Introductory Lectures on Resurgence

Gerald Dunne

University of Connecticut

CERN Summer School: July 2024

Continuum Foundations of Lattice Gauge Theories

Basic Introduction to Resurgence

1.
- Stokes Phenomenon and Trans-series
- Borel Summation basics
- Recovering Non-perturbative Connection Formulas

2.
- Nonlinear Stokes Phenomenon
- Parametric Resurgence & Phase Transitions
- Gross-Witten-Wadia unitary matrix model

3.
- QFT: Euler-Heisenberg and Effective Field Theory
- Resurgence analysis
- Inhomogeneous fields

4.
- Resurgent Extrapolation
- The Physics of Padé Approximation
- Probing the Borel Plane Numerically

- nonlinearity $\Rightarrow$ new phenomena: e.g. "multi-instantons"
- Painlevé = "nonlinear special functions" [P. Clarkson]

	Number of (essential) parameters	Special Function	Number of Parameters	Associated Orthogonal Polynomial	Number of Parameters
P_I	0	–		–	
P_{II}	1	Airy $\mathrm{Ai}(z)$	0	–	
P_{III}	2	Bessel $J_\nu(z)$	1	–	
P_{IV}	2	Parabolic cylinder $D_\nu(z)$	1	Hermite $\mathrm{He}_n(z)$	0
P_V	3	Whittaker $M_{\kappa,\mu}(z)$	2	Associated Laguerre $L_n^{(k)}(z)$	1
P_{VI}	4	Hypergeometric ${}_2F_1(a,b;c;z)$	3	Jacobi $P_n^{(\alpha,\beta)}(z)$	2

- N.B. integrability is NOT important for resurgence

Resurgence in Nonlinear ODEs: e.g. Painlevé II = "nonlinear Airy"

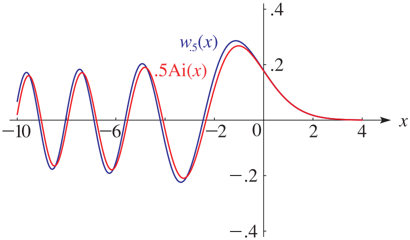
Painlevé II:

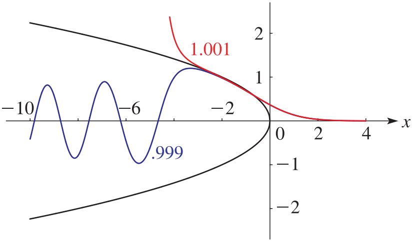
$$y'' = x\, y(x) + 2\, y^3(x)$$

- double-scaling limit in unitary matrix models
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in 2d supergravity
- non-intersecting Brownian motions
- correlators in polynuclear growth; directed polymers (KPZ)
- Tracy-Widom law for statistics of maximum eigenvalue for Gaussian random matrices
- longest increasing subsequence in random permutations
- ... universal !

$$y''(x) = x\,y(x) + 2\,y^3(x)$$

- numerical instability: separatrix

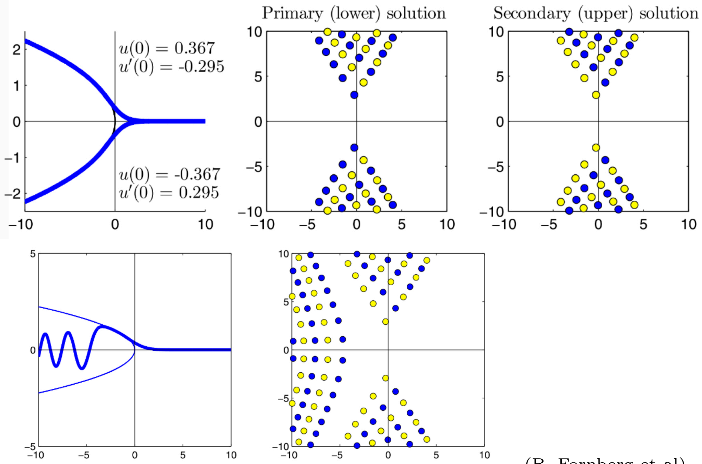




[https://dlmf.nist.gov/32.3.ii]

Resurgence in Nonlinear ODEs: Painlevé II = "nonlinear Airy"

- Hastings-McLeod: $\sigma_+ = 1$ unique real solution on $\mathbb{R}$ that matches $\mathrm{Ai}(x)$ asymptotics as $x \to +\infty$ with $\sqrt{-\frac{x}{2}}$ asymptotics as $x \to -\infty$



(B. Fornberg et al)

$$y''(x) = x\,y(x) + 2\,y^3(x)$$

Exercise 2.1:

1. Show that the general Painlevé II solution has a meromorphic expansion with only poles for moveable singularities (those associated with boundary conditions):

$$y(x) = \frac{1}{x - x_0} - \frac{x_0}{6}(x - x_0) - \frac{1}{4}(x - x_0)^2 + h_0 (x - x_0)^3 + \frac{x_0}{72}(x - x_0)^4 + \dots$$

where all coefficients are expressed in terms of the pair (x_0, h_0), for any pole x_0.

2. Change the nonlinearity of the equation from $y^3(x)$ to $y^4(x)$ and show that this Painlevé integrability condition fails (comment: nevertheless, despite being nonintegrable, all the subsequent resurgent trans-series analysis still holds for such an equation)

- exact integral equation:

$$y(x) = \sigma_+ \,\mathrm{Ai}(x) + 2\pi \int_x^\infty dz\, y^3(z) \left[\mathrm{Ai}(x)\,\mathrm{Bi}(z) - \mathrm{Ai}(z)\,\mathrm{Bi}(x)\right]$$

- iterate $\to$ trans-series ("trans-series parameter": σ_+)

$$y_+(x) \sim \sum_{n=0}^{\infty} \sigma_+^{2n+1}\, Y_{[2n+1]}(x) \qquad , \quad x \to +\infty$$

$$\begin{aligned}
Y_{[1]}(x) &= \mathrm{Ai}(x) \\
Y_{[3]}(x) &= 2\pi \left(\mathrm{Ai}(x) \int_x^\infty \mathrm{Ai}^3(z)\mathrm{Bi}(z)\, dz - \mathrm{Bi}(x) \int_x^\infty \mathrm{Ai}^4(z)\, dz \right) \\
Y_{[5]}(x) &= 6\pi \left(\mathrm{Ai}(x) \int_x^\infty Y_{[3]}(z) \left(Y_{[1]}(z)\right)^2 \mathrm{Bi}(z)\, dz \right. \\
&\qquad \left. - \mathrm{Bi}(x) \int_x^\infty Y_{[3]}(z) \left(Y_{[1]}(z)\right)^2 \mathrm{Ai}(z)\, dz \right) \\
&\;\;\vdots
\end{aligned}$$

Painlevé II: trans-series analysis as $x \to -\infty$

- now consider the opposite direction: $x \to -\infty$

$$y''(x) = x\,y(x) + 2\,y^3(x)$$

- "smoothness" and separatrix as $x \to -\infty \Rightarrow$

$$0 \approx x\,y(x) + 2\,y^3(x) \quad , \quad x \to -\infty$$

- formal series solution

$$y_-(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots\right)$$

- no parameter! $\Rightarrow$ something is missing (non-perturbative corrections)

- non-alternating factorially divergent $\Rightarrow$ something is missing (non-perturbative corrections)

Painlevé II: trans-series analysis as $x \to -\infty$

- first non-perturbative correction "beyond all orders":

$$y''_{[1]} = \left(6\, y_{[0]}^2 + x\right) y_{[1]} \sim \left(-\, 2\, x - \frac{3}{4x^2} + \dots\right) y_{[1]}$$

- exponential ansatz (using Écalle critical variable):

$$y_{[1]}(x) \sim (-x)^\beta e^{-\gamma (-x)^{\frac{3}{2}}} \left(1 + \dots\right)$$

- matching terms $\Rightarrow$

$$y_{[1]}(x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3}(-x)^{3/2}} \left(1 - \frac{\frac{17}{72}}{\sqrt{2}\,\frac{2}{3}(-x)^{3/2}} + \frac{\frac{1513}{10368}}{(\sqrt{2}\,\frac{2}{3}(-x)^{3/2})^2} - \dots\right)$$

- the $\sqrt{2}$ factor is not a misprint !

Resurgence Relation for Painlevé II

- recall formal perturbative series as $x \to -\infty$

$$y_{[0]}(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots\right)$$

- large order growth of coefficients as $n \to \infty$

$$c_n^{[0]} \sim -\frac{1}{\pi}\sqrt{\frac{2}{3\pi}}\frac{\Gamma\left(2n - \frac{1}{2}\right)}{\left(\frac{2\sqrt{2}}{3}\right)^{2n}}\left(1 - \frac{\frac{17}{72}}{(2n - \frac{3}{2})} + \frac{\frac{1513}{10368}}{(2n - \frac{3}{2})(2n - \frac{5}{2})} - \dots\right)$$

Resurgence Relation for Painlevé II

- recall formal perturbative series as $x \to -\infty$

$$y_{[0]}(x) \sim \sqrt{\frac{-x}{2}} \left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10567}{1024(-x)^9} - \dots \right)$$

- large order growth of coefficients as $n \to \infty$

$$c_n^{[0]} \sim -\frac{1}{\pi}\sqrt{\frac{2}{3\pi}} \frac{\Gamma\left(2n-\frac{1}{2}\right)}{\left(\frac{2\sqrt{2}}{3}\right)^{2n}} \left(1 - \frac{\frac{17}{72}}{(2n-\frac{3}{2})} + \frac{\frac{1513}{10368}}{(2n-\frac{3}{2})(2n-\frac{5}{2})} - \dots \right)$$

- compare with fluctuations around the first exponential:

$$y_{[1]}(x) \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\sqrt{2}\frac{2}{3}(-x)^{\frac{3}{2}}} \left(1 - \frac{\frac{17}{72}}{\sqrt{2}\,\frac{2}{3}(-x)^{\frac{3}{2}}} + \frac{\frac{1513}{10368}}{(\sqrt{2}\,\frac{2}{3}(-x)^{\frac{3}{2}})^2} - \dots \right)$$

- large-order/low-order resurgence relation (*cf.* Airy)

Exercise 2.2:

1. Generate many terms, and verify the large order behavior of the coefficients of the formal $x \to -\infty$ series, $y_{[0]}(x)$, for the Painlevé II Hastings-McLeod solution
2. Numerically identify the Stokes constant to high precision

$$0.1466323... = \frac{1}{\pi}\sqrt{\frac{2}{3\pi}}$$

3. Confirm the first large-order/low-order resurgence relation

Transmutation of a trans-series

- different trans-series solutions for $x \to \pm\infty$
- $x \to +\infty$

$$y_+(x) \sim \sum_{k=0}^{\infty} \left(\frac{\sigma_+ \, e^{-\frac{2}{3}\, x^{3/2}}}{2\,\sqrt{\pi}\, x^{1/4}} \right)^{2k+1} \mathcal{F}_{[2k+1]}(x)$$

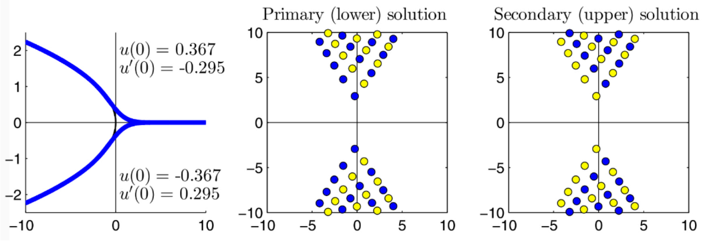
- $x \to -\infty$

$$y_-(x) \sim \sqrt{\frac{-x}{2}} \sum_{k=0}^{\infty} \left(\frac{\sigma_- \, e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}\,(-x)^{1/4}} \right)^{k} \mathcal{Y}_{[k]}(x)$$

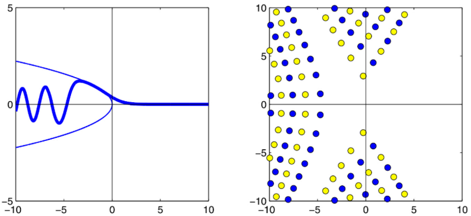
- non-linear Stokes phenomenon
- "condensation of instantons" across the transition
- "trans-asymptotic analysis" describes the transition

Transmutation of a trans-series

- recall the interesting structure in the complex plane ...



- there are other more general solutions ...



- 3rd order phase transition at $N = \infty$, $t = 1$ (universal)

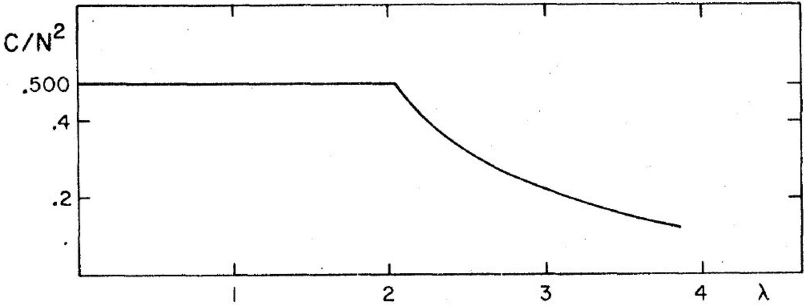


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

Gross-Witten-Wadia Matrix Model

- "order parameter" $\Delta(t, N) \equiv \langle \det U \rangle$ satisfies a nonlinear ODE
- Rossi equation (Painlevé III):

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2}\left(1 - \Delta^2\right) = \frac{\Delta}{1 - \Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

Gross-Witten-Wadia Matrix Model

- "order parameter" $\Delta(t, N) \equiv \langle \det U \rangle$ satisfies a nonlinear ODE
- Rossi equation (Painlevé III):

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2} \left(1 - \Delta^2\right) = \frac{\Delta}{1 - \Delta^2} \left(N^2 - t^2 \left(\Delta'\right)^2\right)$$

- non-perturbative large N effects from the ODE

$$\Delta(t, N) = \sum_n \frac{c_n^{(0)}(t)}{N^{2n}} + e^{-N\, S(t)} \sum_n \frac{c_n^{(1)}(t)}{N^n} + e^{-2N\, S(t)} \sum_n \frac{c_n^{(2)}(t)}{N^n} + \dots$$

- all physical observables inherit this trans-series structure
- phase transition = nonlinear Stokes phenomenon

Exercise 2.3: Consider the GWW model Rossi equation (Painlevé III in Okamoto form):

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

1. Show that in the $t>1$ region this equation linearizes to

   $$t^2\Delta'' + t\Delta' + \frac{N^2}{t^2}\left(1-t^2\right)\Delta \approx 0$$

   and this is solved by the Bessel functions $J_N\left(\frac{N}{t}\right)$, $Y_N\left(\frac{N}{t}\right)$.
2. Hence show that a solution decreasing at large t can be written as an exact integral equation, which can be iterated to generate the $t>1$ large N trans-series.
3. Show that for $t<1$ the dominant large N solution is algebraic, $\Delta(t)\sim\sqrt{1-t}$, from which the formal large N series solution can be generated.

Resurgence: Large N at Strong 't Hooft Coupling

- large N trans-series at strong coupling ($t > 1$)

$$\Delta(t, N) \approx \sigma_{\text{strong}}\, J_N\left(\frac{N}{t}\right) \sim \sigma_{\text{strong}}\, \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\, (t^2 - 1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \dots$$

- strong-coupling large N instanton action

$$S_{\text{strong}}(t) = \text{arccosh(t)} - \sqrt{1 - \frac{1}{t^2}}$$

Resurgence: Large N at Strong 't Hooft Coupling

- large N trans-series at strong coupling ($t > 1$)

$$\Delta(t,N) \approx \sigma_{\text{strong}}\, J_N\left(\frac{N}{t}\right) \sim \sigma_{\text{strong}}\, \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \dots$$

- strong-coupling large N instanton action

$$S_{\text{strong}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$$

- nonlinearity $\Rightarrow$ trans-series with all odd powers of

$$\sigma_{\text{strong}} \frac{e^{-N S_{\text{strong}}(t)}}{\sqrt{S'_{\text{strong}}(t)}}$$

Resurgence: Large N at Weak 't Hooft Coupling

- large N trans-series at weak-coupling ($t < 1$)

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\text{weak}}}{2\sqrt{2\pi N}}\,\frac{t\,e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling large N instanton action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

Resurgence: Large N at Weak 't Hooft Coupling

- large N trans-series at weak-coupling ($t < 1$)

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\rm weak}}{2\sqrt{2\pi N}}\frac{t\, e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling large N instanton action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,{\rm arctanh}\left(\sqrt{1-t}\right)$$

- large-order growth of perturbative coefficients ($\forall\, t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

- (parametric) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

Exercise 2.4:

1. Generate many terms of the formal large N solution for $\Delta(t, N)$ in the small t regime

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(\lambda)}{N^{2n}}$$

2. Derive the form of the first non-perturbative correction to this formal large N expansion, including the first few fluctuation corrections:

$$\Delta_{NP}(t, N) \sim f(t)\, e^{-N\, S_{\text{weak}}(t)} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(\lambda)}{N^n} + \dots$$

3. Show that the subleading corrections to the large n growth of the coefficient functions $d_n^{(0)}(\lambda)$ are associated with the expansion terms $d_n^{(1)}(\lambda)$ of the first non-perturbative correction to the formal large N expansion.

Resurgence in GWW: double-scaling limit = Painlevé II

- uniform limit of Bessel function:

$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \mathrm{Ai}\left(2^{1/3}\kappa\right)$$

- scaling of $J_N(N/t)$ as $t \to 1$: $N \to \infty$ with x fixed

$$t \sim 1 + \frac{x}{(2N^2)^{1/3}} \quad ; \quad \Delta(t, N) = \left(\frac{2t}{N}\right)^{1/3} y(x)$$

Δ PIII equation $\longrightarrow$ $\frac{d^2y}{dx^2} = x\, y(x) + 2\, y^3(x)$ (PII)

Resurgence in GWW: double-scaling limit = Painlevé II

- uniform limit of Bessel function:

$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \mathrm{Ai}\left(2^{1/3}\kappa\right)$$

- scaling of $J_N(N/t)$ as $t \to 1$: $\quad N \to \infty$ with x fixed

$$t \sim 1 + \frac{x}{(2N^2)^{1/3}} \quad ; \quad \Delta(t, N) = \left(\frac{2t}{N}\right)^{1/3} y(x)$$

Δ PIII equation $\longrightarrow$ $\dfrac{d^2y}{dx^2} = x\, y(x) + 2\, y^3(x)$ (PII)

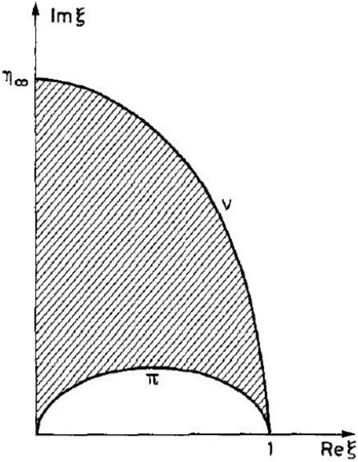
- near $t_c = 1^+$, $S_{\text{strong}} \sim \frac{2\sqrt{2}}{3}(t-1)^{3/2}$

$$N\, S_{\text{strong}} \sim N\, \frac{2\sqrt{2}}{3}\, \frac{x^{3/2}}{\sqrt{2}\, N} = \frac{2}{3}\, x^{3/2}$$

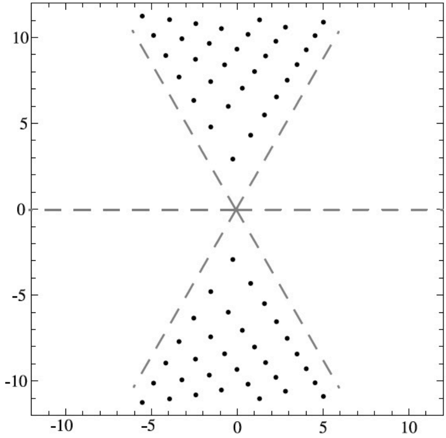
- near $t_c = 1^-$, $S_{\text{weak}} \sim \frac{4}{3}(1-t)^{3/2}$

$$N\, S_{\text{weak}} \sim N\, \frac{4}{3}\, \frac{(-x)^{3/2}}{\sqrt{2}\, N} = \frac{2\sqrt{2}}{3}\, (-x)^{3/2}$$

Lee-Yang: complex zeros of Z pinch the real axis at the phase transition point in the thermodynamic limit



GWW zeros (Kolbig)



Painlevé II (Novokshenov; Huang)