultaneously diagonalizable!

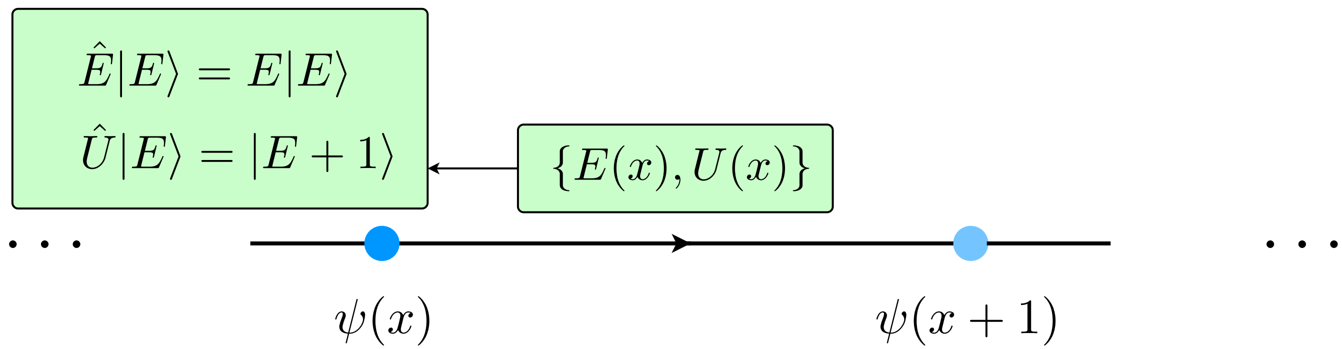
$$[E(x), agA(x)] = \frac{1}{i}$$

or: $$[E, U] = U$$

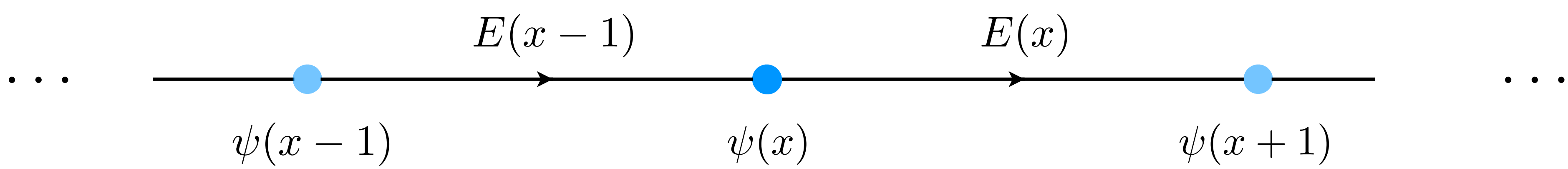


THE BASICS

A discrete infinite-dimensional Hilbert space of a 1D quantum rotor: $E \in \mathbb{Z}$



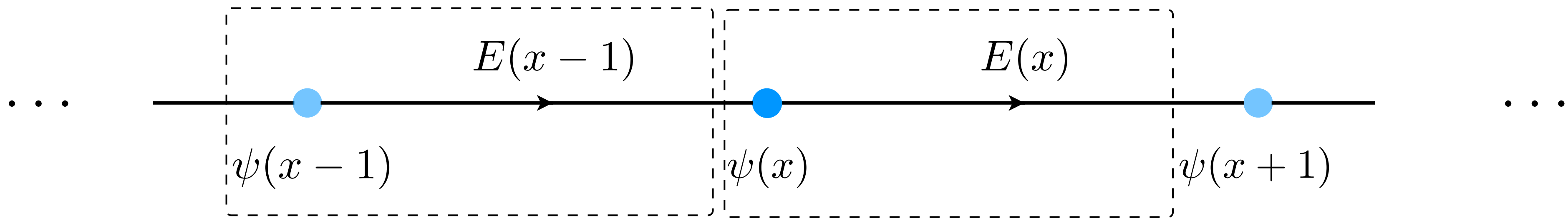
THE BASICS

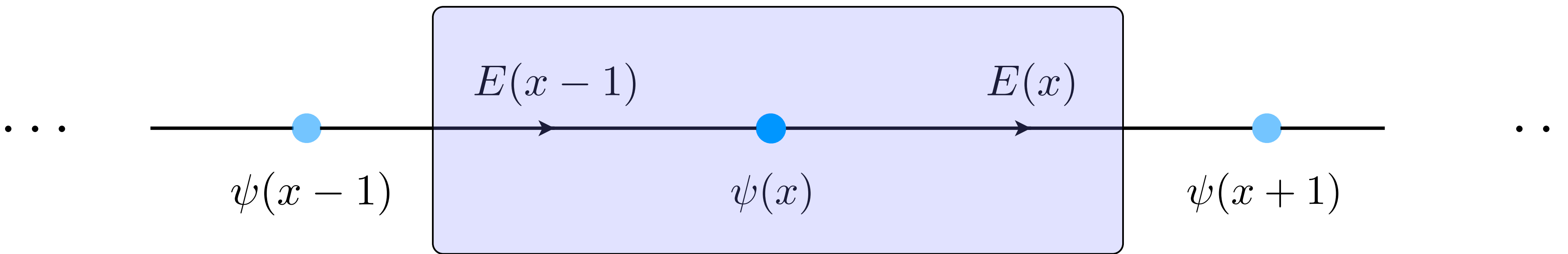


Therefore Hilbert space is spanned by the basis states:

$$\cdots (|f\rangle_{x-1} \otimes |E\rangle_{x-1}) \otimes (|f\rangle_x \otimes |E\rangle_x) \cdots$$

However, not all states are physical!





$$G(x)|\psi\rangle_{\text{phys.}} = 0$$

$$G(x) = E(x) - E(x-1) + \psi^\dagger(x)\psi(x) - \frac{1-(-1)^x}{2}$$

Gauss's law constraint stating that the flux of the electric field is equal to the staggered electric charge.

EXAMPLE

Consider a two-site theory with periodic boundary conditions. Impose a cutoff $\Lambda = 1$ on the electric field such that $E \in [-\Lambda, \Lambda]$.

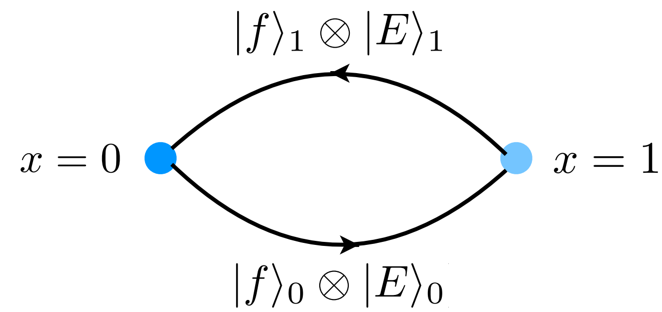
a) How many basis states are there?

b) What are the physical states? Identify the particle content of states.

c) What is the value of the total electric charge for each state?

EXAMPLE

Consider a two-site theory with periodic boundary conditions. Impose a cutoff $\Lambda = 1$ on the electric field such that $E \in [-\Lambda, \Lambda]$.



a) How many basis states are there?

There are $2^2 \times 3^2 = 36$ basis states.

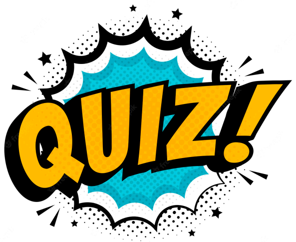
b) What are the physical states? Identify the particle content of states.

There are only 5 states consistent with the Gauss's law:

$(\|0\rangle_0 \otimes \|-1\rangle_0) \otimes (\|1\rangle_1 \otimes \|-1\rangle_1)$	No matter
$(\|0\rangle_0 \otimes \|0\rangle_0) \otimes (\|1\rangle_1 \otimes \|0\rangle_1)$	No matter
$(\|0\rangle_0 \otimes \|1\rangle_0) \otimes (\|1\rangle_1 \otimes \|1\rangle_1)$	No matter
$(\|1\rangle_0 \otimes \|0\rangle_0) \otimes (\|0\rangle_1 \otimes \|1\rangle_1)$	e^- e^+
$(\|1\rangle_0 \otimes \|-1\rangle_0) \otimes (\|0\rangle_1 \otimes \|0\rangle_1)$	e^- e^+

c) What is the value of the total electric charge for each state?

Recall that $Q(x) = -\psi^\dagger(x)\psi(x) + \dfrac{1-(-1)^x}{2}$, so $Q(0) + Q(1) = 0$ for all physical states.



How many different electric-charge sectors exist for lattice Schwinger model with periodic boundary conditions (with no background charges)? What about with open boundary conditions?

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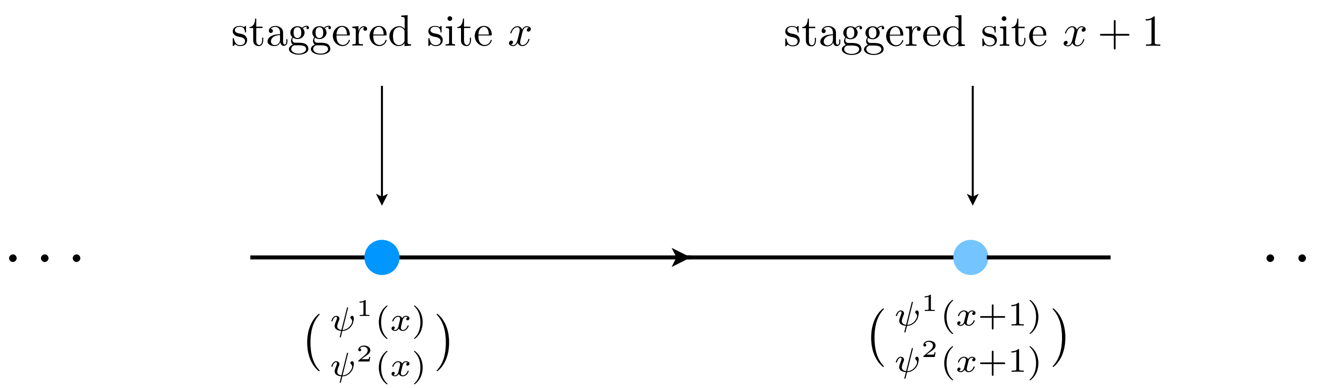
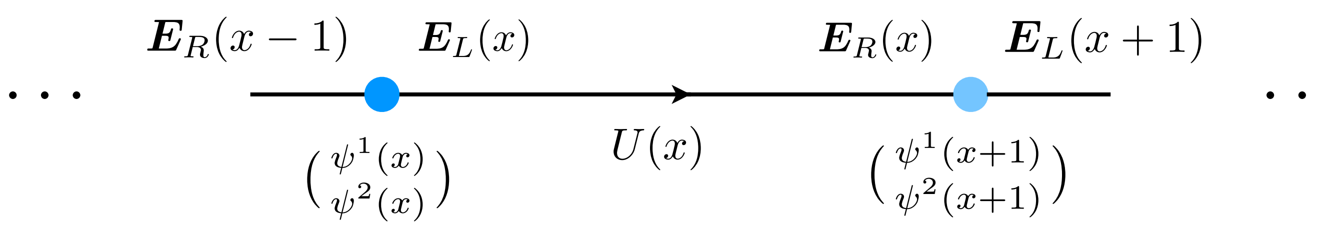
EXAMPLE: LET US CONSIDER THE CASE OF SU(2) LGT IN 1+1 D.

ZD, Raychowdhury, and Shaw, Phys. Rev. D 104, 074505, arXiv:2009.11802 [hep-lat]

THE BASICS


staggered site x
staggered site $x+1$
$\begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$
$\begin{pmatrix} \psi^1(x+1) \\ \psi^2(x+1) \end{pmatrix}$

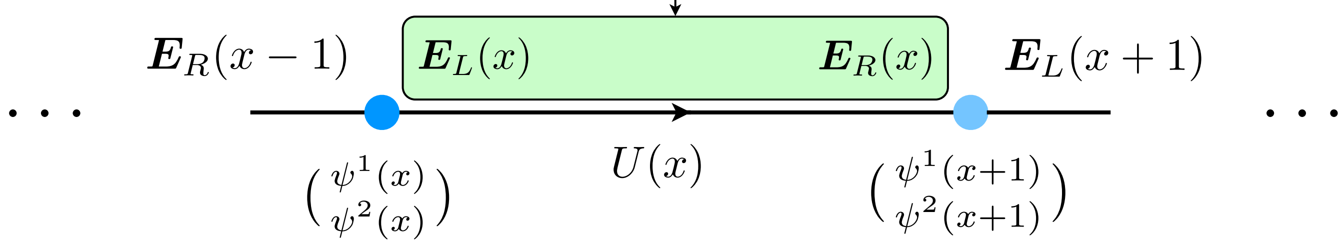
THE BASICS



$\boldsymbol{E}_R(x-1)$
$\boldsymbol{E}_L(x)$
$\boldsymbol{E}_R(x)$
$\boldsymbol{E}_L(x+1)$
$\binom{\psi^1(x)}{\psi^2(x)}$
$U(x)$
$\binom{\psi^1(x+1)}{\psi^2(x+1)}$

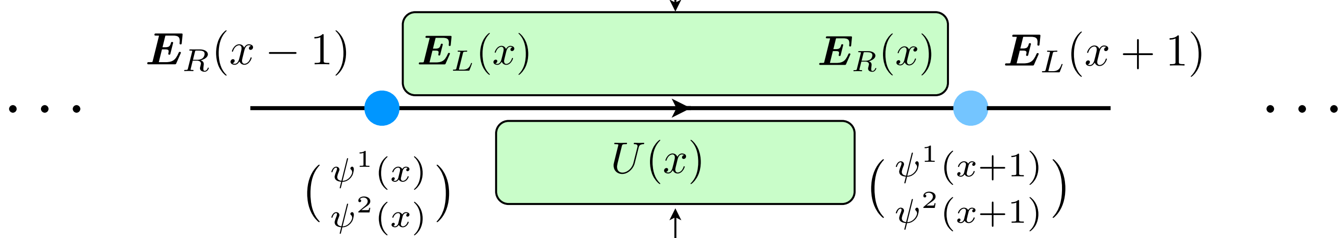
THE BASICS

$$[\hat{E}_L^a, \hat{E}_L^b] = -i\epsilon^{abc}\hat{E}_L^c,$$
$$[\hat{E}_R^a, \hat{E}_R^b] = i\epsilon^{abc}\hat{E}_R^c,$$
$$[\hat{E}_L^a, \hat{E}_R^b] = 0,$$



THE BASICS

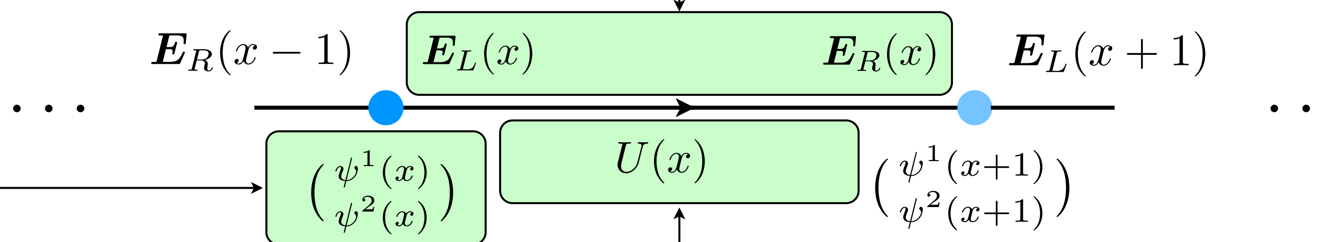
$$[\hat{E}_L^a, \hat{E}_L^b] = -i\epsilon^{abc}\hat{E}_L^c,$$
$$[\hat{E}_R^a, \hat{E}_R^b] = i\epsilon^{abc}\hat{E}_R^c,$$
$$[\hat{E}_L^a, \hat{E}_R^b] = 0,$$



$$[\hat{E}_L^a, \hat{U}] = T^a\hat{U}$$
$$[\hat{E}_R^a, \hat{U}] = \hat{U}T^a$$

THE BASICS

$$[\hat{E}^a_L, \hat{E}^b_L] = -i\epsilon^{abc}\hat{E}^c_L,$$
$$[\hat{E}^a_R, \hat{E}^b_R] = i\epsilon^{abc}\hat{E}^c_R,$$
$$[\hat{E}^a_L, \hat{E}^b_R] = 0,$$

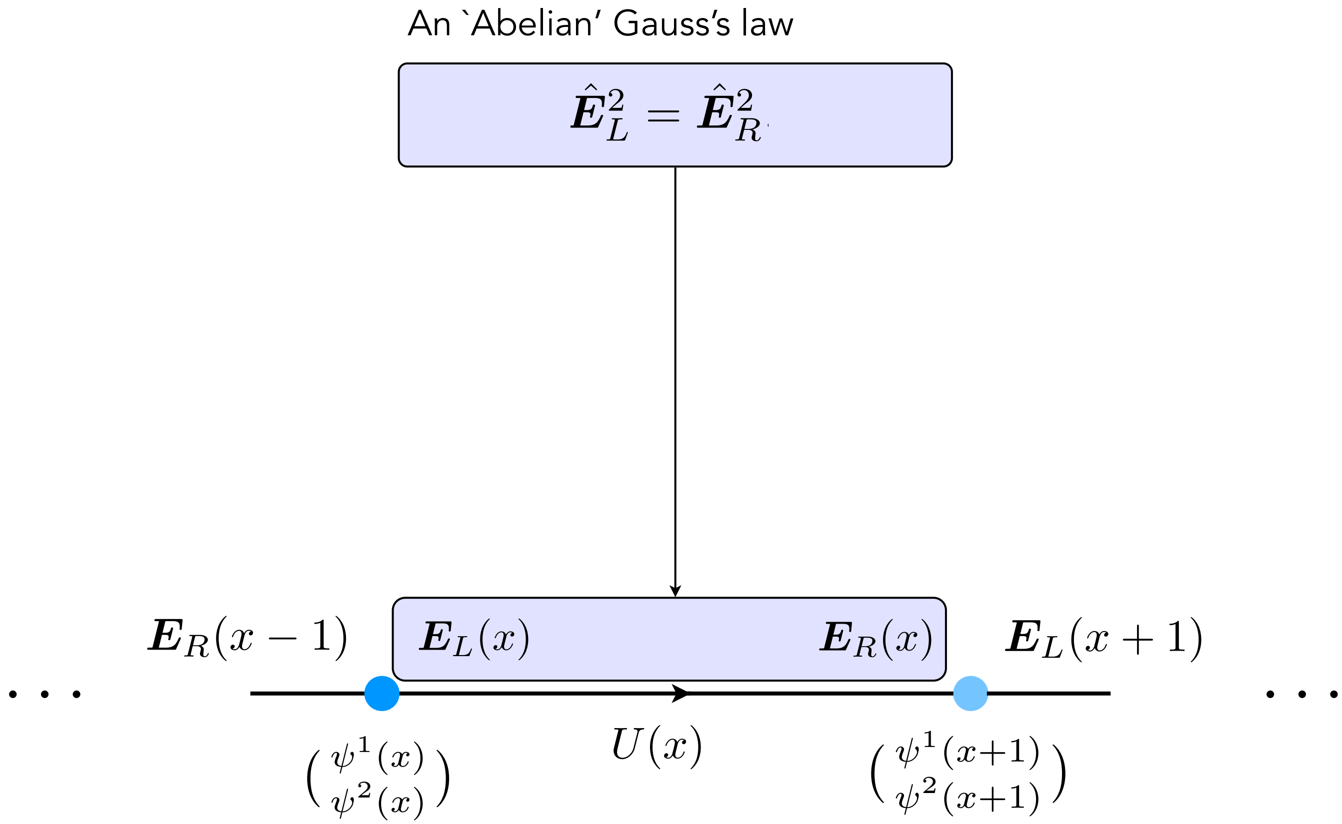


$$\hat{\rho}^a(x) \equiv \psi^\dagger(x) T^a \psi(x),$$
$$[\hat{\rho}^a, \psi] = -T^a\psi.$$

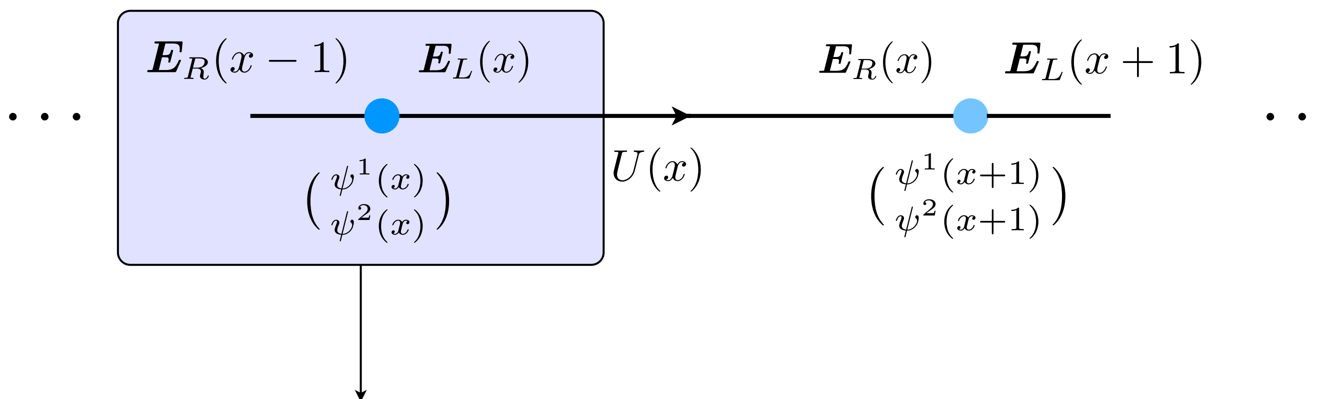
SU(2) fermionic charges

$$[\hat{E}^a_L, \hat{U}] = T^a\hat{U}$$
$$[\hat{E}^a_R, \hat{U}] = \hat{U}T^a$$

THE BASICS


An `Abelian' Gauss's law
$\hat{\boldsymbol{E}}_L^2 = \hat{\boldsymbol{E}}_R^2$
$\boldsymbol{E}_R(x-1)$
$\boldsymbol{E}_L(x)$
$\boldsymbol{E}_R(x)$
$\boldsymbol{E}_L(x+1)$
$\cdots$
$\cdots$
$U(x)$
$\begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$
$\begin{pmatrix} \psi^1(x+1) \\ \psi^2(x+1) \end{pmatrix}$

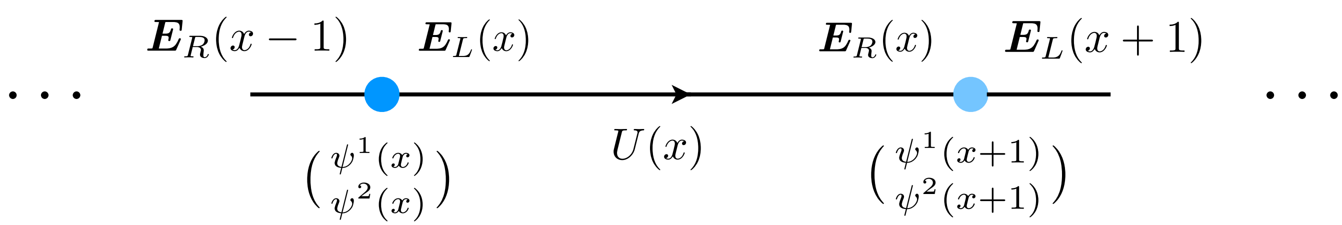
THE BASICS



$$\hat{G}^a(x) = -\hat{E}_L^a(x) + \hat{E}_R^a(x-1) + \hat{\rho}^a(x)$$

Three Gauss's law operators

ANGULAR MOMENTUM BASIS OF SU(2) LGT



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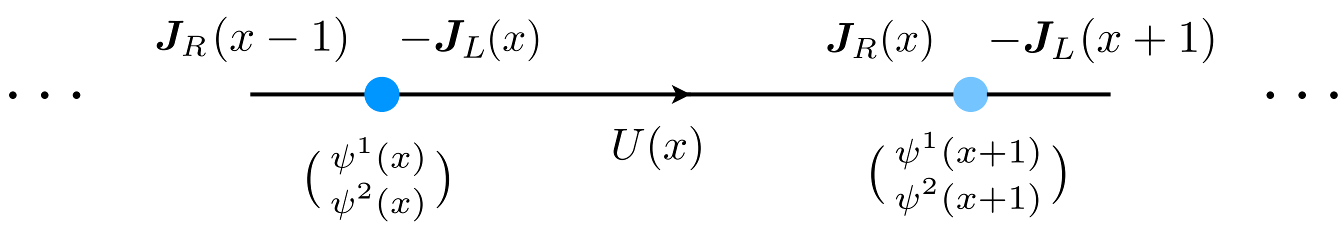
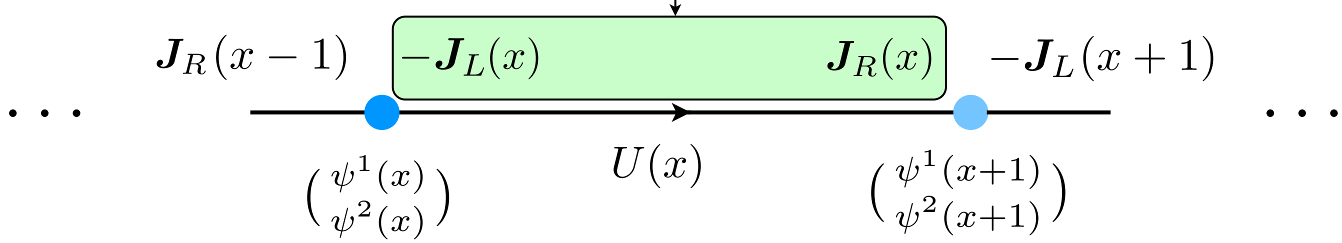


ANGULAR MOMENTUM BASIS OF SU(2) LGT

$$\hat{\boldsymbol{J}}_R^2 \, |J_R, m_R\rangle = J_R(J_R+1) \, |J_R, m_R\rangle$$
$$\hat{\boldsymbol{J}}_L^2 \, |J_L, m_L\rangle = J_L(J_L+1) \, |J_L, m_L\rangle$$

$$\hat{J}_R^3 \, |J_R, m_R\rangle = m_R \, |J_R, m_R\rangle$$
$$\hat{J}_L^3 \, |J_L, m_L\rangle = m_L \, |J_L, m_L\rangle$$



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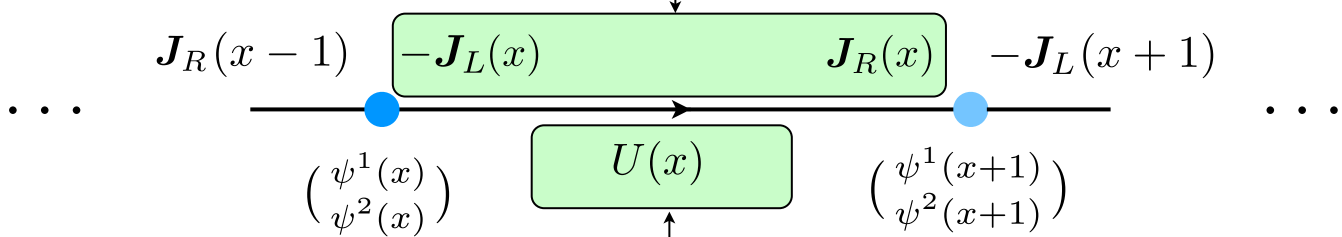
$$\hat{\boldsymbol{J}}_L^2 \, |J_L, m_L\rangle = J_L(J_L+1) \, |J_L, m_L\rangle$$

$$\hat{J}_R^3 \, |J_R, m_R\rangle = m_R \, |J_R, m_R\rangle$$
$$\hat{J}_L^3 \, |J_L, m_L\rangle = m_L \, |J_L, m_L\rangle$$

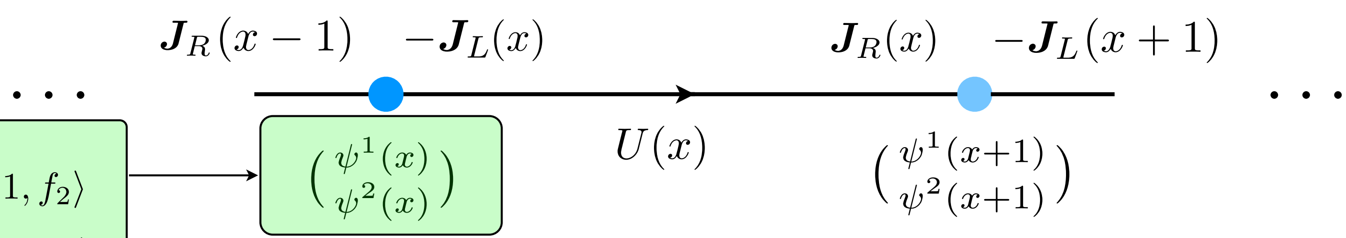


$$\hat{U}^{(\alpha,\beta)}(x) \, |J_L, m_L\rangle^{(x)} \otimes |J_R, m_R\rangle^{(x)} =$$

$$\sum_{j=J\pm\frac{1}{2}} \sqrt{\frac{2J+1}{2j+1}} \, \langle J, m_L; \frac{1}{2}, \alpha | j, m_L+\alpha\rangle \langle J, m_R; \frac{1}{2}, \beta | j, m_R+\beta\rangle \, |j, m_L+\alpha\rangle^{(x)} \otimes |j, m_R+\beta\rangle^{(x)}$$

ANGULAR MOMENTUM BASIS OF SU(2) LGT

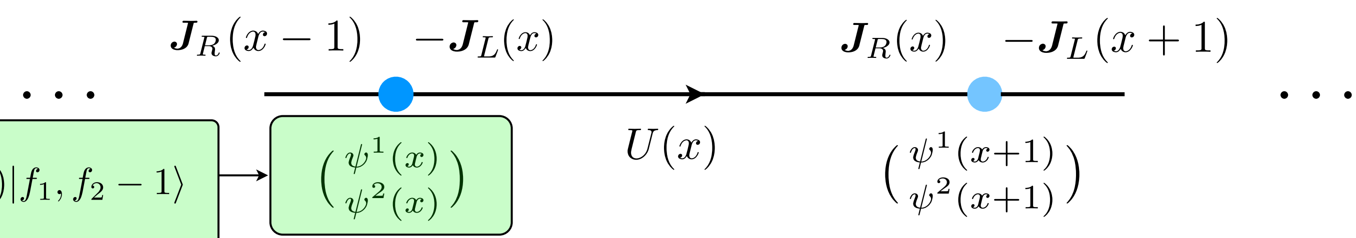
$$\psi^1 \, |f_1, f_2\rangle = (1 - \delta_{f_1,0}) \, |f_1 - 1, f_2\rangle$$
$$\psi^{1^\dagger} \, |f_1, f_2\rangle = (1 - \delta_{f_1,1}) \, |f_1 + 1, f_2\rangle$$



ANGULAR MOMENTUM BASIS OF SU(2) LGT

$$\psi_2|f_1, f_2\rangle = (-1)^{f_1}(1-\delta_{f_2,0})|f_1, f_2 - 1\rangle$$

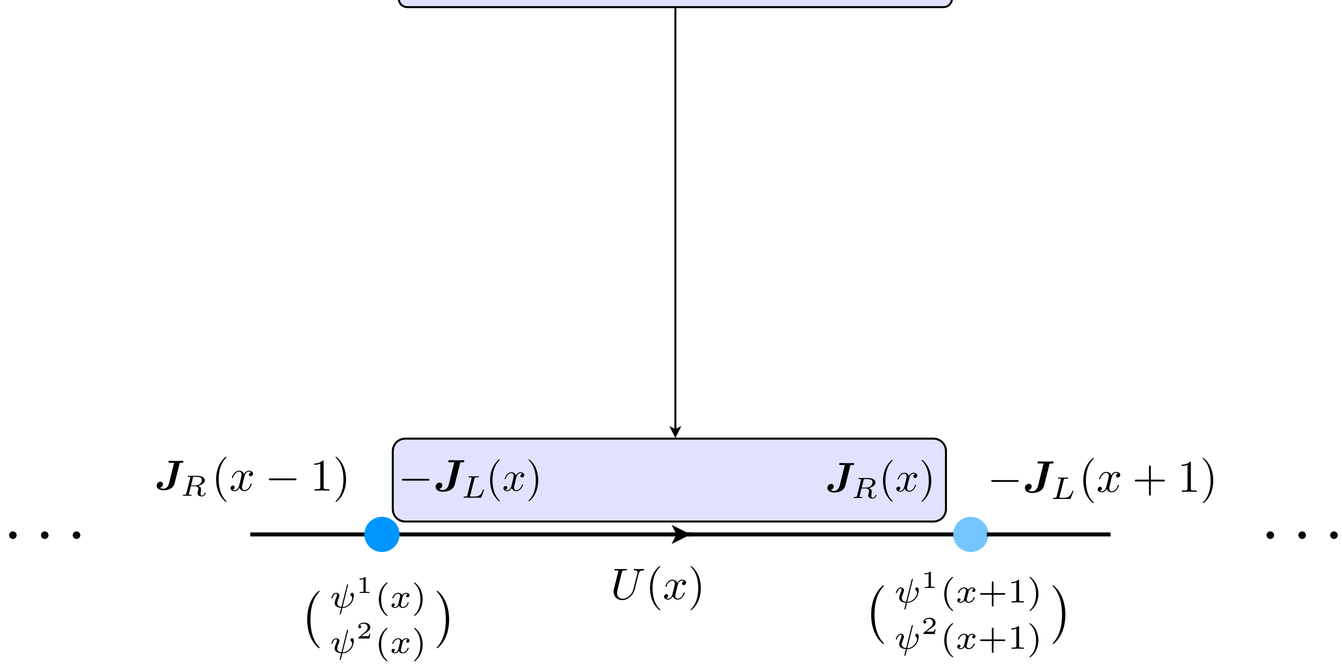
$$\psi_2^\dagger|f_1, f_2\rangle = (-1)^{f_1}(1-\delta_{f_2,1})|f_1, f_2 + 1\rangle$$



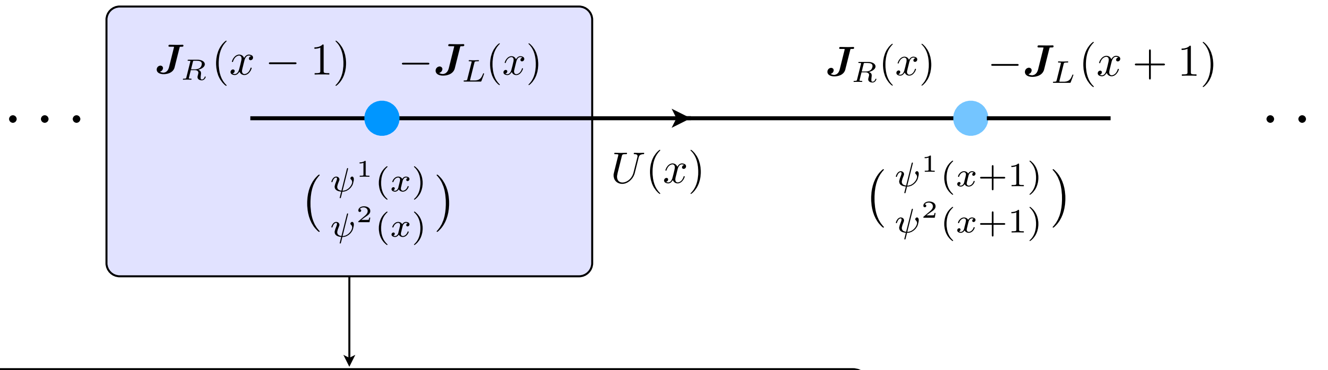
PHYSICAL CONSTRAINTS

An `Abelian' Gauss's law

$$\hat{\boldsymbol{J}}_L^2 = \hat{\boldsymbol{J}}_R^2$$



PHYSICAL CONSTRAINTS

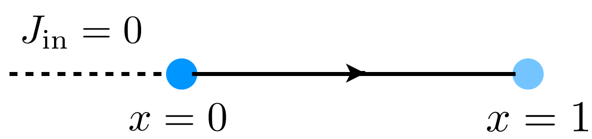


$$\left[\hat{J}^a_L(x) + \hat{J}^a_R(x-1) + \frac{1}{2}\psi^\dagger(x)\tau^a\psi(x)\right]|\phi\rangle_{(\mathrm{KS})} = 0$$

Non-Abelian Gauss's laws

EXAMPLE

Consider a two-site theory with open boundary conditions. Impose a cutoff $\Lambda = 1/2$ on the total angular momentum on each link such that only $J = 0, 1/2$ values are allowed. The incoming angular momentum is set to zero.



a) How many basis states are there?

b) What are the physical states in the sector with $\nu = 1$ where $\nu \equiv \frac{1}{2} \sum_x \psi^\dagger(x)\psi(x)$?

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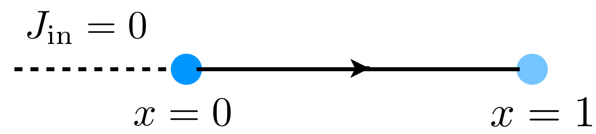

a) How many basis states are there?

There are $4^2 \times 5 = 80$ basis states (4 fermionic states $|f_1, f_2\rangle$ at each site and 5 angular momentum states $|J, m_L\rangle \otimes |J, m_L\rangle$ on the only link.).

b) What are the physical states in the sector with $\nu = 1$ where $\nu \equiv \dfrac{1}{2}\sum_x \psi^\dagger(x)\psi(x)$?

1) $[|0,0\rangle\,|0,0\rangle\,|0,0\rangle]^{(0)} \otimes [|0,0\rangle\,|1,1\rangle\,|0,0\rangle]^{(1)}$

2)
$$\frac{1}{2}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|0,0\rangle\right]^{(1)}$$
$$-\frac{1}{2}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,0\rangle\,|0,0\rangle\right]^{(1)}$$
$$-\frac{1}{2}\left[|0,0\rangle\,|0,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|0,0\rangle\right]^{(1)}$$
$$+\frac{1}{2}\left[|0,0\rangle\,|0,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,0\rangle\,|0,0\rangle\right]^{(1)}$$

4) $[|0,0\rangle\,|1,1\rangle\,|0,0\rangle]^{(0)} \otimes [|0,0\rangle\,|0,0\rangle\,|0,0\rangle]^{(1}$

3)
$$\frac{1}{\sqrt{6}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|1,0\rangle\,|1,-1\rangle\right]^{(1)}$$
$$-\frac{1}{2\sqrt{3}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|1,0\rangle\right]^{(1)}$$
$$-\frac{1}{2\sqrt{3}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,0\rangle\,|1,0\rangle\right]^{(1)}$$
$$+\frac{1}{\sqrt{6}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|0,1\rangle\,|1,1\rangle\right]^{(1)}$$
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$$-\frac{1}{\sqrt{6}}\left[|0,0\rangle\,|0,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|0,1\rangle\,|1,1\rangle\right]^{(1)}$$

TO BE CONTINUED…

QUESTIONS?