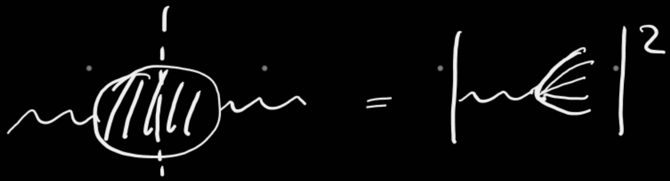
Recapitulation.

2-point function:

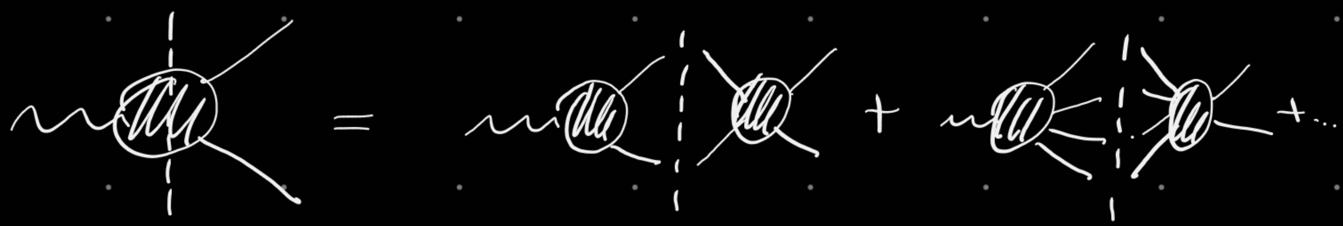


Imaginary part known $\longrightarrow$ can be inserted in the dispersive integral

Subtractions necessary? $\Longrightarrow$

2-point fct. = subtr. polyn. + disp. integral

3-point function.



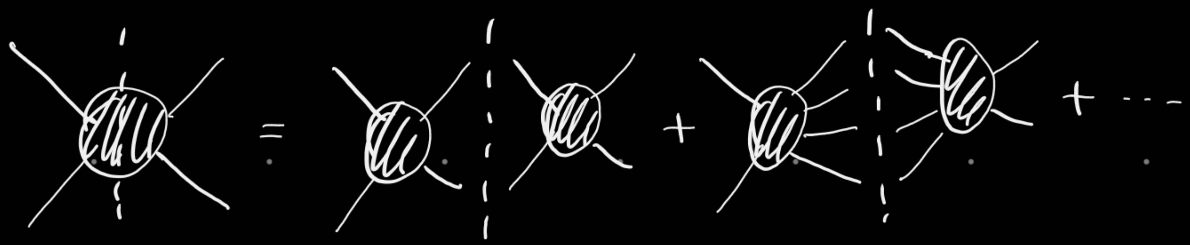
$$F_V^\pi(s) = \underbrace{P(s)}_{=1} \cdot \Omega_1^1(s) \cdot G_\omega(s) \cdot \Omega_{in}(s)$$

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^{\infty} ds' \, \frac{\operatorname{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4$$

$$g_\omega(s) = 1 + \varepsilon_\omega \frac{s}{\left(M_\omega - \frac{i}{2}\Gamma_\omega\right)^2 - s}$$

$\longrightarrow$ Plot of the relative importance of the 3 factors.

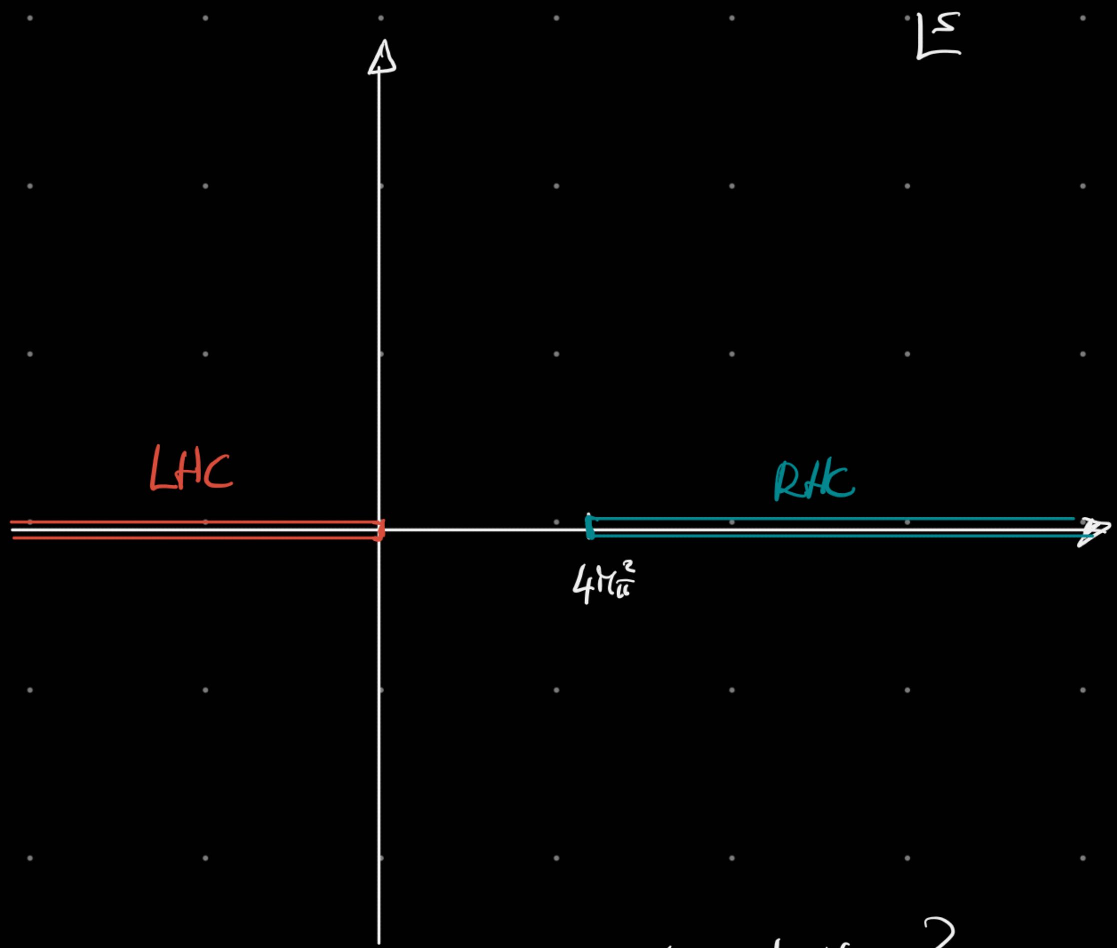
4-point function.



$$\mathrm{Im}\, t(s) = \sigma(s)|t(s)|^2 + (\text{inelastic contr.}) \quad \text{for } s \geq 4M_\pi^2$$

↑ optical theorem

$$t(s) = \frac{\sin\delta(s)}{\sigma(s)} e^{i\delta(s)}$$



What is the origin of the LHC?

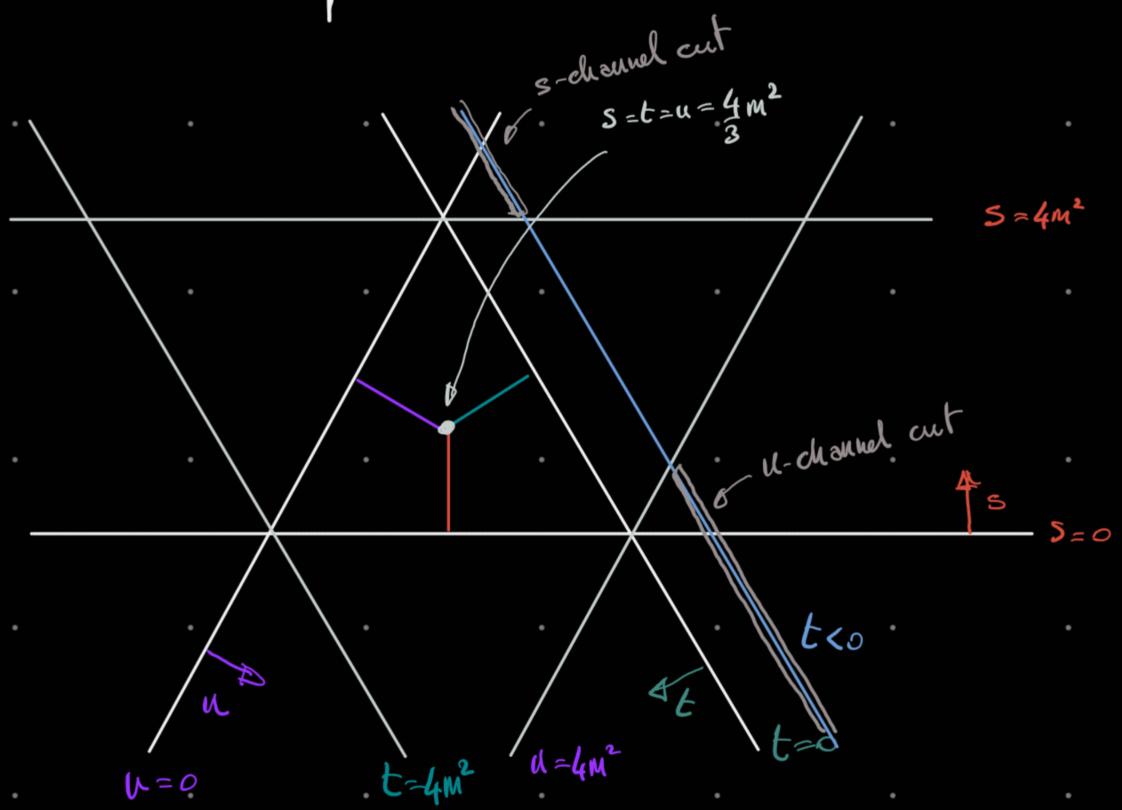
Consider the scattering amplitude of a scalar particle of mass m:

$$ {}_{out}\langle \phi(p_3)\phi(p_4) | \phi(p_1)\phi(p_2)\rangle_{in} \equiv \delta_{fi} + (2\pi)^4 \delta^4(P_f - P_i)\, i\, A(s,t,u) $$

$$ s = (p_1+p_2)^2 \qquad\qquad s+t+u = 4m^2 $$
$$ t = (p_1-p_3)^2 $$
$$ u = (p_1-p_4)^2 $$

The amplitude is symmetric under any permutation of the Mandelstam variables. What is its analytic structure?

In each channel the amplitude develops an imaginary part ($\Rightarrow$ a cut) for $x \geq 4m^2$ $(x = s,t,u)$. To understand the interplay between the different channels let us look at the Mandelstam plane



We can fix t and consider the scattering amplitude as a function of a single variable. For t in the range $|t|<4m^2$ we have two separate cuts and can represent the amplitude with a dispersion relation which takes into account both cuts:

$$A(s,t,u) = C(t) + \frac{s}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} A(s',t,u')}{s'(s'-s)} + \frac{u}{\pi}\int_{4m^2}^{\infty} du' \, \frac{\operatorname{Im} A(s',t,u')}{u'(u'-u)}$$

with $u = 4m^2 - s - t$ and $u' = 4m^2 - s' - t$

Since the amplitude is symmetric in all variables I can write the second integral as:

$$\frac{u}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} A(s',t,u')}{s'(s'-u)}$$

$$\Rightarrow \quad A(s,t) = C(t) + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{ds'}{s'} \left(\frac{s}{s'-s} + \frac{u}{s'-u} \right) \operatorname{Im} A(s',t) \qquad (1)$$

which makes the $s \leftrightarrow u$ crossing symmetry explicit.

Notice that for a fixed-t DR the subtraction constant is a function of t. With this representation the full crossing symmetry among the three variables is not manifest.

In order to determine $C(t)$ I can do the following:

$$A(0,t) = C(t) + \frac{u_0}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} A(s',t)}{s'(s'-u_0)}$$

with $u_0 = 4m^2 - t$

But $A(0,t) = A(t,0)$

$$\Rightarrow A(t,0) = C(0) + \frac{t}{\pi}\int_{4m^2}^{\infty} \frac{ds'}{s'} \, \frac{\operatorname{Im} A(s',0)}{s'-t} + \frac{u_0}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} A(s',0)}{s'(s'-u_0)}$$

which allows us to reexpress $C(t)$ as follows:

$$C(t) = C(0) + \frac{t}{\pi}\int_{4m^2}^{\infty} \frac{ds'}{s'} \, \frac{\operatorname{Im} A(s',0)}{s'(s'-t)} + \frac{u_0}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\operatorname{Im} A(s',0) - \operatorname{Im} A(s',t)}{s'(s'-u_0)}$$

Inserting this back into expression (1) we get a DR which is closer to showing full crossing symmetry, but not quite. What we have gained is an explicit expression of the t-dependence of the subtraction constant in terms of a dispersive integral.

If we insert back $C(t)$ in the DR we get:

$$A(s,t,u) = C(0) + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\left(\frac{s}{s'-s} + \frac{t}{s'-t} + \frac{u}{s'-u}\right)\operatorname{Im}A(s',t)$$

$$+ \frac{t}{\pi}\int_{4m^2}^{\infty} ds'\,\frac{\operatorname{Im}A(s',0) - \operatorname{Im}A(s',t)}{s'(s'-t)} + \frac{u_0}{\pi}\int_{4m^2}^{\infty} ds'\,\frac{\operatorname{Im}A(s',0) - \operatorname{Im}A(s',t)}{s'(s'-u_0)}$$

which shows that full crossing symmetry is almost but not fully manifest.

Partial wave projection

Projection on partial waves is easily obtained:

$$t_\ell(s) = \frac{1}{2}\int_{-1}^{1} dz\, P_\ell(z)\, A(s, t(s,z))$$

with P_ℓ Legendre polynomials and $t(s,z) = \frac{1}{2}(4m^2 - s)(1-z)$.

In view of crossing symmetry the projection operation can also be achieved with

$$t_\ell(s) = \int_{0}^{1} dz\, P_\ell(z)\, A(s, t(s,z))$$

which is convenient because it involves a smaller range of values for t.

One may also wish to distinguish the contribution of individual partial waves to the dispersive integrals. This involves a different projection because the variables involved are now: (s', t).

$$A(s',t) = \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell\left(1 + \frac{2t}{s'-4m^2}\right)t_\ell(s')$$

Inserting this expression for the imaginary parts in the integrand and carrying out the projection on partial waves for the external kinematic variables one can obtain an expression of the form:

$$t_\ell(s) = a_0\,\delta_{\ell 0} + \sum_{\ell'=0}^{\infty} \int_{4M^2}^{\infty} ds'\, K_{\ell\ell'}(s,s')\, \mathrm{Im}\, t_{\ell'}(s')$$

where a_0 is the S-wave scattering length: $t_0(4M^2) \equiv a_0$

if one replaces the subtraction constant $C(0)$ with the value of the amplitude $A(s,t)$ at threshold: $A(4M^2;0) = a_0$.

Isospin.

In the real world pions are not a singlet, but have an additional quantum number, isospin. This additional structure makes things somewhat more complicated, but the analytic structure remains the same.

$$a,b,c,d = 1,2,3$$

$$_{out}\langle \pi^d(p_4)\pi^c(p_3)\,|\,\pi^a(p_1)\pi^b(p_2)\rangle_{in} =$$

$$= \delta_{fi} + (2\pi)^4\delta^4(P_f - P_i)\, i\left[\delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t)\right]$$

with $A(s,t,u)$ symmetric under the exchange of t and u.

If Isospin is a symmetry, we only need one amplitude to describe all possible scattering channels. Out of the isospin-invariant ampl. $A(s,t,u)$ we can then build all amplitudes of interest, in particular those with a fixed isospin in a given channel:

$$T^0(s,t) = 3A(s,t,u) + A(t,u,s) + A(u,s,t)$$

$$T^1(s,t) = A(t,u,s) - A(u,s,t)$$

$$T^2(s,t) = A(t,u,s) + A(u,s,t)$$

It is conventional to define partial waves with the following normalization:

$$T^I(s,t) = 32\pi \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell\left(1 + \frac{2t}{4M_\pi^2 - s}\right) t_\ell^I(s)$$

and the partial waves are parametrized in terms of phase and inelasticity:

$$t_\ell^I(s) = \frac{1}{2i\sigma(s)}\left[\eta_\ell^I(s)\, e^{2i\delta_\ell^I(s)} - 1\right]$$

with $\sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$.

We could now repeat the derivation of the eqs. for the partial waves for the present case. The additional complication here is due to isospin: in each channel I have 3 amplitudes and crossing symmetry has to take into account this additional structure:

$$\vec{T}(s,u) = C_{tu}\, \vec{T}(s,t)$$
$$\vec{T}(t,s) = C_{st}\, \vec{T}(s,t)$$
$$\vec{T}(u,t) = C_{su}\, \vec{T}(s,t)$$

$$C_{tu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad C_{st} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}; \quad C_{su} = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

$$T(s,t) = C_{st}\left[C(t) + (s-u)D(t)\right] +$$

$$+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\left(\frac{s^2}{s'-s} + \frac{u^2}{s'-u} C_{su}\right) \operatorname{Im} T(s',t)$$

Observation:

$$T(0,t) = C_{st}\, T(t,0)$$

$$C_{st}\left[C(t) - u_0 D(t)\right] + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\,\frac{u_0^2}{s'-u_0}\, C_{su} \operatorname{Im} T(s',t)$$

$$T(t,0) = C_{st}\left[C(0) + (t-u_0)D(0)\right] +$$

$$+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\left(\frac{t^2}{s'-t} + \frac{u_0^2}{s'-u_0} C_{su}\right) \operatorname{Im} T(s',0)$$

Solve for $C(t)$ and $D(t)$ (the two can be disentangled because $C(t) = \begin{pmatrix} c^0(t) \\ 0 \\ c^2(t) \end{pmatrix}$ and $D(t) = \begin{pmatrix} 0 \\ d^1(t) \\ 0 \end{pmatrix}$)

and reexpress $C(0)$ and $D(0)$ in terms of a_0^0 and a_0^2.

One ends up with:

$$T(s,t) = \frac{1}{4M_\pi^2}\left(s\mathbb{1} + t\,C_{st} + u\,C_{su}\right) T(4M_\pi^2, 0)$$

$$+ \int_{4M_\pi^2}^{\infty} ds'\, g_2(s,t,s')\, \mathrm{Im}\, T(s',0)$$

$$+ \int_{4M_\pi^2}^{\infty} ds'\, g_3(s,t,s')\, \mathrm{Im}\, T(s',t)$$

where

$$g_2(s,t,s') = -\frac{t}{\pi s'(s'-4M_\pi^2)}\left(u\,C_{st} + s\,C_{st}C_{tu}\right)\left(\frac{\mathbb{1}}{s'-t} + \frac{C_{su}}{s'-u_0}\right)$$

$$g_3(s,t,s') = \frac{-\,su}{\pi s'(s'-u_0)}\left(\frac{1}{s'-s} + \frac{C_{su}}{s'-u}\right)$$

After partial wave projection one ends up with:

$$t_\ell^I(s) = k_\ell^I(s) + \sum_{I'=0}^{2}\sum_{\ell'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds'\, K_{\ell\ell'}^{II'}(s,s')\, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

with

$$k_\ell^I(s) = a_0^I\delta_\ell^0 + \frac{s-4M_\pi^2}{4M_\pi^2}\left(2a_0^0 - 5a_0^2\right)\left(\frac{1}{3}\delta_0^I\delta_\ell^0 + \frac{1}{18}\delta_1^I\delta_\ell^1 - \frac{1}{6}\delta_2^I\delta_\ell^0\right)$$

Kernels of the Roy eqs.:

$$K^{II'}_{\ell\ell'}(s,s') = (2\ell'+1)\int_0^1 dz\, P_\ell(z)\left[g_2^{II'}(s,t_z,s') + g_3^{II'}(s,t_z,s')\, P_{\ell'}\left(1+\frac{2t_z}{s'-4M_\pi^2}\right)\right]$$

with $\quad t_z = \frac{1}{2}(4M_\pi^2 - s)(1-z)$

P_ℓ projects on the external partial wave (s,t)

$P_{\ell'}$ " " internal " " (s',t).

Example:

$$K^{II'}_{\ell\ell'}(s,s') = \frac{\delta^{II'}\delta_{\ell\ell'}}{\pi(s'-s)} + \bar{K}^{II'}_{\ell\ell'}(s,s')$$

$$\bar{K}^{00}_{00}(s,s') = \frac{2}{3\pi(s-4M_\pi^2)}\ln\left(\frac{s+s'-4M_\pi^2}{s'}\right) - \frac{2s+5s'-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}$$

Notice that the integration variable $s' \geq 4M_\pi^2$, so that the logarithm develops a cut for $s \leq 0$, the so-called left-hand cut.