- i.e. perturbative information $\longrightarrow$ non-perturbative information
- now let's go in the other direction:

we start with the integral representation of $Ai(x)$ (analogue of the Borel / Laplace integral) instead of the differential equation

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \; e^{i(xt + t^3/3)}$$

write this in the form of an asymptotic exponential integral

$$\int_\gamma dz \; e^{\frac{1}{\hbar} S(z)}$$

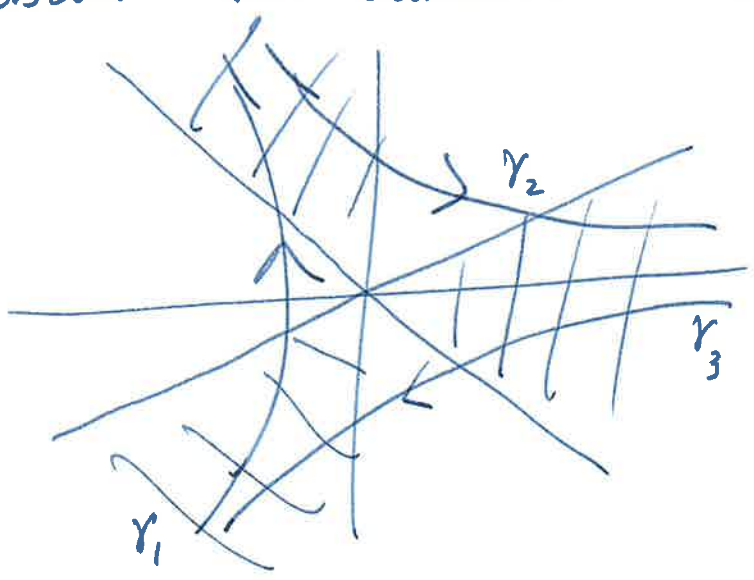
write $x = r e^{i\theta}$ $\quad (r = |x|)$ and $t = -i\sqrt{r}\, z$

$$\Rightarrow \quad Ai(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz \; e^{r^{3/2}\left(e^{i\theta} z - \frac{1}{3} z^3\right)}$$

large, real, positive

we can deform the z contour, but any contour γ for $\int_\gamma dz \; e^{r^{3/2}\left(e^{i\theta} z - \frac{1}{3} z^3\right)}$

must asymptotically end in one of the sectors about the directions $\arg(z) = 0, \; 2\pi/3, \; 4\pi/3$



$\gamma_1 + \gamma_2 + \gamma_3 = 0$

saddle points

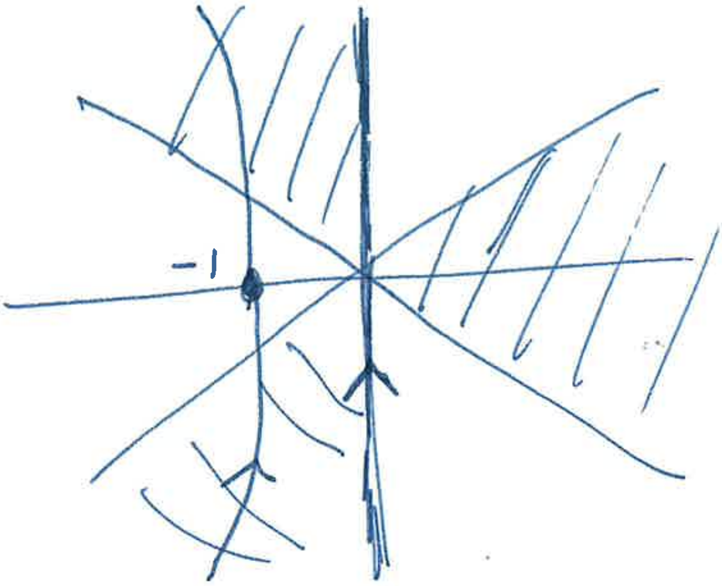
$$S(z;\theta) = e^{i\theta} z - \frac{1}{3} z^3$$

$$\Rightarrow \quad z_\pm = \pm e^{i\theta/2}$$

$$S_\pm \equiv S(z_\pm) = \pm \frac{2}{3} e^{3i\theta/2}$$

$\theta = 0 \Rightarrow z_- = -1$, $S_- = -2/3$

↑ ∴ decaying sol.ⁿ for large r



- original contour (imag. axis) can be smoothly deformed to γ_1
- leading saddle analysis

$\Rightarrow$ Gaussian fluctuations

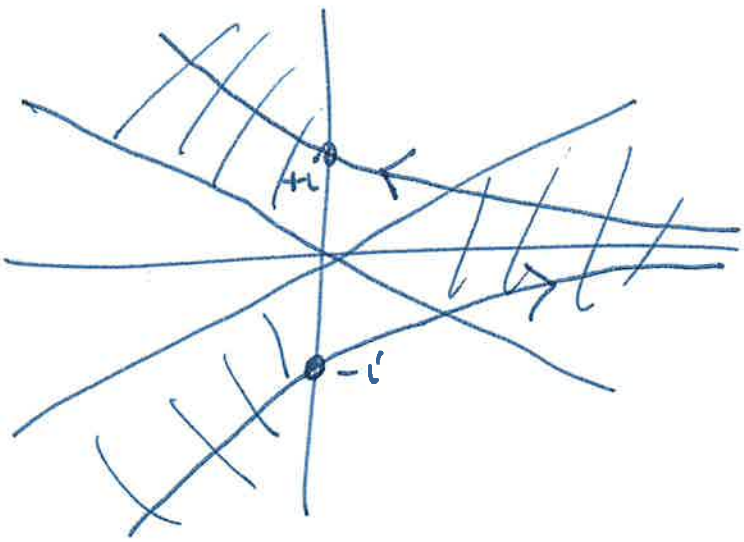
require γ to go through $z_1 = -1$ in the vertical direction

$$\Rightarrow \quad Ai(x) \sim \frac{1}{\sqrt{4\pi}\, x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}} \quad \text{as} \quad x \to \infty$$

(i.e. $\theta = 0$)

$\theta = \pi$: $\Rightarrow z_\pm = \pm i$, $S_\pm = \mp \frac{2i}{3}$

$\Rightarrow e^{r^{3/2} S_\pm}$ equal magnitude



deform contour to $-\gamma_3 - \gamma_2$

- go through $z_- = -i$ at angle $\pi/4$
- go through $z_+ = +i$ at angle $3\pi/4$

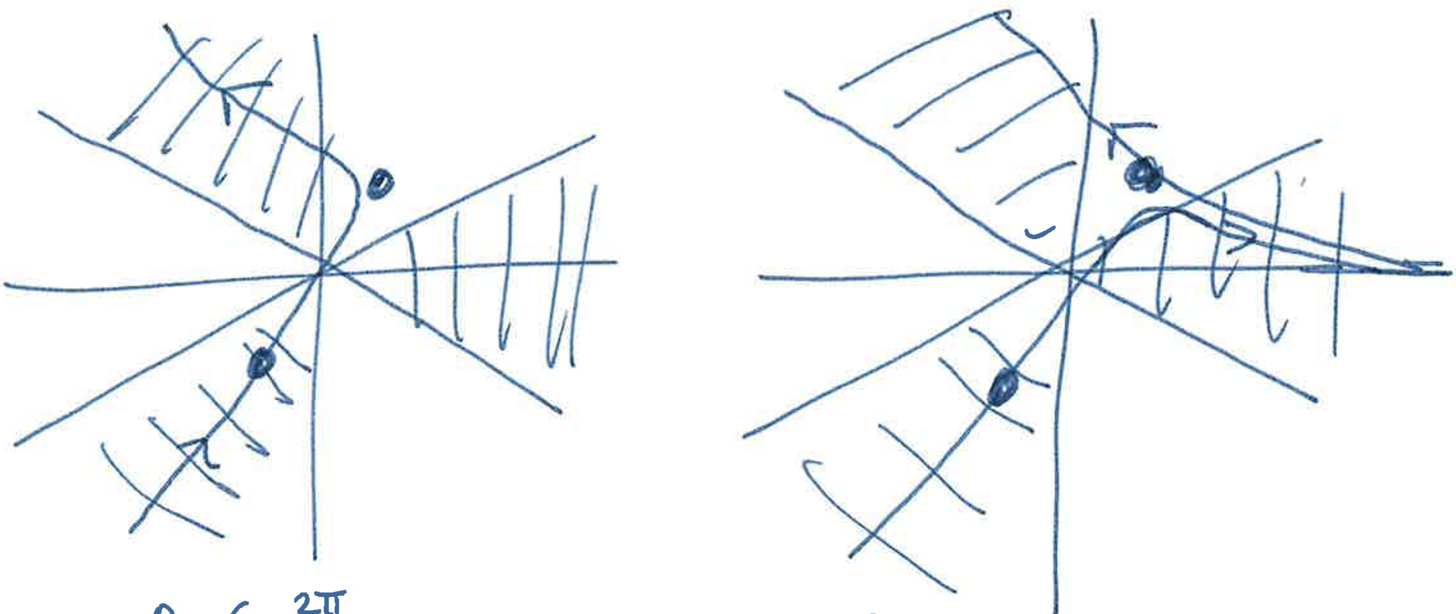
combine (note: $|S_+| = |S_-|$)

$$Ai(x) \sim \frac{1}{\sqrt{\pi}\,(-x)^{3/2}} \cos\left(\frac{2}{3}(-x)^{3/2} - \pi/4\right)$$

↑ phase shift!

Stokes phenomenon

- as θ varies $(x \equiv r e^{i\theta})$ the saddle points $(z_\pm = \pm e^{i\theta/2})$ rotate, and the steepest descent contours are deformed
- there is a jump at $\theta = \frac{2\pi}{3}$ (and $\theta = \frac{4\pi}{3}$)



$\theta \lesssim \frac{2\pi}{3}$ $\qquad\qquad$ $\theta \gtrsim \frac{2\pi}{3}$

There is a sudden jump in the structure of the steepest descent contours

$\Rightarrow$ a new contour contributes now

- a steepest descent contour is one for which the imaginary part of the "action" $S(z;\theta)$ remains constant (recall the Cauchy-Riemann eqs.)

$z \equiv u + iv$

- for Airy: consider $\theta = 0$

$$\Rightarrow \quad S = (u+iv) - \frac{1}{3}(u+iv)^3$$
$$= \left(u + uv^2 - \tfrac{1}{3}u^3\right) + i\left(v - u^2 v + \tfrac{1}{3}v^3\right)$$

All orders steepest descent

$$S(z_- = -1) = -\tfrac{2}{3} \;\Rightarrow\; \mathrm{Im}\left(S(z_- = -1)\right) = 0$$

so $\mathrm{Im}(S) = 0$ on the steepest descent contour

$$\Rightarrow \quad v - u^2 v + \tfrac{1}{3} v^3 = 0$$

$$u = \pm\sqrt{1 + v^2/3}$$

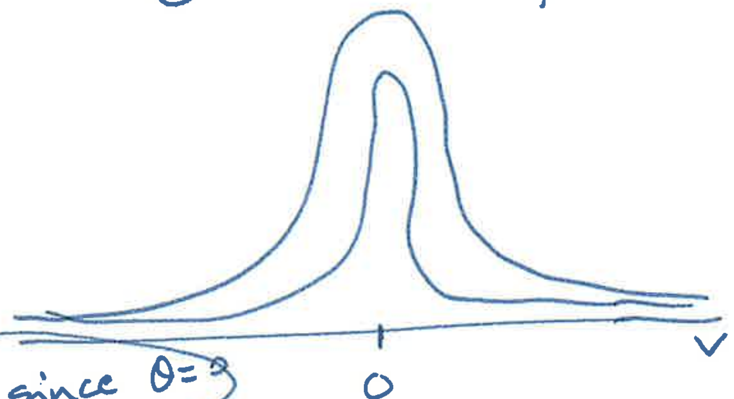
choose $-$ sign to have $v = 0 \Longleftrightarrow z_- = -1$

$$\therefore \quad z = -\sqrt{1 + v^2/3} + iv$$

on the s.d. contour

$$S(v) = -\tfrac{2}{3}\left(1 + \tfrac{4}{3}v^2\right)\sqrt{1 + v^2/3}\,, \quad v \in (-\infty, \infty)$$

plot $e^{x^{3/2} S(v)}$ for various x



$$dz = \left(i - \frac{v}{3\sqrt{1 + v^2/3}}\right) dv$$

$x = \phi$ since $\theta = 0$

odd function vanishes

$$Ai(x) = \frac{\sqrt{x}}{2\pi i}\int_{-\infty}^{\infty} dv \, \underbrace{\left(i - \cancel{\frac{v}{3\sqrt{1+\frac{v^2}{3}}}}\right)}_{\text{Jacobian}} e^{-\frac{2}{3}x^{3/2}\left(1+\frac{4}{3}v^2\right)\sqrt{1+\frac{v^2}{3}}}$$

$$= \frac{\sqrt{x}}{2\pi}\, e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dv\, e^{-x^{3/2}\left[\overset{-v^2}{\cancel{v^2}} - \frac{2}{3}\left(1+\frac{4}{3}v^2\right)\sqrt{1+\frac{v^2}{3}}\right]}$$

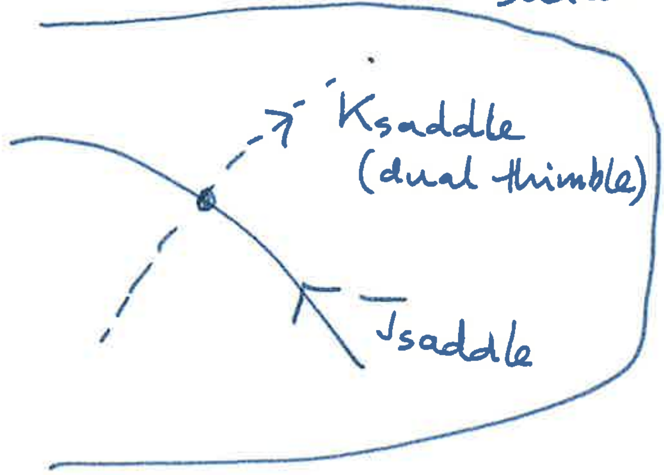
- exact, and easy to integrate by Monte Carlo (exercise!)
- alternatively: expand integrand/exponent beyond the quadratic order $\longrightarrow$ yields asymptotic expansion

Picard - Lefschetz

generally

signed intersection number of original contour Γ with the dual thimble K

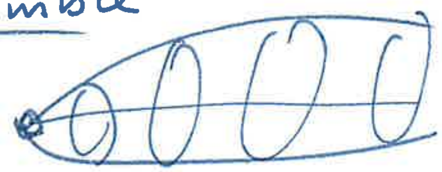
$$\int_\Gamma e^{\frac{1}{\hbar}S(z)} = \sum_{\text{saddles}} \langle \Gamma, K_{\text{saddle}} \rangle \, e^{\text{Im}\left(\frac{1}{\hbar}S_{\text{saddle}}\right)} \times$$



$$\int_{J_{\text{saddle}}} dz \; \mathcal{J}_{\text{saddle}} \; e^{\text{Re}\left(\frac{1}{\hbar}S_{\text{saddle}}\right)}$$

Jacobian

in multidimensions, the curve becomes a "thimble"



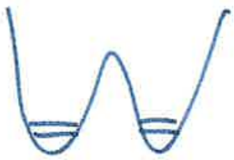
with one direction leading to the saddle pt., and transverse directions also

---

- the Airy example illustrates many important features of resurgence, thimbles and the Stokes phenomenon, but it is simple because there are just 2 saddle points
- remarkably, this resurgent behavior has been observed in many multi-dimensional and infinite dimensional examples!

QM example with an infinite number of saddles (i.e. multi-instanton expansion)

- consider the paradigmatic cases of QM instantons



$V(x) = (x^2-1)^2$ $\qquad$ $V(x) = \cos(x)$

$$E_0(\hbar) \sim \sum_n a_n \hbar^n \pm \frac{e^{-S_I/\hbar}}{\sqrt{\hbar}} \sum_n b_n \hbar^n + \frac{e^{-2S_I/\hbar}}{\hbar}\left[\sum_n (c_n \hbar^n) + \ln\hbar \sum_n d_n \hbar^n\right] + O\left(e^{-3S_I/\hbar}\right)$$

perturbation theory

$$a_n \sim \frac{n!}{(2S_I)^n}$$

S_I = instanton action (tunneling between neighboring wells)

note: $2S_I$ not S_I

but E_0 must be real!

1-instanton sector: exponentially small splitting

fluctuations: $b_n \sim \dfrac{n! \ln n}{(2S_I)^n}$

- 2-instanton sector is unstable because instantons and anti-instantons attract ($\Rightarrow$ "instanton gas picture" is unstable). Regulate by $\hbar \to \hbar \pm i\varepsilon$
- amazing cancellation (Bogomolny, Zinn-Justin, ...)

$$\left.\begin{array}{l} \text{pert. theory} \longrightarrow \text{Borel} \rightarrow \; i\, e^{-2S_I/\hbar} \\ \ln\hbar \text{ term at } O(e^{-2S_I/\hbar}) \longrightarrow -i\, e^{-2S_I/\hbar} \end{array}\right\} \text{cancel!}$$

occurs to all orders in trans-series

- this correlates various terms in the trans-series, ensuring reality of the entire trans-series

[Aniceto & Schiappa : reality relations for trans-series
1308.1115]

- in fact, the situation is even more interesting — include also the level number N (labeling the unperturbed harmonic level)

$$E(\hbar, N) \sim E_{pert}(\hbar, N) \pm \frac{1}{N!} \frac{1}{\hbar^{N+\frac{1}{2}}} e^{-S_I/\hbar} \mathcal{F}_{(1)}(\hbar, N)$$

$$+ O\left(e^{-2S_I/\hbar}\right)$$

fluctuations about the 1-instanton sector

Then (Alvarez - Casares, Dunne - Ünsal, ...)
J Phys A 33 (2000), 35 (2002)

$$\boxed{\mathcal{F}_{(1)}(\hbar, N) \sim \frac{\partial E_{pert}}{\partial N} \, e^{S_I \int_0^\hbar \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{pert}}{\partial N} - \hbar + \frac{\hbar^2}{S_I}(N+\frac{1}{2})\right)}}$$

exact to all orders in $\hbar$!!!

- similar expressions propagate through the entire trans-series

$$E_{pert} \approx \sum_n c_n(N) \hbar^n$$

↑ polynomials

(to leading order)

- see exercise 1.5 where you verify this for the periodic (Mathieu) potential
- high orders match also

(1401.5202
1603.04924)

$\Rightarrow$ perturbation theory encodes the fluctuations about <u>all</u> multi-instanton sectors, to <u>all</u> (infinite #) order in the multi-instanton expansion!

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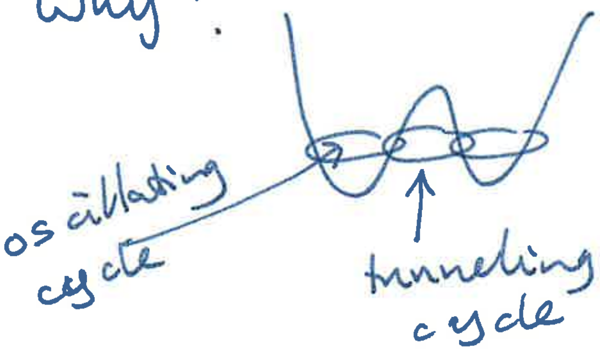
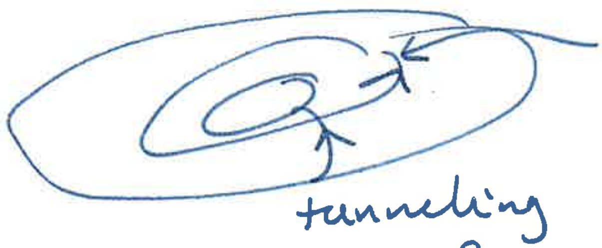
why?



$$H = p^2 + V(x)$$
$$\downarrow$$
$$p^2 = E - V(x)$$

if $V(x)$ (in suitable variable x) is $\leq$ quadratic, this defines an elliptic curve

(phase space $\longrightarrow$ torus)



$$a_0(E) = \oint_\alpha \sqrt{E-V}\ , \qquad a_0^{Dual}(E) = \oint_\beta \sqrt{E-V}$$

<u>Bohr-Sommerfeld:</u> $\quad a_0(E) = 2\pi\hbar\,(N + 1/2)$

∘ expand at low E and <u>invert</u> series

$$\longrightarrow \quad E = E(\hbar, N)$$

extend to all $\hbar$ orders

$$a_0(E) \longrightarrow a(E,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} a_n(E) = \oint_\alpha \sqrt{E-V} + \hbar^2 \oint_\alpha \frac{(V')^2}{(E-V)^{5/2}} + \dots$$

$$a_0^D(E) \longrightarrow a^D(E,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} a_n^D(E) = \oint_\beta \sqrt{E-V} + \hbar^2 \oint_\beta \frac{(V')^2}{(E-V)^{5/2}} + \dots$$

all-orders Bohr-Sommerfeld

$$a(E,\hbar) \overset{!}{=} 2\pi\hbar\,(N+1/2)$$

expand at small E and invert

$$\longrightarrow E(\hbar,N) \sim \sum_{n=0}^{\infty} \hbar^n p_n(N)$$

asymptotic expansion w/ coefficients being polynomials in N

<u>identical to ordinary perturbation theory</u> !

<u>but</u> : we are missing non-perturbative effects

$\rightarrow$ exact quantization condition (monodromy condition)

defined by the potential

$\rightarrow$ transcendental relation between $a(E,\hbar)$ and $a^D(E,\hbar)$

Legendre, Riemann, Picard, Fuchs, ...

(i) $a_0(E)$ determines $a_0^D(E)$ (Riemann bilinear)

(ii) $a_n(E) = \mathcal{D}_E^{(n)}\, a_0(E) \quad \forall n$

$\uparrow$ differential operator w.r.t. E

i.e. <u>all</u> action coefficients expressed in terms of <u>classical</u> $a_0(E)$

(iii) $a_n^D(E) = \mathcal{D}_E^{(n)}\, a_0^D(E) \quad \forall n$

$\uparrow$ same operators since it is an identity for the integrands

$\therefore$ $\boxed{a_n(E) \text{ determines } a_n^D(E) \quad \forall n}$

"non-pert. physics is encoded in perturbative physics" classical Picard-Fuchs eqn

$$a_0(E)\,\frac{da_0^D(E)}{dE} - a_0^D(E)\,\frac{da_0(E)}{dE} = \text{constant},\ c$$

$$\Big\downarrow \text{ all } \hbar \text{ orders}$$

$$\left(a(E,\hbar) - \hbar\frac{\partial}{\partial\hbar}a(E,\hbar)\right)\frac{\partial a^D}{\partial E} - \left(a^D(E,\hbar) - \hbar\frac{\partial}{\partial\hbar}a^D(E,\hbar)\right)\frac{\partial a}{\partial E} = c$$

"quantum Wronskian equation"

Nonlinear Airy equation : Painlevé II

- in a nonlinear ODE, if a term e^{-x} is generated, then generically all powers e^{-kx} will also be generated
  $\Rightarrow$ we expect an infinite order trans-series

$$\sum_n \sum_k \sum_\ell c_{nk\ell} \frac{1}{x^n} e^{-k/x} \underbrace{(\ln x)^\ell}_{\text{possibly present (cf. Frobenius)}}$$

- we will study

$$y'' = xy \; \underbrace{+ 2y^3}_{\text{nonlinearity}}$$

in order to illustrate how this happens

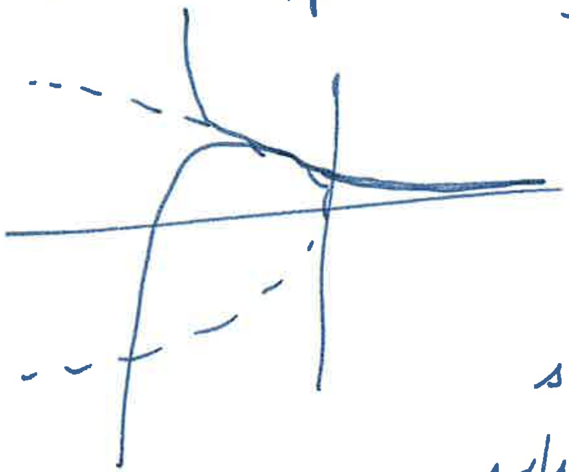
- this is the Painlevé II equation (a special case thereof.)
- Painlevé: classified all 2^{nd} order nonlinear ODEs whose moveable singularities (ie. those associated with b.c.'s) are only poles ($\Rightarrow$ meromorphic solutions)
- P II is universal in the sense that Airy is universal (local behavior near a linear turning point). PII local behavior across large class of phase transitions. Applications in physics, biology, statistics, math, combinatorics, ...

• see slides for separatrix behavior

$x \to +\infty \qquad y'' = xy + 2y^3$ exponentially small

$\therefore \quad y \sim \sigma_+ \mathrm{Ai}(x) + \dots$

$x \to \infty$: exponentially sensitive to b.c.'s



separatrix $y = \pm\sqrt{\frac{-x}{2}}$

from balancing

$xy \approx -2y^3$

special "Hastings-McLeod" solution hugs the separatrix

HM solution only has poles in 2 $\pi/3$ sectors – see slides

$x \to +\infty$ trans-series

$$y'' = xy + 2y^3 \equiv y(x) = \sigma_+ \mathrm{Ai}(x) + 2\pi \int_x^\infty dz\, y^3(z) \left(\mathrm{Ai}(x)\mathrm{Bi}(z) - \mathrm{Ai}(z)\mathrm{Bi}(x) \right)$$

(exercise: check!)

iterate ($\mathrm{Ai}(x)$ expon. small as $x \to +\infty$)

$$y_+(x) \sim \sum_{k=0}^{\infty} \sigma_+^{2k+1}\, Y_{(2k+1)}(x)$$

note: odd powers only

(see slides)

$$\sim \sum_{k=0}^{\infty} \left(\frac{\sigma_+ e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\pi}\, x^{1/4}} \right)^{2k+1} \underbrace{F_{(2k+1)}(x)}_{\text{fluctuations}}$$

$x \to -\infty$ trans-series

ansatz : $y_-(x) \sim \sqrt{\frac{-x}{2}} \sum_{n=0}^{\infty} \frac{c_n}{(-x)^{3n/2}}$

ODE $\Rightarrow$ $y_-(x) \sim \sqrt{\frac{-x}{2}}\left(1 - \frac{1}{8}\frac{1}{(-x)^3} - \frac{73}{128}\frac{1}{(-x)^6} - \frac{10567}{1024}\frac{1}{(-x)^9} + \cdots\right)$

note: • even powers $\frac{1}{(-x)^{3n}}$

• non-alternating $[(-x)>0]$

• factorially divergent coefficients

• no parameter (!) $\Rightarrow$ something missing

write: $y = y_{pert} + \varepsilon \underbrace{y_{non\text{-}pert}}_{\text{"beyond all orders"}}$

ODE $\Rightarrow$ linearize in ε to get leading non-pert. behavior

$\cancel{y''_{pert}} + \varepsilon y''_{nonpert} = \cancel{x y_{pert}} + \varepsilon x y_{nonpert} + \cancel{2 y^3_{pert}}$
$+ 2 \cdot 3 \varepsilon y^2_{pert}(x) \, y_{nonpert}(x) + \cdots$

$O(\varepsilon)$: $y''_{np} = (6 y_p^2 + x) y_{np}$ linear + homog.

$\sim \left(6 \cdot \left(-\frac{x}{2}\right) + x + O\left(\frac{1}{x^2}\right)\right) y_{np}$

$\sim (-2x + \cdots) \, y_{np}$

Airy with a different sign and scale!

$\Rightarrow$ try $y_{np} \sim (-x)^{\beta} e^{-\gamma(-x)^{3/2}} \sum \frac{d_n}{(-x)^{3n/2}}$

$\Rightarrow$ $\beta = -1/4$, $\gamma = \frac{2\sqrt{2}}{3}$ note: $\sqrt{2}$ factor is not a misprint. Trace it to

(30)

ODE $\Rightarrow$

$$y_{np} \sim \frac{\sigma_-}{(-x)^{1/4}} e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}} \left(1 - \frac{17/72}{\sqrt{2}\cdot\frac{2}{3}(-x)^{3/2}} + \frac{1513/10368}{\left(\frac{\sqrt{2}\,2}{3}(-x)^{3/2}\right)^2} - \cdots \right)$$

recall perturbative series for $x \to -\infty$

$$y_p \sim \sqrt{\frac{-x}{2}} \left(1 - \frac{1}{8(-x)^3} - \frac{73}{128(-x)^6} - \frac{10367}{1024(-x)^9} - \cdots \right)$$

coefficients

$$c_n^{(p)} \sim -\frac{1}{\pi}\sqrt{\frac{2}{3\pi}} \frac{\Gamma(2n - 1/2)}{\left(\frac{2\sqrt{2}}{3}\right)^{2n}} \left(1 - \frac{17/72}{(2n - 3/2)} + \frac{1513/10368}{(2n-\frac{3}{2})(2n - 5/2)} \cdots \right)$$

$\Rightarrow$ large-order / low-order resurgence relation, like for Airy

- this happens throughout the full trans-series (here just the leading one).

summary

$x \to +\infty$ transseries
$$y_+(x) \sim \sum_{k=0}^{\infty} \left(\frac{\sigma_+ e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\pi}\, x^{1/4}} \right)^{2k+1} f_{(2k+1)}(x)$$

$x \to -\infty$ trans-series
$$y_-(x) \sim \sqrt{\frac{-x}{2}} \sum_{k=0}^{\infty} \left(\frac{\sigma_- e^{-\frac{2\sqrt{2}}{3}(-x)^{3/2}}}{2\sqrt{\pi}(-x)^{1/4}} \right)^k y_{(k)}(x)$$

- non-linear Stokes phenomenon
- asymptotic expansion crosses $\arg(x) = \pi/3$ and becomes a meromorphic exp.^{n} w/ poles! Condensation of instantons

"trans-asymptotics": resum all instanton orders (Costin)

- cross from the $x \to -\infty$ into the pole region and there is a condensation of <u>different instantons</u>!
- note $(x \leftrightarrow -x)$ distortion of poles in the pole region

---

<u>physics application</u> : Gross-Witten-Wadia unitary matrix model

$\equiv$ 2 dim. $U(N)$ lattice gauge theory

$$Z(t,N) = \int_{\substack{N\times N \\ U(N)}} dU \; e^{\frac{N}{t} \operatorname{tr}(U + U^{+})}$$

random matrix theory

$t = Ng^2$, 't Hooft coupling

$N \to \infty$, $g \to 0$
t fixed (but $t<1$ or $t>1$)

$$= \det\left(I_{j-k}\left(\tfrac{N}{t}\right)\right)_{j,k=1\ldots N}$$

but large determinants are difficult to analyze

$\Rightarrow$ another approach ...

define $\Delta(t,N) = \langle \det U \rangle$ "order parameter"

- all thermodynamic quantities can be derived from $\Delta(t,N)$
- $\therefore$ study its trans-series structure

$$\Delta^2(t,N) = 1 - \frac{Z(t,N-1)\,Z(t,N+1)}{Z(t,N)^2}$$ "Toda eq^n"

$\Delta(t,N)$ satisfies a nonlinear ODE in the Painlevé III class: ("Rossi Equation")

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}(1-\Delta^2) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2(\Delta')^2\right)$$

large N :

$$\Delta(t,N) \overset{?}{=} \sum_n \frac{C_n^{(0)}(t)}{N^{2n}} + e^{-NS(t)}\sum_n \frac{C_n^{(1)}(t)}{N^n}$$

$$+ e^{-2NS(t)}\sum_n \frac{C_n^{(2)}(t)}{N^n} + \dots$$

- all thermodynamic observables (free energy, specific heat, ...) inherit this basic structure from $\Delta(t,N)$
- at $N=\infty$, the GWW has a 3rd order phase transition at $t_c=1$ (kink in the specific heat - see slides for figure)
- for $t>1$, Δ is very small so we can linearize the Rossi eq^n:

$$t^2\Delta'' + t\Delta' + N^2\left(\frac{1}{t^2}-1\right)\Delta \approx 0$$

$$\Rightarrow \quad \Delta = J_N\left(\frac{N}{t}\right) \quad \text{or} \quad Y_N\left(\frac{N}{t}\right)$$

at large N, need <u>uniform</u> asymptotics of Bessel functions

"Debye expansion" for $t>1$

$$\Delta(t,N) \approx \sigma_{strong}\, J_N\left(\frac{N}{t}\right)$$

recursively generated

$$\sim \sigma_{strong} \sqrt{t}\, \frac{e^{-N S_{strong}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \dots$$

where $S_{strong}(t) = \mathrm{arccosh}(t) - \sqrt{1-\frac{1}{t^2}}$

- but the equation is non-linear, so we find a trans-series with all odd powers of

$$\left(\sigma_{strong} \frac{e^{-N S_{strong}(t)}}{\sqrt{S'_{str}(t)}}\right)$$

---

side comment: this solution diverges at $t=1$, analogous to how WKB wavefunctions diverge at a turning point. There is a "better" form of $\Delta(t,N)$ given by the uniform expansion in terms of Airy functions: see DLMF 10.20.4

$$J_N\left(\frac{N}{t}\right) \sim \left(\frac{4\zeta}{1-1/t^2}\right)^{\frac{1}{4}} \frac{Ai(N^{2/3}\zeta)}{N^{1/3}} \sum_n \frac{A_n(\zeta)}{N^{2n}} + \dots$$

where: $\frac{2}{3}\zeta^{3/2} = S_{strong}(t)$

this large N form is smooth as $t \to 1^+$

large N trans-series for $t<1$:

dominant terms are those multiplying N^2 in the ODE:

$$\longrightarrow \quad \frac{\Delta}{t^2}\left(1-\Delta^2\right) \sim \frac{\Delta}{1-\Delta^2}$$

$$\Rightarrow \quad \left(1-\Delta^2\right) \sim t^2 \longrightarrow \quad \Delta \sim \sqrt{1-t} + \dots$$

formal series

$$\Delta_{\text{pert.}} \approx \sqrt{1-t}\, \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}}$$

trans-series:

$$\Delta(t,N) \sim \Delta_{\text{pert}}(t,N) - \frac{\sigma_{\text{weak}}}{2\sqrt{2\pi N}}\, \frac{t\, e^{-N S_{\text{weak}}(t)}}{(1-t)^{1/4}} \sum_n \frac{d_n^{(1)}(t)}{N^n}$$

$$+ \dots$$

where ODE $\Rightarrow$ $S_{\text{weak}} = \dfrac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$

<u>note</u>: the expansion coefficients $d_n^{(0)}(t)$ and $d_n^{(1)}(t)$ are non-trivial functions of t. We still find a <u>resurgent</u> large order / low order resurgence relation – see the slides

large order (n) behavior of $d_n^{(0)}(t)$ is directly related to the low orders of $d_n^{(1)}(t)$

... and also for higher terms in the trans-series "double scaling limit"

in the vicinity of the phase transition at $t=1$, we can rescale as follows

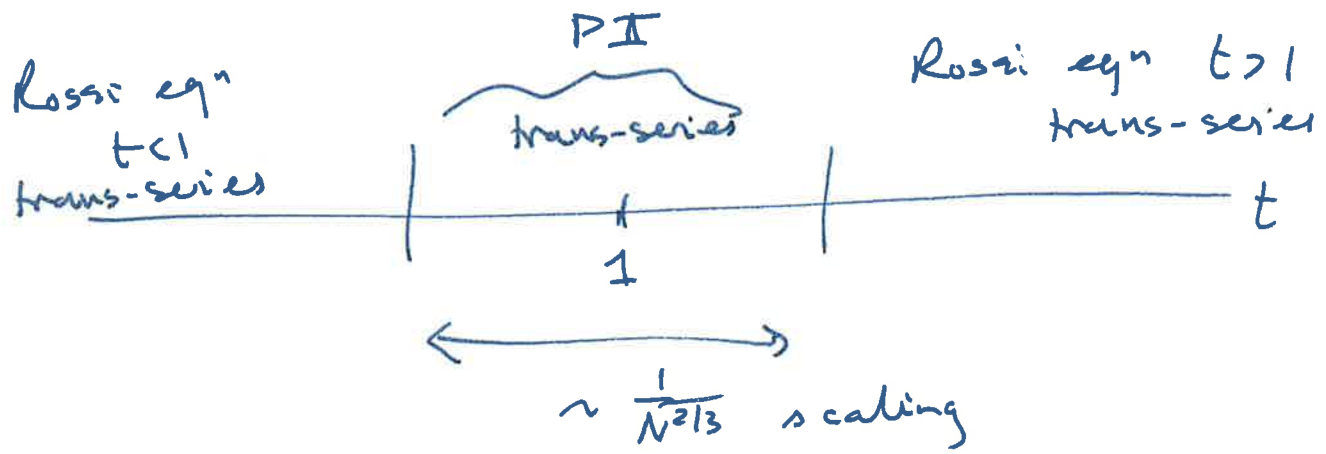
$$t \sim 1 + \frac{x}{(2N^2)^{1/3}}$$

(note: $t>1 \Leftrightarrow x>0$
$t<1 \Leftrightarrow x<0$)

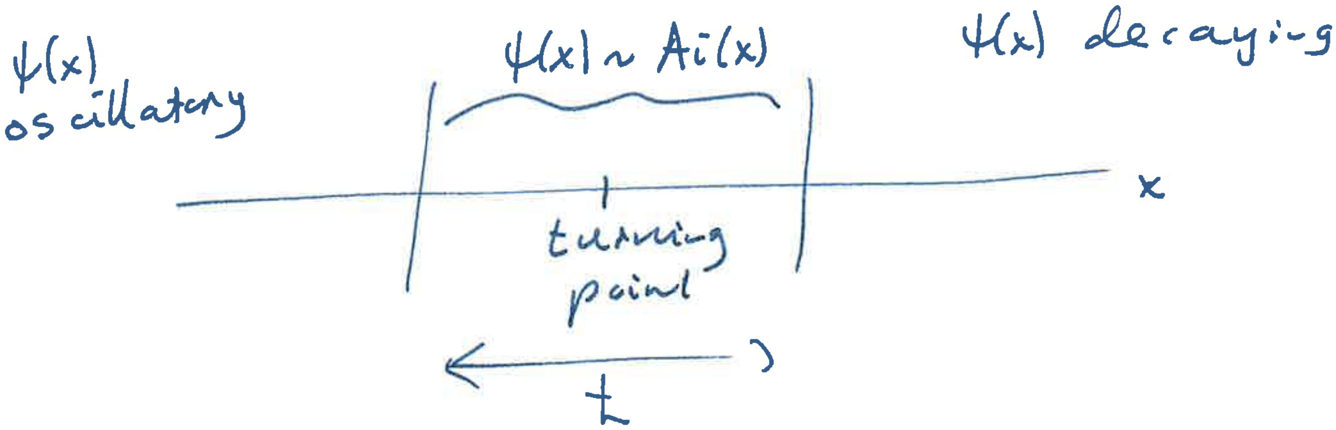
$$\Delta(t,N) = \left(\frac{2t}{N}\right)^{1/3} y(x)$$

then Rossi eq^n $\xrightarrow[N\to\infty]{}$ $\dfrac{d^2y}{dx^2} = x\,y(x) + 2y^3(x)$

Painlevé II



compare with (linear!) Airy function near a turning pt. in QM



• non-linear connection problem!

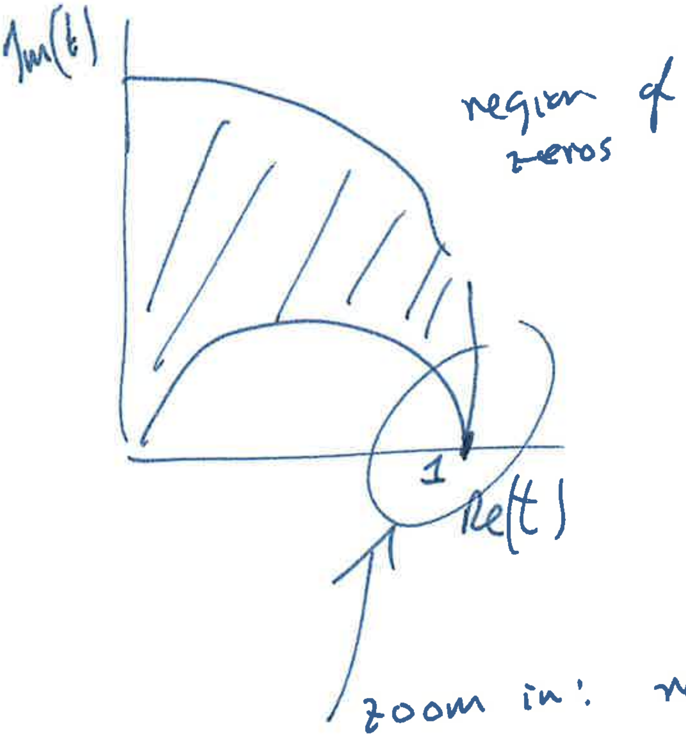
so $P\,\mathrm{III} \to P\,\mathrm{II}$ is "universal"

relation to Lee-Yang zeros

Lee-Yang: at finite N, the partition fn. $Z(t,N)$ has zeros in the complex plane. In the thermodynamic limit these pinch the real axis at the location of the phase transition.

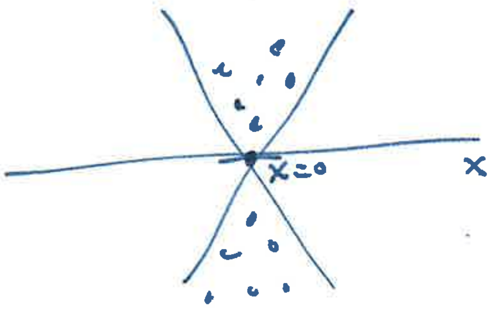
That is exactly what happens here in the GWW model - see slides

(note: zeros of Z $\Rightarrow$ singularities of $\ln Z$)



zoom in: recall $t \sim 1 + \frac{x}{(2N^2)^{1/3}}$

as $N \to \infty$, $t \to 1$ scales to $x \to 0$



poles of Painlevé II Hastings-McLeod solution!