3. Neural Networks f. Inverse Problems.

$f(x) = \phi^{(L)}(x)$, where $\phi^{(L)}(x) \in \mathbb{R}^{n_L}$ is the output of a feed-forward NN.

NNs provide a flexible, unbiased parametrization of f

3.1 Feed-forward NN

NN architecture : layers $l = 0 \dots L$

$l = 0$, $\phi_i^{(0)}(x_\delta) = x_{\delta i}$ input layer, $i = 1 \dots n_0$

$$\phi_{i\delta}^{(l+1)} = \sum_{j=1}^{n_l} w_{ij}^{(l+1)} \rho\left(\phi_{j\delta}^{(l)}\right) + b_i^{(l+1)}, \quad i = 1 \dots n_{l+1}$$

ρ is the activation fn.

$w_{ij}^{(l)}$, $b_i^{(l)}$ are the parameters.

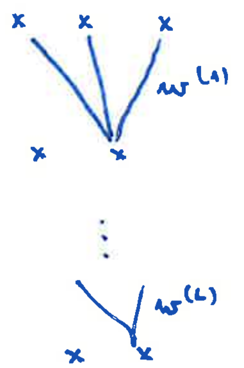
↳ compute the output $\phi_i^{(L)}(x_\delta)$ for all x_δ, $i = 1 \dots n_L$.

$l = 0$ $\qquad$ $i = 1 \dots n_0$

$l = 1$ $\qquad$ $i = 1 \dots n_1$

$\vdots$

$l = L$ $\qquad$ $i = 1 \dots n_L$



$\phi^{(L)}: \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}$

output of neuron $\underline{i}$ in layer l : $\rho\left(\phi_{i\delta}^{(l)}\right) = \rho_{i\delta}^{(l)}$

$\phi^{(l)}$ is called the preactivation fn

$\delta \in \mathcal{D}$: set of data, $x_\delta \in \mathbb{R}^{n_0}$ is the input.

3.2 Statistical Ensembles of NN

$w_{ij}^{(\ell)} \sim \mathcal{N}\left(0, \frac{C_w^{(\ell)}}{n_{\ell-1}}\right)$, $\quad b_i^{(\ell)} \sim \mathcal{N}\left(0, C_b^{(\ell)}\right)$. at initialization.

$$\hookrightarrow \quad \phi_{i\delta}^{(\ell)} = \sum_{j=1}^{n_{\ell-1}} w_{ij}^{(\ell)} \rho_{j\delta}^{(\ell-1)} + b_i^{(\ell)}$$

$\uparrow$ $n_{\ell-1}$ terms in the sum.

Hence $\quad E\left[w_{ij}^{(\ell)}\right] = 0$

$$\mathrm{Cov}\left[w_{i_1 j_1}^{(\ell)}, w_{i_2 j_2}^{(\ell)}\right] = \delta_{i_1 i_2} \delta_{j_1 j_2} \frac{C_w^{(\ell)}}{n_{\ell-1}}$$

$$E\left[b_i^{(\ell)}\right] = 0$$

$$\mathrm{Cov}\left[b_{i_1}^{(\ell)}, b_{i_2}^{(\ell)}\right] = \delta_{i_1 i_2} C_b^{(\ell)}$$

$\phi_{i\delta}^{(\ell)}$, for $\delta \in \mathcal{D}$ is a stochastic variable,

We are interested in the joint prob. distribution

$$p\left(\phi^{(L)} \dots \phi^{(1)} \mid \mathcal{D}\right) =$$

$$= p\left(\phi^{(L)} \mid \phi^{(L-1)}\right) \dots p\left(\phi^{(1)} \mid \mathcal{D}\right)$$

and in particular

$$p\left(\phi^{(\ell+1)} \mid \mathcal{D}\right) = \int d\phi^{(\ell)}\, p\left(\phi^{(\ell+1)} \mid \phi^{(\ell)}\right) p\left(\phi^{(\ell)} \mid \mathcal{D}\right).$$

$\hookrightarrow$ recursion relation for $p\left(\phi^{(\ell)} \mid \mathcal{D}\right)$ (Forward Eq.).

* $p\left(\phi^{(\ell+1)} \mid \phi^{(\ell)}\right)$ is "easy".

$$\phi_{i\delta}^{(\ell+1)} = \sum_{j=1}^{n_\ell} \underset{\text{stoch.}}{w_{ij}^{(\ell+1)}} \underset{\text{fixed}}{\rho_{j\delta}^{(\ell)}} + \underset{\text{stoch.}}{b_i^{(\ell+1)}} \quad \Rightarrow \quad p\left(\phi^{(\ell+1)} \mid \phi^{(\ell)}\right) \text{ is Gaussian}$$

$$p(f^{(\ell+1)}|f^{(\ell)}) = \int dw^{(\ell+1)} db^{(\ell+1)}\, p(w^{(\ell+1)})\, p(b^{(\ell+1)})\, \delta\left(f^{(\ell+1)} - w^{(\ell+1)} \rho^{(\ell)} - b^{(\ell+1)}\right).$$

$$= \frac{1}{|2\pi \hat{G}^{(\ell+1)}|^{n_{\ell+1}/2}} \exp\left\{-\frac{1}{2} f^{(\ell+1)}_{\delta_1} \cdot f^{(\ell+1)}_{\delta_2}\, \hat{G}^{(\ell+1)}(f^{(\ell)})^{-1}_{\delta_1\delta_2}\right\}.$$

$$f^{(\ell+1)} \cdot f^{(\ell+1)}\, \hat{G}^{(\ell+1)}(f^{(\ell)})^{-1} =$$

$$= \sum_{i=1}^{n_{\ell+1}} \sum_{\delta_1,\delta_2} f^{(\ell+1)}_{i\delta_1} f^{(\ell+1)}_{i\delta_2} \left(\hat{G}^{(\ell+1)}(f^{(\ell)})^{-1}\right)_{\delta_1\delta_2}.$$

$$\hat{G}^{(\ell+1)}(f^{(\ell)})_{\delta_1\delta_2} = \frac{C_W^{(\ell+1)}}{n_\ell} \sum_{j=1}^{n_\ell} \rho(f^{(\ell)}_{j\delta_1})\, \rho(f^{(\ell)}_{j\delta_2}) + C_b^{(\ell+1)}$$

$$= O(1) \quad \text{as } n_\ell \to \infty$$

Note : the dependence on $\delta_1 \delta_2$ through $f^{(\ell)}$

$O(n_{\ell+1})$ symmetry under

$$f^{(\ell+1)}_{i\delta} \mapsto R_{ij} f^{(\ell+1)}_{j\delta}$$

- Now let us analyse the recursion

- the first layer is Gaussian.

$$f^{(1)}_{i\delta} = w^{(1)}_{ij}\, \rho(x_{j\delta}) + b^{(1)}_i \quad \to \quad f^{(1)} \text{ is Gaussian.}$$

↑ always fixed (not stochastic)

$$E[f^{(1)}_{i\delta}] = 0$$

$$E[f^{(1)}_{i_1\delta_1} f^{(1)}_{i_2\delta_2}] = \sum_{j_1 j_2} E[w^{(1)}_{i_1 j_1} w^{(1)}_{i_2 j_2}]\, \rho(x_{j_1\delta_1})\, \rho(x_{j_2\delta_2}) + E[b^{(1)}_{i_1} b^{(1)}_{i_2}]$$

$$= \delta_{i_1 i_2} \left\{ \frac{C_W^{(1)}}{n_0} \sum_{j=1}^{n_0} \rho(x_{j\delta_1})\, \rho(x_{j\delta_2}) + C_b^{(1)} \right\}.$$

$$= \delta_{i_1 i_2}\, \hat{G}(f^{(0)})_{\delta_1\delta_2}.$$

- second layer is no longer Gaussian.

$$\phi^{(2)}_{i\delta} = \sum_{j=1}^{n_1} \underbrace{w^{(2)}_{ij}\, \rho(\phi^{(1)}_{j\delta})}_{\text{this product is not a Gaussian var.}} + b^{(2)}_i$$

- similarly for all layers w. $l \geq 2$.

3.3 Effective Action

In the spirit of Statistical Field Theory, we write

$$p(\phi^{(l)}|\mathcal{D}) = \frac{1}{Z^{(l)}}\, e^{-S(\phi^{(l)})}.$$

then the recursion relation becomes

$$\frac{e^{-S(\phi^{(l+1)})}}{Z^{(l+1)}} = \int d\phi^{(l)}\, \frac{e^{-S(\phi^{(l)})}}{Z^{(l)}}\, \frac{e^{-\frac{1}{2}\phi^{(l+1)}\cdot\phi^{(l+1)}\, \hat{G}^{(l+1)}(\phi^{(l)})^{-1}}}{|2\pi\, \hat{G}^{(l+1)}(\phi^{(l)})|^{n_{l+1}/2}}$$

generating fn.

$$Z(\eta^{(l+1)}) = \int d\phi^{(l+1)}\, e^{-S(\phi^{(l+1)}) + i\,\eta^{(l+1)}\cdot\phi^{(l+1)}}$$

$$\eta^{(l+1)}\cdot\phi^{(l+1)} = \sum_{j=1}^{n_{l+1}} \sum_{\delta\in\mathcal{D}} \eta^{(l+1)}_{i\delta}\, \phi^{(l+1)}_{i\delta}$$

As usual

$$\langle \phi^{(l+1)}_{i_1\delta_1} \cdots \phi^{(l+1)}_{i_n\delta_n} \rangle = \frac{1}{Z^{(l+1)}} \left(\frac{1}{i}\frac{\delta}{\delta\eta^{(l+1)}_{i_1\delta_1}}\right) \cdots \left(\frac{1}{i}\frac{\delta}{\delta\eta^{(l+1)}_{i_n\delta_n}}\right) Z(\eta^{(l+1)}) \Big|_{\eta^{(l+1)}=0}$$

$$W(\eta^{(l+1)}) = \log Z(\eta^{(l+1)})$$

$$\langle \phi^{(l+1)}_{i_1\delta_1} \cdots \phi^{(l+1)}_{i_n\delta_n} \rangle_c = \left(\frac{1}{i}\frac{\delta}{\delta\eta^{(l+1)}_{i_1\delta_1}}\right) \cdots\; W(\eta^{(l+1)}) \Big|_{\eta^{(l+1)}=0}$$

$\hookrightarrow$ connected correlators, cumulants.

EFT approach: include in $S(\phi^{(u)})$ <u>all</u> terms that are compatible w. the symmetry of the problem,

i.e. powers of $\phi^{(u)}_{\delta_1} \cdot \phi^{(u)}_{\delta_2}$

Hence:

$$S(\phi^{(u)}) = \frac{1}{2} \gamma^{(2)}_{\delta_1 \delta_2} (\phi^{(u)}_{\delta_1} \cdot \phi^{(u)}_{\delta_2}) + \frac{1}{8} \gamma^{(4)}_{\delta_1 \delta_2, \delta_3 \delta_4} (\phi^{(u)}_{\delta_1} \cdot \phi^{(u)}_{\delta_2})(\phi^{(u)}_{\delta_3} \cdot \phi^{(u)}_{\delta_4}) + \cdots$$

<u>Edgeworth Expansion.</u>

$$e^{W(\eta^{(u)})} = \int d\hat{\phi}^{(u)} \, e^{-S(\hat{\phi}^{(u)}) + i \hat{\phi}^{(u)} \cdot \eta^{(u)}}$$

$$\Rightarrow \int d\eta^{(u)} \, e^{W(\eta^{(u)}) - i \phi^{(u)} \cdot \eta^{(u)}} =$$

$$= \int d\hat{\phi}^{(u)} \, e^{-S(\hat{\phi}^{(u)})} \underbrace{\int d\eta^{(u)} \, e^{i \eta^{(u)} \cdot (\hat{\phi}^{(u)} - \phi^{(u)})}}_{\delta(\hat{\phi}^{(u)} - \phi^{(u)})}$$

$$= e^{-S(\phi^{(u)})}$$

↳ action as a fn. of the <u>cumulants</u>.

$$W(\eta^{(u)}) = \sum_{r=1}^{\infty} \frac{i^r}{r!} \langle \phi^{(u)}_{i_1 \delta_1} \cdots \phi^{(u)}_{i_r \delta_r} \rangle_c \, \eta^{(u)}_{i_1 \delta_1} \cdots \eta^{(u)}_{i_r \delta_r} .$$

$$\Rightarrow \int d\eta^{(u)} \, e^{W(\eta^{(u)}) - i \phi^{(u)} \cdot \eta^{(u)}} =$$

$$= \exp\left\{ \sum_{r=3}^{\infty} \frac{i^r}{r!} \langle \phi^{(u)}_{i_1 \delta_1} \cdots \phi^{(u)}_{i_r \delta_r} \rangle_c \left(i \frac{\partial}{\partial \phi^{(u)}_{i_1 \delta_1}} \right) \cdots \left(i \frac{\partial}{\partial \phi^{(u)}_{i_r \delta_r}} \right) \right\} \times$$

$$\times \int d\eta^{(u)} \exp\left\{ -\frac{1}{2} \langle \phi^{(u)}_{i_1 \delta_1} \phi^{(u)}_{i_2 \delta_2} \rangle_c \, \eta^{(u)}_{i_1 \delta_1} \eta^{(u)}_{i_2 \delta_2} + \right.$$

$$\left. + i \left(\langle \phi^{(u)}_{i \delta} \rangle - \phi^{(u)}_{i \delta} \right) \eta^{(u)}_{i \delta} \right\} .$$

$$= e^{-S(\phi^{(u)})} .$$

3.4 "Cumulants Calculations"

- $\langle \phi^{(l)}_{i\delta} \rangle = 0$ for symmetry reasons.

- $\langle \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2} \rangle_c = \langle \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2} \rangle$.

$$= \int d\phi^{(l+1)} \frac{e^{-S(\phi^{(l+1)})}}{Z^{(l+1)}} \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2}$$

$$= \int d\phi^{(l)} \frac{e^{-S(\phi^{(l)})}}{Z^{(l)}} \underbrace{\int d\phi^{(l+1)} \frac{e^{-\frac{1}{2}\phi^{(l+1)}\cdot\phi^{(l+1)}\hat{G}^{(l+1)-1}}}{|2\pi\hat{G}^{(l+1)}|^{n_{l+1}/2}} \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2}}_{\text{Wick's thm}}$$

$$= \int d\phi^{(l)} \frac{e^{-S(\phi^{(l)})}}{Z^{(l)}} \left[\delta_{i_1 i_2} \hat{G}^{(l+1)}(\phi^{(l)})_{\delta_1\delta_2} \right].$$

$$= \delta_{i_1 i_2} E_{\phi^{(l)}}\left[\hat{G}^{(l+1)}(\phi^{(l)})_{\delta_1\delta_2} \right].$$

$$= \delta_{i_1 i_2} \left\{ \underbrace{\frac{C_W^{(l+1)}}{n_l} \sum_{j=1}^{n_l} E_{\phi^{(l)}}\left[\rho^{(l)}_{j\delta_1} \rho^{(l)}_{j\delta_2} \right]}_{O(1) \text{ in powers of } \frac{1}{n_l}\,!} + C_b^{(l+1)} \right\}$$

$$= \delta_{i_1 i_2} G^{(l+1)}_{\delta_1\delta_2}.$$

- $\langle \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2} \phi^{(l+1)}_{i_3\delta_3} \phi^{(l+1)}_{i_4\delta_4} \rangle_c = \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle - \langle \phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle -$

$- \langle \phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle - \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle$.

$$(*) = \int d\phi^{(l)} \frac{e^{-S(\phi^{(l)})}}{Z^{(l)}} \int d\phi^{(l+1)} \frac{e^{-\frac{1}{2}\phi^{(l+1)}\cdot\phi^{(l+1)}(\hat{G}^{(l+1)})^{-1}}}{|2\pi\hat{G}^{(l+1)}|^{n_{l+1}/2}} \phi^{(l+1)}_{i_1\delta_1} \phi^{(l+1)}_{i_2\delta_2} \phi^{(l+1)}_{i_3\delta_3} \phi^{(l+1)}_{i_4\delta_4}$$

↳ computed using Wick's thm.

$$(\ldots) = \int d\phi^{(\ell)} \frac{e^{-S(\phi^{(\ell)})}}{Z^{(\ell)}} \left\{ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \right\}$$

$$\{\cdots\} = \delta_{i_1 i_2}\delta_{i_3 i_4}\, \hat{G}^{(\ell+1)}(\phi^{(\ell)})_{\delta_1\delta_2}\, \hat{G}^{(\ell+1)}(\phi^{(\ell)})_{\delta_3\delta_4} +$$

$$+ \delta_{i_1 i_3}\delta_{i_2 i_4} \ \cdots$$

$$+ \delta_{i_1 i_4}\delta_{i_2 i_3} \ \cdots$$

Each of these contribution gets a subtraction from the disconnected term, e.g.

(2) $\langle \phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle = \delta_{i_1 i_2}\delta_{i_3 i_4}\, G^{(\ell+1)}_{\delta_1\delta_2}\, G^{(\ell+1)}_{\delta_3\delta_4}$.

Contribution to $\langle\phi_1\phi_2\phi_3\phi_4\rangle_c$, proportional to $\delta_{i_1 i_2}\delta_{i_3 i_4}$

$$E_{\phi^{(\ell)}}\left[\hat{G}^{(\ell+1)}(\phi^{(\ell)})_{\delta_1\delta_2}\, \hat{G}^{(\ell+1)}_{i_1 i_2}(\phi^{(\ell)})_{\delta_3\delta_4}\right] - G^{(\ell+1)}_{\delta_1\delta_2}\, G^{(\ell+1)}_{\delta_3\delta_4} = E_{\delta_1\delta_2,\delta_3\delta_4}$$

Exercise : set $C_b^{(\ell+1)} = 0$ and show that

$\langle\phi_1\cdots\phi_4\rangle_c = O\left(\frac{1}{n_\ell}\right)$! see (*)

3.5 Couplings of the EFT

From EE : $\gamma^{(\ell)}_{\delta_1\delta_2} = \left(G^{(\ell)}\right)^{-1}_{\delta_1\delta_2} + O\left(\frac{1}{n_{\ell-1}}\right)$

$$\gamma^{(\ell)}_{\delta_1\delta_2,\delta_3\delta_4} = E^{(\ell)}_{\delta_1\delta_2,\delta_3\delta_4} = \frac{1}{n_{\ell-1}} V_{\delta_1\delta_2,\delta_3\delta_4}.$$

$\vdots$

higher order couplings $\phi^6, \phi^8 \ldots$ are suppressed by higher powers of $\frac{1}{n_\ell}$

For wide NNs, $n_\ell \to \infty$, there emerges a hierarchy of couplings.

(*) Conjecture.

$$C(x_1 \dots x_m) = \sum_{\mu_1 \dots \mu_{k_m}} \Delta^{(\pi)}_{\mu_1 \dots \mu_{k_m}} E\left[T_{\mu_1 \dots \mu_{k_1}}(x_1) \dots T_{\mu_{k_{m-1}+1} \dots \mu_{k_m}}(x_m) \right].$$

$$\begin{cases} \Delta^{(\pi)}_{\mu_1 \dots \mu_{k_m}} = \delta_{\mu_{\pi(1)} \mu_{\pi(2)}} \cdots \delta_{\mu_{\pi(k_m-1)} \mu_{\pi(k_m)}} \\ T_{\mu_1 \dots \mu_k}(x) = \dfrac{\partial^k}{\partial \theta^{\mu_1} \dots \partial \theta^{\mu_k}} f(x) \end{cases}$$

$G_c(V,E)$ is a graph w. vertices $V = \{v_1 \dots v_m\}$

and edges $E = \{ (v_i, v_j) \mid T(x_i) T(x_j) \text{ contracted} \}$.

G_c has n_e connected components w. even # of vertices

n_o " " odd "

Then $C(x_1 \dots x_n) = O(n^{s_c})$

$$s_c = n_e + \frac{n_o}{2} - \frac{m}{2}$$

$\Rightarrow \langle f_{i_1 \delta_1} \cdots f_{i_{2n} \delta_{2n}} \rangle \sim O(1), \quad \forall n$

Not true for the connected correlator (aka cumulants).

At leading order, $O(1)$, $\phi^{(a)}_{is}$ is a Gaussian Process!

3.6 Generating Functional at $O(1/n)$

Dropping all indices.

$$S(\phi) = \frac{1}{2}\gamma(\phi\cdot\phi) - \frac{1}{8n}v(\phi\cdot\phi)(\phi\cdot\phi)$$

use the identity

$$1 = \frac{1}{Z_\sigma}\int d\sigma\, e^{-\frac{n}{2}v^{-1}\left(\sigma+\frac{1}{2n}v(\phi\cdot\phi)\right)\left(\sigma+\frac{1}{2n}v(\phi\cdot\phi)\right)} \qquad : (23.1)$$

$$\Rightarrow Z(\eta) = \int d\phi\, e^{-S(\phi)+i\eta\cdot\phi}$$

$$= \int d\phi\, d\sigma\, \frac{e^{-S(\sigma,\phi)+i\eta\cdot\phi}}{Z_\sigma}$$

where $S(\sigma,\phi) = S(\phi) + \frac{n}{2}v^{-1}(\ldots)(\ldots)$ as in Eq. (23.1).

$S(\sigma,\phi)$ is quadratic in ϕ, the quartic term is exactly cancelled by the exponent of (23.1).

$$S(\sigma,\phi) \supset \frac{1}{2}(\gamma+\sigma)(\phi\cdot\phi).$$

Perform the Gaussian integral, yields

$$Z(\eta) = \int d\sigma\, \frac{\left|2\pi(\gamma+\sigma)^{-1}\right|^{n/2}}{Z_\sigma}\exp\left\{-\frac{1}{2}(\gamma+\sigma)^{-1}(\eta\cdot\eta) - \frac{n}{2}v^{-1}\sigma\sigma\right\}$$

$$= \frac{1}{Z_\sigma}\int d\sigma \exp\left\{-\frac{1}{2}(\gamma+\sigma)^{-1}(\eta\cdot\eta) - \mathcal{S}(\sigma)\right\}$$

$$\mathcal{S}(\sigma) = \frac{n}{2}\left[v^{-1}\sigma\sigma + \mathrm{tr}\log(\gamma+\sigma)\right]$$

for large-n, saddle-pt exp. around $\bar\sigma$:

$$\left.\frac{\partial\mathcal{S}}{\partial\sigma}\right|_{\bar\sigma} = 0$$

Set $\sigma = \bar\sigma + \sigma'$

$$\mathrm{tr}\log(\gamma+\bar\sigma+\sigma') = \mathrm{tr}\log(\gamma+\bar\sigma) + \mathrm{tr}\log(1+\bar g\sigma')$$

(where $\bar g^{-1} = \gamma+\bar\sigma$

$$= \mathrm{tr}\log(\gamma+\bar\sigma) + \mathrm{tr}\,\bar g\sigma' - \frac12\,\mathrm{tr}\,\bar g\sigma'\bar g\sigma' + O(\sigma'^3).$$

cancel when expanding around $\bar\sigma$

$$S(\bar\sigma+\sigma') = S(\bar\sigma) + \frac n2\, v^{-1}\sigma'\sigma' - \frac n4\,\mathrm{tr}\,\bar g\sigma'\bar g\sigma'$$

$$= S(\bar\sigma) + \frac n2\,\bar v^{-1}\sigma'\sigma'$$

$$\mathrm{tr}\,\bar g\sigma'\bar g\sigma' = \bar g_{\delta_4\delta_1}\bar g_{\delta_2\delta_3}\sigma'_{\delta_1\delta_2}\sigma'_{\delta_3\delta_4}$$

$$\Rightarrow\ \bar v^{-1}_{\delta_1\delta_2;\delta_3\delta_4} = v^{-1}_{\delta_1\delta_2,\delta_3\delta_4} - \frac12\,\bar g_{\delta_4\delta_1}\bar g_{\delta_2\delta_3}.$$

$$Z(\eta) = \frac{1}{Z_\sigma}\int d\sigma'\, e^{-\frac12(\eta\cdot\eta)(\bar g^{-1}+\sigma')^{-1} - S(\bar\sigma) - \frac n2\bar v^{-1}\sigma'\sigma' + O(\sigma'^3)}$$

$$= \frac{1}{Z_\sigma}\exp\left\{-\frac12(\eta\cdot\eta)\,\bar g\left(1+\frac{\partial}{\partial J}\bar g\right)^{-1} + O\left(\frac{\partial^3}{\partial J^3}\right)\right\} e^{-\frac{1}{2n}\bar v(J\cdot J)}\Big|_{J=0}$$

where we introduced a source term for σ', called J.

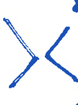
$$= e^{-\frac12\bar g(\eta\cdot\eta)}\left[1 - \frac12(\eta\cdot\eta)\,\bar g\frac{\partial}{\partial J}\bar g\frac{\partial}{\partial J}\bar g + \frac18\left((\eta\cdot\eta)\,\bar g\frac{\partial}{\partial J}\bar g\right)^2\right] e^{-\frac{1}{2n}\bar v(J\cdot J)}\Big|_{J=0}$$

$$+\, O\left(\frac{1}{n^2}\right)$$

$= e^{-\frac12}$  $\Big[\,1 - \frac12$  $+ \frac18$  $\Big] + O\left(\frac{1}{n^2}\right)$

• η_δ

δ_1 δ_2 $\quad \bar g_{\delta_1\delta_2}$

$\delta_2 \ \delta_4 \ \delta_1 \ \delta_3$  $= \frac1n\,\bar v_{\delta_1\delta_2,\delta_3\delta_4}$

→ Feynman diagrams repr. of correlations.

Summarizing

$$Z(\eta) = e^{-\frac{1}{2}\,\bullet\!\!-\!\!\bullet}\left[1 - \frac{1}{2}\,\Omega + \frac{1}{8}\,\rangle\!\langle\,\right] + O\!\left(\frac{1}{n^2}\right).$$

$\bullet \quad \eta_\delta$

$\delta_1 \,\frac{\qquad}{}\, \delta_2 \qquad \bar{g}_{\delta_1\delta_2} = \left(\gamma^{(2)} - 1\right)_{\delta_1\delta_2}$

$\begin{matrix}\delta_2 & & \delta_4 \\ & \rangle\!\langle & \\ \delta_1 & & \delta_3\end{matrix} \qquad \frac{1}{n}\,\bar{v}_{\delta_1\delta_2,\delta_3\delta_4}$

$$\langle \phi_{i_1\delta_1}\,\phi_{i_2\delta_2}\rangle = -\frac{\delta}{\delta\eta_{i_1\delta_1}}\frac{\delta}{\delta\eta_{i_2\delta_2}} Z(\eta)\Big|_{\eta=0}$$

$$= -\frac{\delta}{\delta\eta_{i_1\delta_1}}\frac{\delta}{\delta\eta_{i_2\delta_2}}\left(1 - \frac{1}{2}\,\bullet\!\!-\!\!\bullet + \dots\right)\left(1 - \frac{1}{2}\,\Omega + \dots\right)$$

$$= -\!\!-\!\!- + \Omega + O\!\left(\frac{1}{n^2}\right)$$

$$= \delta_{i_1i_2}\left[\bar{g}_{\delta_1\delta_2} + g_{\delta_1\eta_1}\, g_{\eta_2\eta_3}\, g_{\eta_4\delta_2}\,\frac{\bar{v}_{\eta_1\eta_2,\eta_3\eta_4}}{n} + O\!\left(\frac{1}{n^2}\right)\right]$$

$$\langle \phi_{i_1\delta_1}\cdots\phi_{i_4\delta_4}\rangle = \frac{\delta}{\delta\eta_{i_1\delta_1}}\cdots\frac{\delta}{\delta\eta_{i_4\delta_4}} Z(\eta)\Big|_{\eta=0}$$

$$= \frac{\delta}{\delta\eta_1}\cdots\frac{\delta}{\delta\eta_4}\left(1 - \frac{1}{2}\,-\!\!- + \frac{1}{8}\,=\right)\left(1 - \frac{1}{2}\,\Omega + \frac{1}{8}\,\rangle\!\langle\right)$$

$$= \frac{\delta}{\delta\eta_1}\cdots\frac{\delta}{\delta\eta_4}\left\{\frac{1}{8}\,= + \frac{1}{4}\,\underline{\Omega} + \frac{1}{8}\,\rangle\!\langle\right\} + O\!\left(\frac{1}{n^2}\right)$$

$$= \delta_{i_1i_2}\,\delta_{i_3i_4}\left\{\bar{g}_{\delta_1\delta_2}\,\bar{g}_{\delta_3\delta_4} + g_{\delta_1\eta_1}\, g_{\eta_2\eta_3}\, g_{\eta_4\delta_2}\,\frac{\bar{v}_{\eta_1\eta_2,\eta_3\eta_4}}{n}\, g_{\delta_3\delta_4} + \dots\right\} + \dots$$