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Lecture 3 : CERN Summer School July 2024

Resurgence for Euler-Heisenberg & effective field theory

- QED vacuum = dielectric of virtual e^+e^- pairs

  (Bohr, Heisenberg, Euler, Weisskopf, Uehling, ...)

- probe QED vacuum with:
  - (i) external b.c.'s (Casimir)
  - (ii) external electromagnetic fields
  - (iii) external gravitational —
  - (iv) temperature

  ⋮

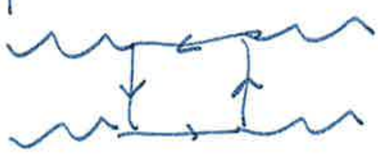
- key quantity: 1-loop effective action

$$\langle 0_{out} | 0_{in} \rangle \equiv e^{\frac{i}{\hbar} \Gamma[A]}$$

polarization tensor $\quad \Pi_{\mu\nu} \propto \dfrac{\delta^2 \Gamma}{\delta A_\mu \, \delta A_\nu}$

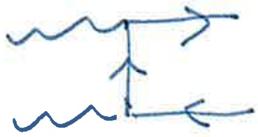
→ dispersive effects : e.g. • vacuum birefringence
• light-light scattering

$Re(\Gamma)$

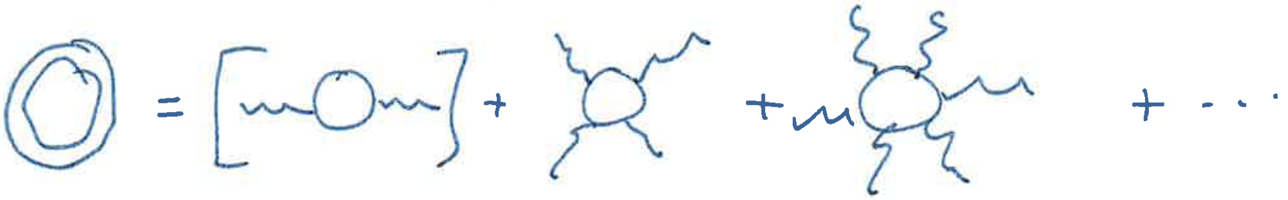


→ absorptive effects: e.g. • vacuum pair production

$Im(\Gamma)$



Γ = generating function of perturbative 1-loop amplitudes



only even # of external photon lines (Furry)

- but there are also <u>non-perturbative</u> effects, beyond perturbation theory (e.g. vacuum pair production)

- QED path integral

$$\int \mathcal{D}\psi \int \mathcal{D}A \; e^{\frac{i}{\hbar}\int\left(\bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F_{\mu\nu}^2\right)}$$

$\longrightarrow$ "integrate out the fermions" $\longrightarrow \int \mathcal{D}A \; e^{\frac{i}{\hbar}\int\left(-\frac{1}{4}F_{\mu\nu}^2\right)} \det(i\not{D}-m)$

$$= \int \mathcal{D}A \; e^{\frac{i}{\hbar}\left(\text{Maxwell} - i\hbar \underbrace{\ln\det(i\not{D}-m)}_{\Gamma[A]}\right)}$$

Schwinger: $\ln\det\left(1 - G_+^{(0)} e\gamma_\mu A^\mu\right)$

$$G_+^{(0)} = \frac{1}{(i\not{\partial}-m)+i\varepsilon}$$

expand in e $\longrightarrow$ perturbation theory

Q: does the perturbative expansion converge?

Q: where are the non-perturbative effects?

Euler + Heisenberg: consider first constant (but stated EFT expansion including derivatives)

probe fields: $\vec{E}$, $\vec{B}$

(Lorentz invariants; $\vec{E}^2 - \vec{B}^2$, $\vec{E}\cdot\vec{B}$)

- if $\vec{E} \not\perp \vec{B}$ then can go to frame where $E \parallel B$
  $\Rightarrow$ use E, B

(in general use $a^2 - b^2 = \vec{E}^2 - \vec{B}^2$, $ab = \vec{E}\cdot\vec{B}$, $a \Leftrightarrow E$, $b \Leftrightarrow B$)

$$\mathcal{L} = \frac{e^4}{360\pi^2 m^4}\left((\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2 \right) + O(F^6)$$

to all orders

$$\mathcal{L} = \frac{-1}{2\pi}\int_0^\infty dt\, e^{-\left(\frac{m^2c^3}{\hbar}\right)t}\frac{1}{t^3}\left(\frac{(eBt)(eEt)}{\tanh(eBt)\tan(eEt)} - 1 - \frac{e^2t^2}{3}(\vec{B}^2 - \vec{E}^2)\right)$$

- this is a Borel-Laplace integral!!!
  $\Rightarrow$ all-orders (+ non-perturbative) result

how? • choose gauge (Fock-Schwinger)

$$A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu$$

constant

$$\Rightarrow (i\not{D} - m) \rightarrow ((\not{D})^2 - m^2)$$

indep.

$2\times$ harmonic oscillators

solvable (zeta fn. method) $\Leftarrow$ with frequency: B, iE

Landau "degeneracy": $\frac{eB}{2\pi}$, $\frac{eE}{2\pi}$

... to produce nonlinear and non-perturbative effects of QED

- relativistic enhancement of the electric field in the electron rest frame

N.B. this strong-field regime of QED has not been explored in detail

- wide open theoretically and experimentally

(high intensity is different from high energy)

- relevant for astrophysics, heavy ion collisions, neutron stars, magnetars, ... Hawking radiation, ...

for our purposes here:

the Euler-Heisenberg result provides a simple + explicit demonstration of resurgence in QFT

- how do non-perturbative effects arise in effective field theory?

. it is also a beautiful realization of Dyson's argument ...

F. Dyson : Phys. Rev. 85, (1952) 631

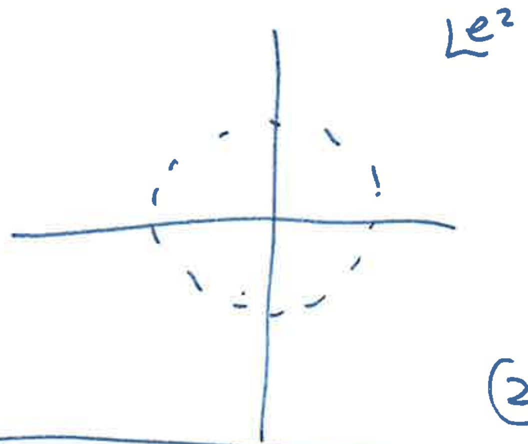
- physical argument (not a proof!) that the α expansion of QED perturbation theory should be divergent

comment: Dyson retracted here his "proof"/"argument" for the convergence of QED perturbation theory from 1951

$\Rightarrow$ the problem is non-trivial (still unsolved)

consider a physical quantity (e.g. (g-2)) computed in pert. theory

$$F(e^2) = a_0 + a_1 e^2 + a_2 e^4 + a_3 e^6 + \cdots$$



(1) suppose $F(e^2)$ convergent

$\Rightarrow$ small region near origin in complex e^2 plane that is free of singularities

(2) "$\Rightarrow$" $e^2 < 0$ regime is "smoothly connected to" $e^2 > 0$ regime

(3) but usual $e^2 > 0$ QED vacuum "would be unstable" if $e^2 < 0$

Dyson did not cite Euler + Heisenberg - unaware?

tentative, vague, imprecise ... but probably correct!

B field

$$\mathcal{L}_B = -\frac{e^2B^2}{8\pi^2}\int_0^\infty dt\, e^{-\frac{m^2c^3}{eB\hbar}t}\,\overbrace{\frac{1}{t^2}\left(\coth(t)-\frac{1}{t}-\frac{t}{3}\right)}^{\text{Borel transform}}$$

free field subtraction

log term ≡ charge renormalization

E field

$$\mathcal{L}_E = \frac{e^2E^2}{8\pi^2}\int_0^\infty dt\, e^{-\frac{m^2c^3}{eE\hbar}t}\,\underbrace{\frac{1}{t^2}\left(\cot(t)-\frac{1}{t}+\frac{t}{3}\right)}_{\text{Borel transform}}$$

$\frac{1}{t}\left(\cot(t) - \frac{1}{t} + \frac{t}{3}\right)$ has simple poles at

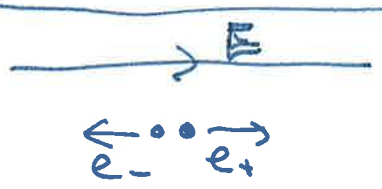
$$t = \pm k\pi \quad , k = 1, 2, 3, \dots$$

$$\Rightarrow \quad \mathrm{Im}\,\mathcal{L}_E \sim \frac{e^2E^2}{8\pi^3}\, e^{-\frac{m^2\pi c^3}{eE\hbar}}$$

$$|\langle 0_{out} | 0_{in} \rangle|^2 = e^{-2V\,\mathrm{Im}(\mathcal{L})} \approx 1 - 2V\,\mathrm{Im}(\mathcal{L})$$

$$= 1 - (\text{prob. of pair production})$$

<u>Bohr, Sauter, Heisenberg, Euler</u>



work done by field on scale of Compton wavelength $2eE\frac{\hbar}{2mc} \sim 2mc^2$

$\rightarrow E_c \sim \frac{m^2c^3}{e\hbar}$

desired extensions of Euler-Heisenberg

(1) • strong-field limit

Borel integral

$$\Rightarrow \quad \mathcal{L} \sim \frac{(eB)^2}{24\pi^2} \ln\left(eB/m^2\right) + \dots$$

$$= \underbrace{\frac{e^2}{4\pi}}_{\alpha} \cdot \left(-\frac{B^2}{2}\right) \cdot \underbrace{\frac{1}{3\pi}}_{1^{st}\ \beta\text{-function coefficient}} \ln\left(m^2/eB\right)$$

(Weisskopf, Heisenberg, Pauli, ...)

(see lecture 4 for numerical extension from small B to large B)

(2) <u>very</u> inhomogeneous fields → (GD and Z. Harris 2212.04599)

• the only conceivable earth-based way to make precision studies of high intensity regime of QED is using high intensity lasers, for which $\vec{E}(x)$ and $\vec{B}(x)$ are <u>extremely</u> inhomogeneous

$$\Longrightarrow \text{EFT at high intensity ?}$$

$E_c \sim \frac{m^2c^3}{e\hbar} \sim 10^{16}$ V/cm "Schwinger critical field"

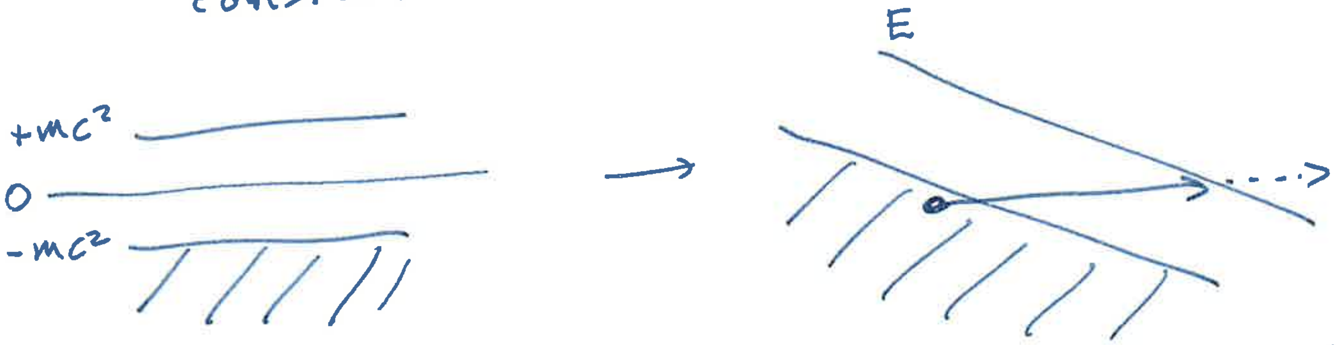
$\rightsquigarrow$ laser intensity $\sim 10^{29}$ W/cm^2

current max. intensity $\sim 10^{22}$ W/cm^2

"Schwinger effect": QED vacuum is unstable to e^+e^- pair production in the presence of an electric field (not a magnetic field)

"tunneling from the Dirac sea to the continuum"

constant E $\longrightarrow$ linear potential



cf. atomic Stark effect (ionization)

$2mc^2 \sim$ binding energy

- Zeeman effect: no tunneling ionization in a magnetic field

- Schwinger effect has not been seen directly. But current experiments in development at SLAC (FACET II) and DESY (LUXE) to combine high energy electron beams with high intensity lasers

- see slides for WKB analysis of the case of a monochromatic time-dependent field $E(t) = \mathcal{E}\cos(\omega t)$

$\uparrow$ $\uparrow$ new scale!

- Keldysh (atomic ionization by a laser) 1960's

$\downarrow$ key inhomogeneity parameter γ relates photon energy to binding energy

- Brézin-Itzykson (1972, PRD)
  Marinov-Popov ($\sim$ 1972)
  adapted Keldysh analysis to QED vacuum pair production

here: $\gamma \equiv \frac{m\omega}{e\mathcal{E}}$

small γ : "static"
large γ : high intensity, rapid time variation

- leading exponential contribution to the pair production rate

$$\operatorname{Im}\Gamma \sim e^{-\frac{m^2\pi}{e\mathcal{E}}\cdot g(\gamma)}$$

$$g(\gamma) = \frac{4}{\pi\gamma^2}\left((1+\gamma^2)K(-\gamma^2) - E(-\gamma^2)\right)$$

$$\longrightarrow \begin{cases} 1-\frac{\gamma^2}{8}, & \gamma\to 0 \\ \frac{4}{\pi\gamma}\ln\gamma, & \gamma \gg 1 \end{cases}$$

$$\downarrow$$

$$\longrightarrow \begin{cases} e^{-\frac{m^2\pi}{e\mathcal{E}}}, & \text{tunneling} \\ \left(\frac{e\mathcal{E}}{m\omega}\right)^{\frac{4mc^2}{\hbar\omega}}, & \text{multiphoton} \end{cases}$$

World line instanton method

- Feynman, Cecile Morette
- Alvarez, Affleck, Manton
- GD & C. Schubert

(illustrate for scalar QED for simplicity)

$$\boxed{\ln \mathcal{O} := -\int_0^\infty \frac{dt}{t} e^{-\mathcal{O}t}}$$

Feynman: $\ln \det\left(-D_\mu^2 + m^2\right)$
Morette

$$= \mathrm{tr} \ln \left(-D_\mu^2 + m^2\right)$$

$$= -\int_0^\infty \frac{dT}{T} e^{-m^2 T} \underbrace{\mathrm{tr}\left(e^{-(-D_\mu^2)T}\right)}_{\text{interpret as a "QM" path integral in "4d" } x_\mu(\tau)}$$

proper-time

$$= -\int_0^\infty \frac{dT}{T} e^{-m^2 T}\, \mathrm{tr} \int_{x_\mu(0) = x_\mu(T)} \mathcal{D}x \; e^{-\int_0^T d\tau \underbrace{\left(\dot{x}_\mu^2 + e\,\dot{x}_\mu A^\mu(x(\tau))\right)}}$$

relativistic Lagrangian for a charged particle moving in the field $F^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu$

- compute imaginary part by a 2-step saddle approximation
- (1) saddle of the path integral: solution to classical eq^n of motion

$$\boxed{\ddot{x}^\mu(\tau) = F_{\mu\nu}(x(\tau))\,\dot{x}_\nu}$$

- seek a closed loop solution of period T
- given $x^\mu_{\text{classical}}(\tau)$, evaluate the action

$$\longrightarrow \quad S_{\text{saddle}}(T)$$

- calculate fluctuation det using Gelfand-Yaglom

hep-th/0602176
- GD, Gies, Schubert, Wang
- Dietrich & GD 0706.4006

- fluctuation operator

$$\left.\frac{\delta^2 S}{\delta x_\mu \delta x_\nu}\right|_{x_\mu = x_\mu^{\text{classical}}(\tau)} = \text{differential operator w.r.t. } \tau$$

- Gelfand-Yaglom is a way to evaluate the determinant of an operator (in 1 variable!) i.e. τ without computing eigenvalues! $\Rightarrow$ det (fluctuation operator)

- review: 0711.1178
  + Saalburg lectures 2009
    saalburg.aei.mpg.de → Lecture Notes

- but we still need to do the T (proper time) integral
  - use saddle pt. method again

    T_c from $\frac{\partial S_{cl}[T]}{\partial T} = -m^2$

    (cf. Legendre transform in classical mechanics)

$\Rightarrow$ seek "worldline instanton" solutions of fixed "energy" m^2

→ universal semiclassical expression for time dependent electric field

- some limited examples in higher dims.
- agrees with WKB but can be pushed further

the "worldline instanton" (more properly "worldline saddle") method should be revisited now that we have learned a lot about how resurgence operates for QM models

- open problems of interest
  - use resurgence to connect WLI method with Lefschetz thimble Monte Carlo
- goal: "quantum control" i.e. design a pulsed focussed external field that maximizes the yield of pair production (or, more generally, interesting strong-field QED effects); or at least leads to a detectable momentum spectrum
- WLI is one of the few multidimensional semiclassical methods

3 higher loop order

- Euler-Heisenberg is the <u>1-loop</u> QED effective action
- Ritus (~1975): heroic computation of 2-loop QED EH effective action

→ double integral (<u>not</u> in Borel form)

- resurgence structure studied recently (GD & Z. Harris 2101.10409)

- Ritus - Narozhny conjectures
  - approximate a plane-wave background field by a constant crossed field (CCF) $E \perp B$ and E, B constant

"quantum nonlinearity parameter"

$$\chi = \frac{e\hbar}{m^3c^4}\sqrt{-(F_{\mu\nu}p^\nu)^2}$$

(amplitude of field in electron rest frame)

RN conjecture: for a CCF field, QED perturbation theory is an expansion in $\propto \chi^{2/3}$ at large χ ($\frac{2}{3}$ comes from Airy!)