Introductory Lectures on Resurgence

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Continuum Foundations of Lattice Gauge Theories

1. - Stokes Phenomenon and Trans-series
   - Borel Summation basics
   - Recovering Non-perturbative Connection Formulas
2. - Nonlinear Stokes Phenomenon
   - Parametric Resurgence & Phase Transitions
   - Gross-Witten-Wadia unitary matrix model
3. - QFT: Euler-Heisenberg and Effective Field Theory
   - Resurgence analysis
   - Inhomogeneous fields
4. - Resurgent Extrapolation
   - The Physics of Padé Approximation
   - Probing the Borel Plane Numerically

- the first (non-perturbative) QFT computation

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

($\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.

$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2}$ = „Kritische Feldstärke“.)

- this is a Borel integral

Euler's thesis 1936: the first effective field theory

- constant E and B fields:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -a\eta \operatorname{ctg} a\eta \cdot b\eta \operatorname{Ctg} b\eta + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\}$$

- nonlinear photon-photon interaction

$$L = \frac{\mathfrak{E}^2 - \mathfrak{B}^2}{2} + \frac{1}{90\pi} \frac{\hbar c}{e^2} \frac{1}{E_0^2} [(\mathfrak{E}^2 - \mathfrak{B}^2)^2 + 7(\mathfrak{E}\mathfrak{B})^2]$$

- philosophy of effective field theory:

One thus expects that along with the Maxwellian energy of the individual light quanta there is a mutual interaction between the light quanta of the form:

$$\bar{U}_1 = \frac{\hbar c}{e^2} \frac{1}{E_0^2} \int \left[FFFF + \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) FF + \cdots \right] dV \,. \tag{1.3}$$

Dyson's Argument

$$F(e^2) = a_0 + a_2\, e^2 + a_4\, e^4 + \ldots \tag{2}$$

Alternative B: All the information that can in principle be obtained from the formalism of quantum electrodynamics is contained in the coefficients a_0, a_2, a_4, *..., of series such as (2). In this case the quantity* $F(e^2)$ *is neither physically well-defined nor mathematically calculable, except in so far as the asymptotic expansion (2) gives some workable approximation to it.*

Exercise 3.1: Analytically continue the integral representation of the Hurwitz zeta function dlmf.25.11.vii ($Re(s) > 1$)

$$\zeta_H(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, e^{-z\,t} \frac{t^{s-1}}{1-e^{-t}}$$

into the region $Re(s) > -2$ to obtain

$$\begin{aligned} \zeta_H(s,z) &= \frac{z^{1-s}}{s-1} + \frac{z^{-s}}{2} + \frac{s\,z^{-1-s}}{12} \\ &\quad + \frac{2^{s-1}}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t^{1-s}}\, e^{-2z\,t} \left(\coth(t) - \frac{1}{t} - \frac{t}{3}\right) \end{aligned}$$

Hence show that $\zeta_H(-1,z) = -\frac{1}{12} + \frac{z}{2} - \frac{z^2}{2}$, and

$$\begin{aligned} \zeta_H'(-1,z) &= \frac{1}{12} - \frac{z^2}{4} - \zeta_H(-1,z)\,\ln z \\ &\quad - \frac{1}{4} \int_0^{\infty} \frac{dt}{t^2}\, e^{-2z\,t} \left(\coth(t) - \frac{1}{t} - \frac{t}{3}\right) \end{aligned}$$

Exercise 3.2: In the zeta function method we define

$$\ln \det(\text{operator}) := -\zeta'(0) \qquad \text{where} \qquad \zeta(s) := \sum_{\text{spectrum}\,\lambda} \frac{1}{\lambda^s}$$

Given that the eigenvalues of the Dirac operator in a constant magnetic field B are given by the Landau level result

$$\lambda_n^{\pm} = m^2 + p_\perp^2 + eB(2n+1\pm 1) \qquad , \qquad n = 0, 1, 2, \ldots$$

with the Landau degeneracy factor $\frac{eB}{2\pi}$, derive the Euler-Heisenberg effective action $\mathcal{L}$ by showing that

$$\begin{aligned}\mathcal{L} \quad &= \quad \frac{e^2B^2}{2\pi^2}\left\{\zeta_H'\left(-1, \frac{m^2}{2eB}\right) + \zeta_H\left(-1, \frac{m^2}{2eB}\right)\ln\left(\frac{m^2}{2eB}\right)\right. \\ &\qquad \left. -\frac{1}{12} + \frac{1}{4}\left(\frac{m^2}{2eB}\right)^2\right\}\end{aligned}$$

Exercise 3.3:

1. Show that the Euler-Heisenberg effective action can be expressed in terms of the log of the Barnes gamma function (https://dlmf.nist.gov/5.17)

$$\begin{aligned}\mathcal{L}(b) \quad = \quad & \frac{b^2}{2\pi^2}\left[-\log(A)+\frac{1}{16b^2}+\left(-\frac{1}{8b^2}+\frac{1}{4b}-\frac{1}{12}\right)\log\left(\frac{1}{2b}\right)\right. \\ & \left.-\log\left(G\left(\frac{1}{2b}\right)\right)-\left(1-\frac{1}{2b}\right)\log\left(\Gamma\left(\frac{1}{2b}\right)\right)\right]\end{aligned}$$

   where $b \equiv \frac{eB}{m^2}$.

2. Hence study the small b and large b expansions, showing that the strong field expansion has a finite radius of convergence.

Exercise 3.4: For scalar QED the spectrum of the Klein-Gordon operator in a constant B field has no spin projection term, so it is given by

$$\lambda_n = m^2 + p_\perp^2 + eB(2n+1) \qquad , \qquad n = 0, 1, 2, \ldots$$

1. Hence use the zeta function method to show that the EH effective action for scalar QED has the integral representation

$$\mathcal{L}_{\text{scalar}} = \frac{e^2 B^2}{16\pi^2} \int_0^\infty ds\, e^{-m^2 s/(eB)} \frac{1}{s^2} \left(\frac{1}{\sinh(s)} - \frac{1}{s} + \frac{s}{6} \right)$$

2. Generate the asymptotic weak field expansion of the scalar QED effective action and comparee the large-order behavior of the expansion coefficients with the spinor QED case.

3. Compute the leading strong-field behavior by inspection of the Borel integral representation, and relate this to the scalar QED beta function.

Effective Action in Monochromatic Electric Field: Brézin/Itzykson (1970); Popov/Marinov (1971)

- monochromatic field: $\mathcal{E}(t) = \mathcal{E}\,\cos(\omega\,t)$
- effective action: $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}, \gamma)$
- Keldysh inhomogeneity parameter $\gamma = \frac{m\,c\,\omega}{e\,\mathcal{E}} \sim \frac{mc}{eA}$
- semiclassical imaginary part: (recall $\mathcal{E}_{\rm cr} = \frac{m^2 c^3}{e\,\hbar}$)

$$\mathrm{Im}\,\Gamma \approx e^{-\frac{\mathcal{E}_{\rm cr}}{\mathcal{E}} g(\gamma)} \sim \begin{cases} e^{-\frac{\pi \mathcal{E}_{\rm cr}}{\mathcal{E}}} & , \quad \gamma \ll 1 \\ e^{-\frac{\pi \mathcal{E}_{\rm cr}}{\mathcal{E}} \frac{4}{\pi\gamma}\,\ln\gamma} = \left(\frac{eA}{mc}\right)^{\frac{4mc^2}{\hbar\,\omega}} & , \quad \gamma \gg 1 \end{cases}$$

- Stokes transition: tunneling $\leftrightarrow$ multi-photon

Exercise 3.5: Consider an electric field, directed in the z direction, with a one-dimensional cosine inhomogeneity: $E(t) = E\cos(\omega t)$.

1. Compute the LCFA effective action by integrating the Euler-Heisenberg effective action over one period of the field, with the constant field replaced by its time-dependent form.
2. Show that the coefficients of the weak field expansion grow factorially with perturbative order, and with subleading corrections of both power-law and exponential form. Compute the first few power-law correction terms.
3. Demonstrate the resurgence relation by showing that the power-law corrections in the previous part are related to the fluctuations about the instanton factors for the imaginary part of the effective action.

Inhomogeneous Fields: further divergence

- EFT expansion grows rapidly: one of <u>many</u> pages at 6th order

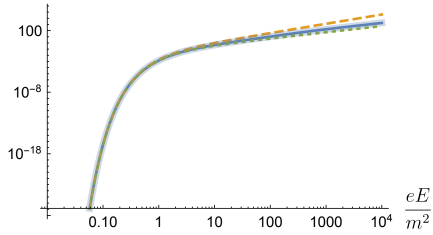
$$
\begin{aligned}
&-\frac{89}{693}F_{\kappa\lambda}F_{\mu\nu\rho}F_{\sigma\rho\lambda}F_{\mu\kappa\sigma\nu}-\frac{89}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu\rho}F_{\mu\rho\sigma}F_{\sigma\nu}-\frac{89}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\nu\sigma}F_{\rho\sigma\mu}\\
&-\frac{94}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\kappa\sigma}F_{\rho\nu}F_{\sigma\mu}+\frac{97}{3080}F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\kappa}F_{\rho\sigma}F_{\nu\mu}F_{\sigma\rho}+\frac{100}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\rho\nu}F_{\sigma\mu\kappa}\\
&+\frac{101}{1386}F_{\kappa\lambda\mu}F_{\nu\mu\lambda}F_{\kappa\rho\sigma}F_{\nu\sigma\rho}-\frac{101}{6930}F_{\kappa\lambda\mu}F_{\mu\nu\rho}F_{\rho\lambda\sigma}F_{\sigma\nu\kappa}-\frac{101}{6930}F_{\kappa\lambda\mu}F_{\nu\kappa\rho}F_{\sigma\nu\mu}F_{\lambda\sigma\rho}\\
&+\frac{101}{9240}F_{\kappa\lambda\mu}F_{\nu\rho\sigma}F_{\kappa\mu\lambda}F_{\nu\sigma\rho}-\frac{103}{630}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\kappa}F_{\mu\rho\sigma}F_{\sigma\rho}-\frac{103}{693}F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\mu\sigma\rho\kappa}F_{\sigma\nu}\\
&+\frac{103}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu\rho}F_{\rho\sigma}F_{\kappa\sigma\nu}-\frac{103}{1386}F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\nu\rho}F_{\rho\sigma}F_{\sigma\mu}-\frac{103}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma}F_{\mu\nu\kappa}F_{\sigma\rho}\\
&-\frac{103}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\nu\mu}F_{\rho\sigma\kappa}-\frac{109}{1260}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\sigma\rho}F_{\kappa\nu\mu}+\frac{109}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\rho\nu}F_{\sigma\mu\kappa}\\
&-\frac{115}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\kappa\sigma}F_{\rho\mu}F_{\sigma\nu}-\frac{122}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\nu\sigma}F_{\rho\kappa}F_{\sigma\mu}-\frac{122}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\rho\nu\sigma}F_{\rho\sigma\kappa}\\
&-\frac{124}{3465}F_{\kappa\lambda}F_{\mu\nu\rho}F_{\rho\mu\lambda\sigma}F_{\sigma\nu\kappa}-\frac{128}{693}F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\mu\kappa}F_{\rho\sigma}F_{\sigma\nu}+\frac{128}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\kappa\rho}F_{\mu\nu\sigma}F_{\sigma\rho}\\
&-\frac{128}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\sigma}F_{\kappa\sigma\nu}F_{\rho\mu}+\frac{130}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\kappa\rho}F_{\nu\sigma}F_{\mu\sigma\rho}-\frac{134}{3465}F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\kappa\rho\sigma}F_{\nu\sigma\rho}\\
&+\frac{151}{13860}F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\sigma}F_{\nu\mu}F_{\sigma\rho}+\frac{152}{3465}F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma\nu\mu}F_{\sigma\rho\lambda\kappa}+\frac{157}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\rho}F_{\mu\sigma}F_{\sigma\rho\kappa}\\
&-\frac{163}{1155}F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\mu\sigma}F_{\sigma\rho\nu\kappa}-\frac{163}{1155}F_{\kappa\lambda}F_{\mu\nu\lambda\rho}F_{\nu\mu\sigma}F_{\sigma\rho\kappa}-\frac{163}{3465}F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\rho\mu\kappa\sigma}F_{\sigma\nu}\\
&+\frac{163}{5544}F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\sigma}F_{\sigma\rho}F_{\nu\mu}+\frac{164}{3465}F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\kappa\sigma}F_{\rho\mu}F_{\sigma\nu}-\frac{166}{3465}F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\mu\rho\sigma}F_{\sigma\nu\kappa}\\
&-\frac{166}{3465}F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\nu\sigma}F_{\sigma\rho\mu\kappa}-\frac{166}{3465}F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\mu\sigma}F_{\rho\kappa\sigma\nu}-\frac{166}{3465}F_{\kappa\lambda}F_{\mu\nu\lambda\rho}F_{\nu\mu\sigma}F_{\kappa\sigma\rho}\\
&+\frac{169}{6930}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\nu\kappa}F_{\mu\sigma\rho}-\frac{178}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho\sigma}F_{\mu\kappa}F_{\nu\sigma\rho}+\frac{179}{1386}F_{\kappa\lambda}F_{\mu\nu\lambda\kappa}F_{\nu\rho\sigma}F_{\mu\sigma\rho}\\
&+\frac{181}{2772}F_{\kappa\lambda}F_{\mu\lambda\kappa}F_{\nu\rho\sigma}F_{\nu\mu\sigma\rho}+\frac{181}{2772}F_{\kappa\lambda}F_{\mu\nu\rho\sigma}F_{\nu\sigma\rho}F_{\mu\lambda\kappa}+\frac{212}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\kappa\rho}F_{\sigma\nu\mu}F_{\sigma\rho}\\
&+\frac{218}{3465}F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\mu\rho}F_{\rho\sigma}F_{\sigma\nu}+\frac{218}{3465}F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\kappa\rho}F_{\nu\sigma}F_{\sigma\rho}-\frac{218}{3465}F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho\sigma}F_{\nu\mu\kappa\sigma\rho}\\
&-\frac{218}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\rho\sigma}F_{\sigma\kappa}F_{\rho\mu}+\frac{221}{1980}F_{\kappa\lambda}F_{\mu\nu\rho}F_{\sigma\rho\nu}F_{\sigma\mu\lambda\kappa}+\frac{227}{6930}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu\rho\sigma}F_{\nu\kappa}F_{\sigma\rho}\\
&-\frac{233}{83160}F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma}F_{\lambda\kappa}F_{\nu\mu}F_{\sigma\rho}-\frac{235}{2772}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\kappa}F_{\nu\rho\sigma}F_{\sigma\rho}F_{\nu\mu}-\frac{236}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\nu\kappa}F_{\rho\sigma\mu}\\
&+\frac{256}{3465}F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\nu\rho}F_{\mu\sigma}F_{\sigma\rho}-\frac{262}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\mu\rho}F_{\nu\kappa\sigma}F_{\sigma\rho}+\frac{263}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\rho\kappa}F_{\sigma\nu\mu}\\
&+\frac{269}{3465}F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho\kappa\sigma}F_{\rho\sigma\nu}+\frac{269}{3465}F_{\kappa\lambda}F_{\mu\nu\rho}F_{\mu\sigma\rho\lambda}F_{\sigma\nu\kappa}-\frac{274}{3465}F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma\mu\kappa}F_{\sigma\rho\nu}\\
&-\frac{284}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\kappa}F_{\nu\rho\sigma}F_{\nu\sigma\rho}-\frac{284}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma}F_{\mu\kappa}F_{\nu\sigma\rho}-\frac{289}{2310}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho\sigma}F_{\nu\kappa}F_{\sigma\rho}\\
&-\frac{326}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho}F_{\nu\kappa\sigma}F_{\sigma\rho}+\frac{331}{6930}F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma\lambda\kappa}F_{\sigma\rho\nu\mu}+\frac{349}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\rho\mu\sigma}F_{\kappa\sigma\nu}\\
&-\frac{349}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\mu\kappa}F_{\nu\rho\sigma}F_{\sigma\rho}-\frac{356}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho}F_{\sigma\nu\kappa}F_{\sigma\rho\mu}+\frac{358}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\rho\mu\sigma}F_{\sigma\nu\kappa}\\
&-\frac{368}{3465}F_{\kappa\lambda}F_{\mu\nu\rho}F_{\lambda\sigma\rho\nu}F_{\kappa\sigma\mu}-\frac{376}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu\rho}F_{\rho\sigma}F_{\mu\sigma\nu}-\frac{397}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\sigma\nu}F_{\kappa\rho\mu}
\end{aligned}
$$

Fliegner et al, hep-th/9707189

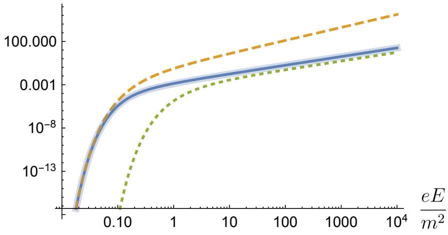
Resurgent Extrapolation for Inhomogeneous Background Fields

- analytic continuation: $B \to i\,E$ and $\lambda \to i\,\tau$
- weak B field to strong E field (+ strong inhomogeneity)
- input: just 15 perturbative input terms

$\mathrm{Im}\,S(E, \gamma = 0.1)$



$\mathrm{Im}\,S(E, \gamma = 10)$



- accurate agreement over many orders of magnitude
- far superior to WKB or LCFA

Exercise 3.6: Consider the classical Euclidean equations of motion for scalar QED in a (generally inhomogeneous) background electromagnetic field :

$$\ddot{x}_\mu = 2ieF_{\mu\nu}(x)\,\dot{x}_\nu$$

where the dots refers to derivatives wrt the proper-time and x_μ is the 4 dim spacetime coordinate.

1. Show that for any solution $\dot{x}_{\rm cl}^2$ is a constant of motion.
2. Show that the closed trajectory (with period T) for a constant E field is a circle, and evaluate the classical action.
3. Show that for a time dependent (but spatially constant) linearly polarized electric field, with Euclidean vector potential $A_3 = -i\frac{E}{\omega}\,f(\omega\,x_4)$, where ω is a frequency scale parameter, the classical action can be expressed as

$$S[x_{\rm cl}](T) = -\frac{\dot{x}_{\rm cl}^2}{4}\,T + \frac{1}{2}\int_0^T d\tau\,(\dot{x}_4^{\rm cl})^2$$

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$$\mathcal{L}_{\mathrm{EH}} = \sum_{l=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^l \mathcal{L}_{\mathrm{EH}}^{(l)}$$

- 1-particle irreducible strong-field limit (Ritus)

$$\mathcal{L}^{(1)} \sim \frac{B^2}{2}\,\alpha\,\beta_1 \ln\left(\frac{eB}{m^2}\right) + \ldots$$

$$\mathcal{L}^{(l)} \sim \frac{B^2}{2}\,(\alpha\,\beta_1)^l\,\frac{\beta_2/\beta_1^2}{l-1}\left(\ln\left(\frac{eB}{m^2}\right)\right)^{l-1} + \ldots \quad , \quad l \geq 2$$

- 1-particle <u>reducible</u> strong-field limit (Karbstein)

$$\mathcal{L}^{(l)} \sim \frac{B^2}{2}\left(\alpha\,\beta_1 \ln\left(\frac{eB}{m^2}\right)\right)^l + \ldots \quad , \quad l \geq 2$$

- all orders

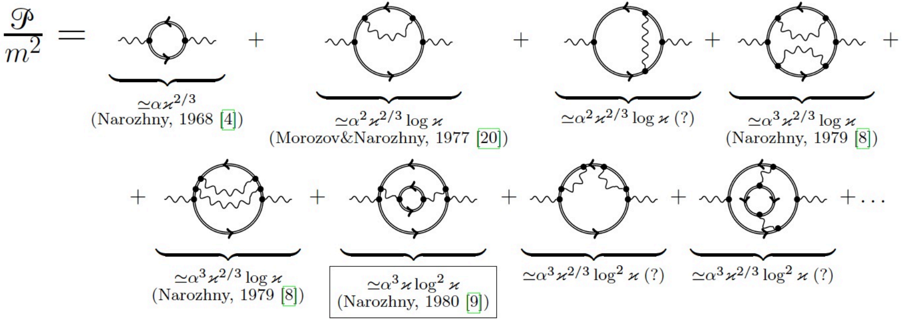
$$\mathcal{L} = -\frac{B^2}{2} + \frac{B^2}{2}\,\alpha^{1-\mathrm{loop}}(eB)\,\beta_1 \ln\left(\frac{eB}{m^2}\right) + \ldots$$

- reducible diagrams $\rightarrow$ correct running coupling

- quantum non-linearity parameter= amplitude of $\mathcal{E}$ field in electron rest frame:

$$\chi = \frac{e\,\hbar}{m^3\,c^4}\sqrt{-\left(F_{\mu\nu}\,p^\nu\right)^2}$$

- in constant-crossed-field approximation



- Ritus-Narozhny conjectures; review, Fedotov (2019)
  - conjecture:QED expansion parameter is $\alpha\,\chi^{2/3}$ at large χ