Roy equations solutions and the σ-resonance pole

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School on continuum foundations, CERN, 25.7.24

Outline

- Roy equations
  - Meaning of $\bar{\ell}_3$ and $\bar{\ell}_4$
  - Current determination of a_0^0, a_0^2 and $\delta_0^0(0.8\mathrm{GeV})$
- Model-independent σ-pole determination
- The σ in $\gamma\gamma \to \pi\pi$

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Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

$$\begin{aligned}
\text{Re}\, t_0^0(s) &= k_0^0(s) + \int_{4M_\pi^2}^{s_0} ds'\, K_{0\,0}^{0\,0}(s,s')\, \text{Im}\, t_0^0(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds'\, K_{0\,1}^{0\,1}(s,s')\, \text{Im}\, t_1^1(s') \\
&+ \int_{4M_\pi^2}^{s_0} ds'\, K_{0\,0}^{0\,2}(s,s')\, \text{Im}\, t_0^2(s') + f_0^0(s) + d_0^0(s) \\
k_0^0(s) &= a_0^0 + \frac{s-4M_\pi^2}{12M_\pi^2}(2a_0^0 - 5a_0^2) \\
f_0^0(s) &= \sum_{I'=0}^{2} \sum_{\ell'=0}^{1} \int_{s_0}^{s_3} ds'\, K_{0\ell'}^{0I'}(s,s')\, \text{Im}\, t_{\ell'}^{I'}(s') \\
d_0^0(s) &= \text{all the rest} \qquad [\sqrt{s_0} = 0.8\text{GeV} \quad \sqrt{s_3} = 2\text{GeV}]
\end{aligned}$$

Roy equations

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

S.M. Roy (71)

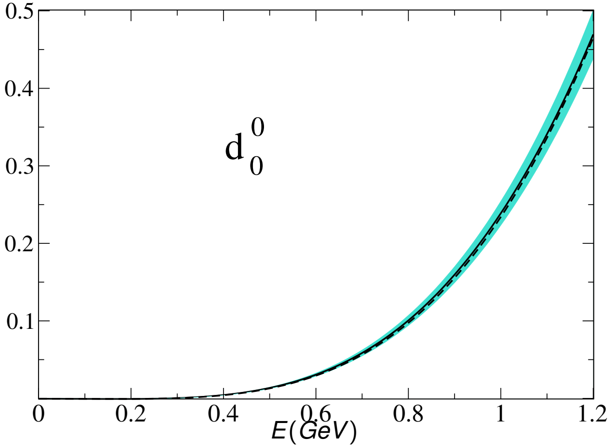
Numerical solutions of the Roy equations

Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)
Kamiński, Peláez and Ynduráin (08)
Garcia-Martin, Kamiński, Peláez, Ruiz de Elvira, Ynduráin (11)

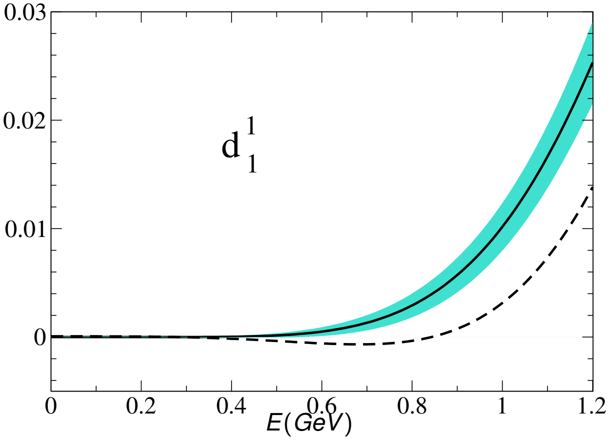
Input: S- and P-wave imaginary parts above 0.8 GeV
imaginary parts of all higher waves
two subtraction constants, *e.g.* a_0^0 and a_0^2

Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: a_0^0, a_0^2 inside the universal band $\Rightarrow$ the solution is unique

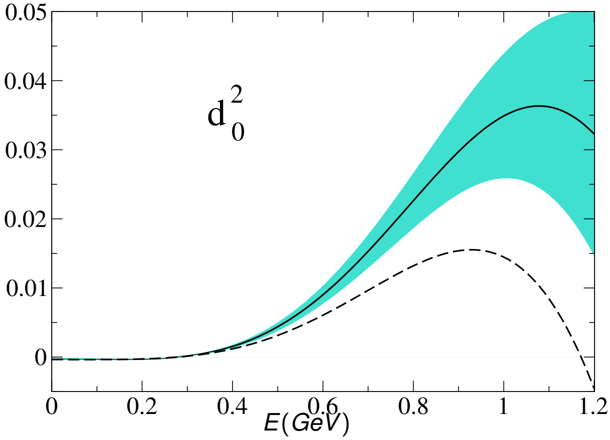
Driving terms



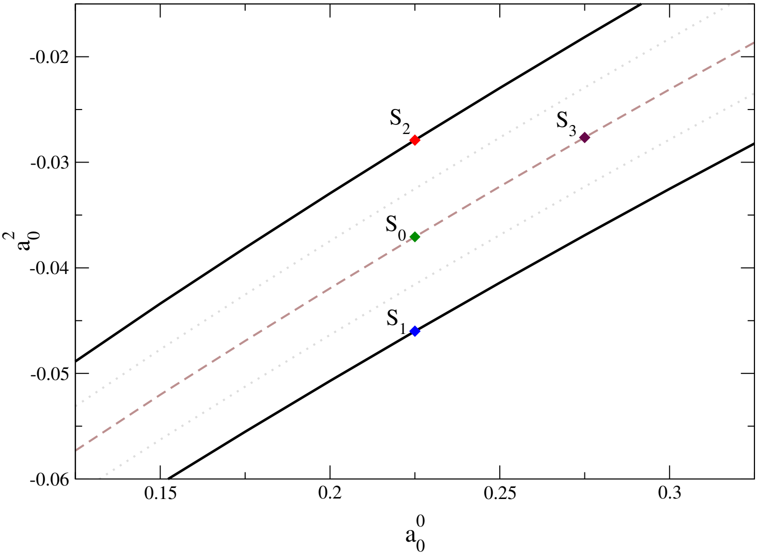
Driving terms



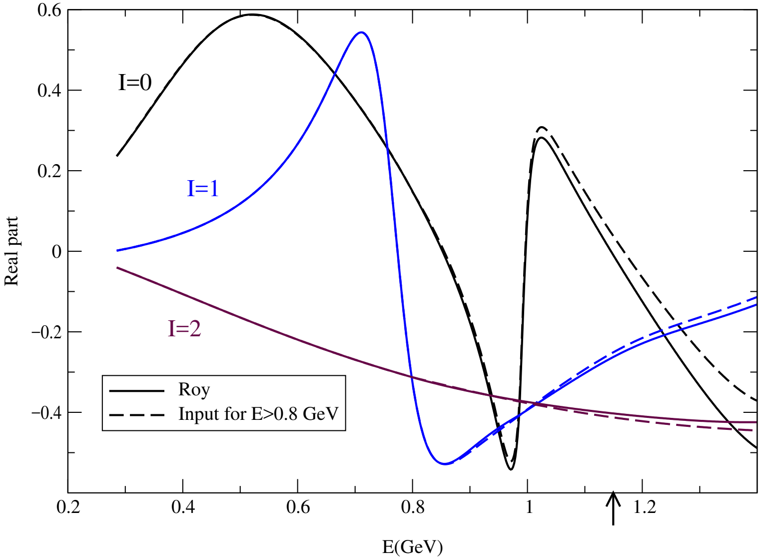
Driving terms



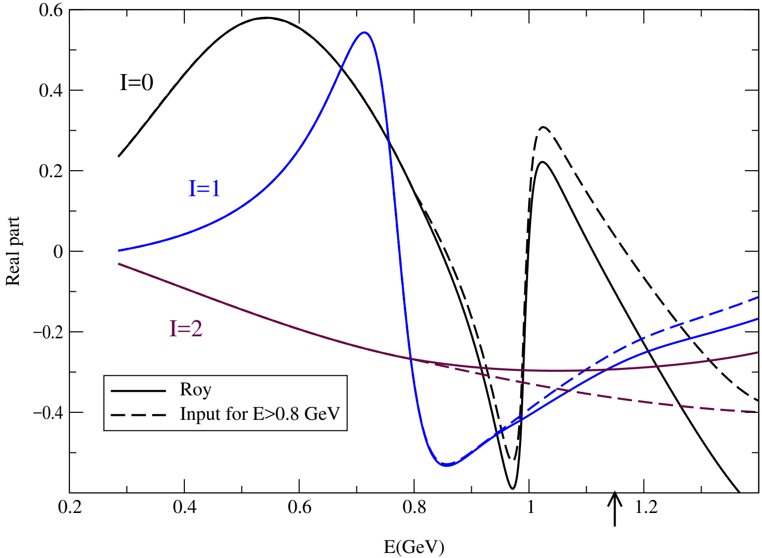
Numerical solutions



Numerical solutions



Numerical solutions



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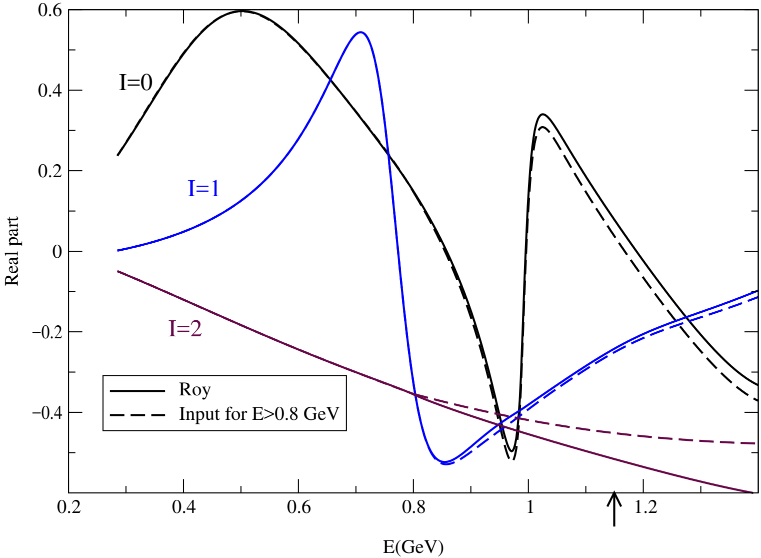
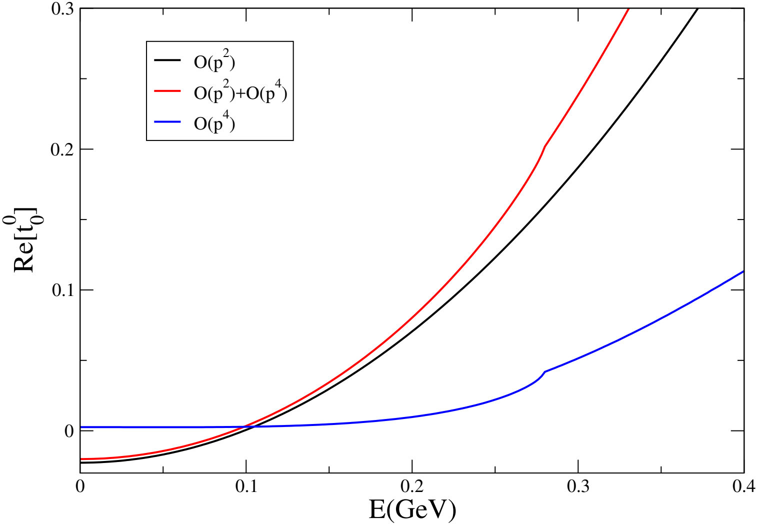


Roy + ChPT

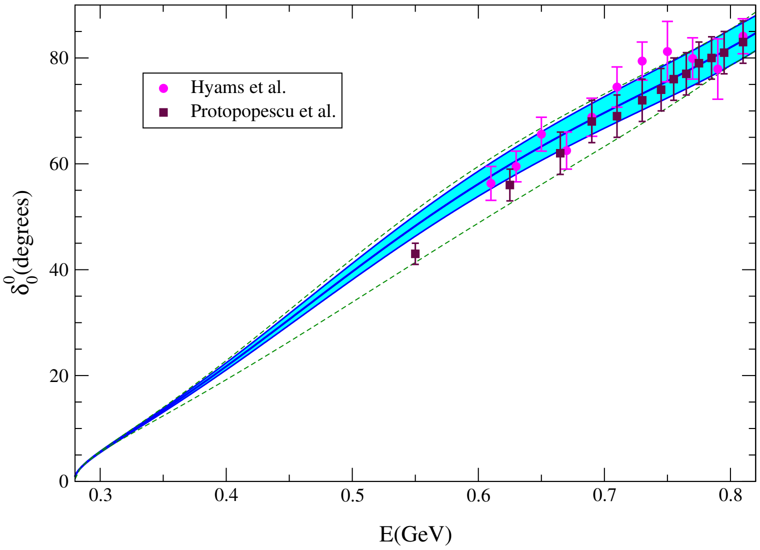
- at fixed input above 0.8 GeV, the only free parameters in the Roy equations are the two S-wave scattering lengths;
- chiral perturbation theory predicts these
- actually the most reliable prediction is for the $\pi\pi$ amplitude below threshold
- we have fixed the two subtraction constants in this way

Roy + ChPT



Roy+ChPT: final results

GC, Gasser and Leutwyler (01)



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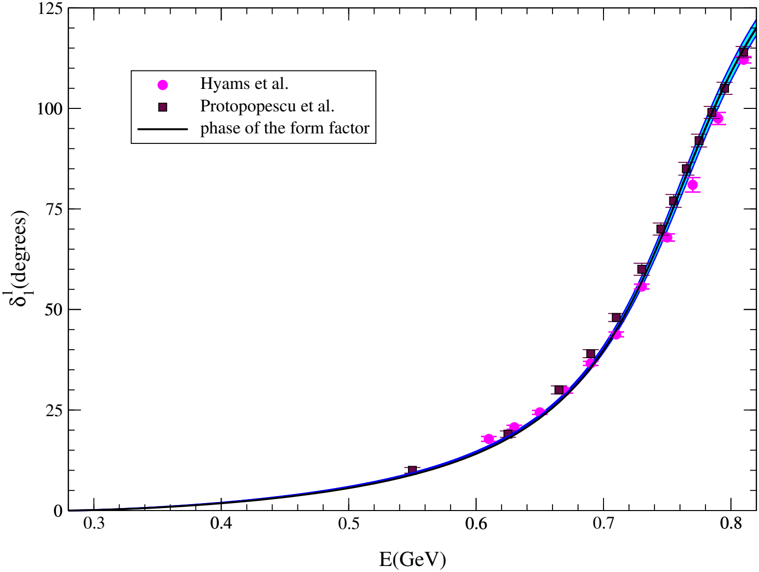

Roy+ChPT: final results

GC, Gasser and Leutwyler (01)

Scattering lengths

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3 \end{aligned}$$

where $\bar{\ell}_4 = 4.4 + \Delta\ell_4$ $\qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$

Adding errors in quadrature $\qquad [\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= 0.265 \pm 0.004 \end{aligned}$$

Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \quad F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0} = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection (I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

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All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=2} = A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=2} = \frac{-s + 2M_\pi^2}{F_\pi^2}$$

S wave projection (I=2)

$$t_0^2(s) = \frac{2M_\pi^2 - s}{32\pi F_\pi^2} \qquad a_0^2 = t_0^2(4M_\pi^2) = \frac{-M_\pi^2}{16\pi F_\pi^2} = -0.045$$

Chiral predictions for a_0^0 and a_0^2

Quark mass dependence of M_π and F_π:

$$M_\pi^2 = M^2\left(1 - \frac{M^2}{32\pi^2F^2}\bar\ell_3 + O(M^4)\right)$$

$$F_\pi = F\left(1 + \frac{M^2}{16\pi^2F^2}\bar\ell_4 + O(M^4)\right)$$

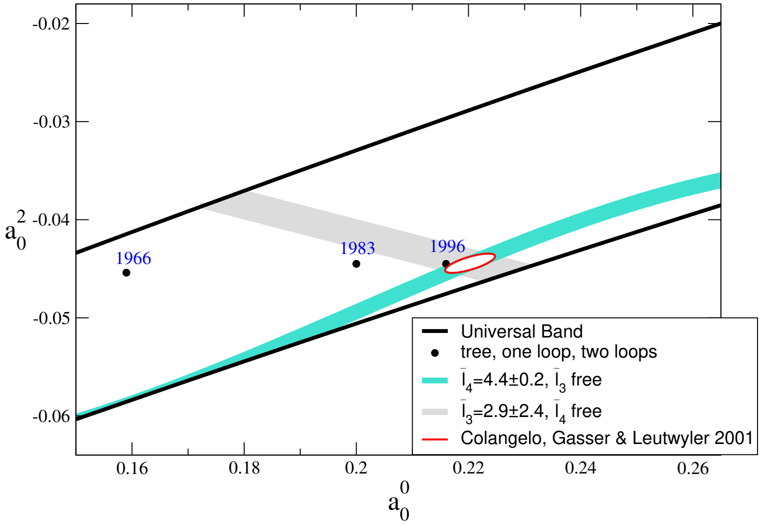
Phenomenological determinations (indirect):

$$\bar\ell_3 = 2.9 \pm 2.4$$ Gasser & Leutwyler (84)

$$\bar\ell_4 = 4.4 \pm 0.2$$ GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly

Chiral predictions for a_0^0 and a_0^2



Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ determines the NLO quark mass dependence of the pion mass

$$M_\pi^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_\pi^2}\bar{\ell}_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \qquad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$

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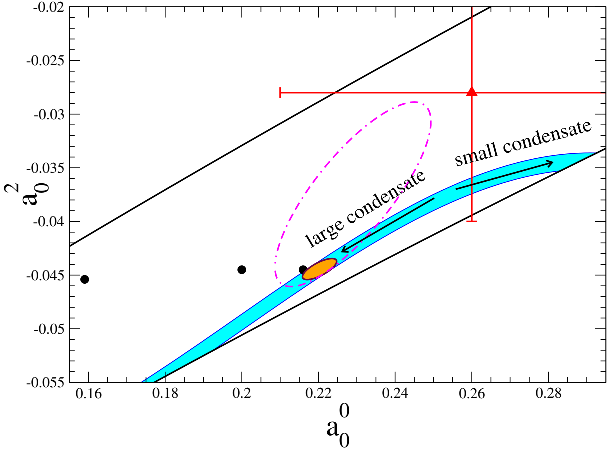
Its size tells us what fraction of the pion mass is given by the Gell-Mann–Oakes–Renner term

$$M_{\text{GMOR}}^2 \equiv 2B\hat{m}$$

or how large is the quark condensate, the order parameter of chiral symmetry breaking.
Jan Stern and collaborators have emphasized this since long!

Sensitivity to the quark condensate



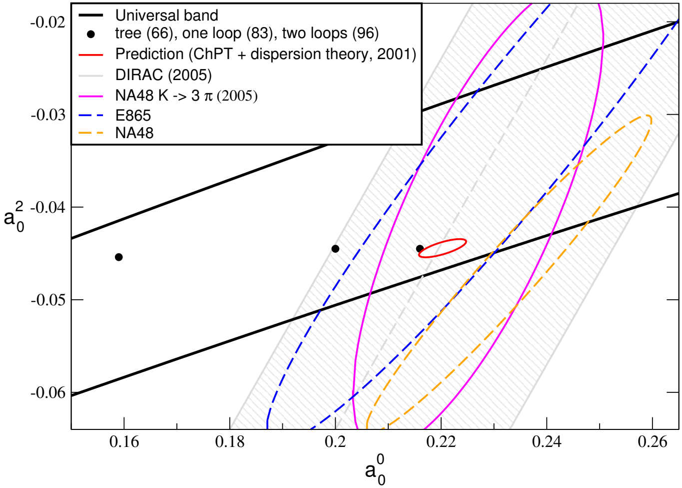
The E865 data on $K_{\ell 4}$ imply that

GC, Gasser and Leutwyler PRL (01)

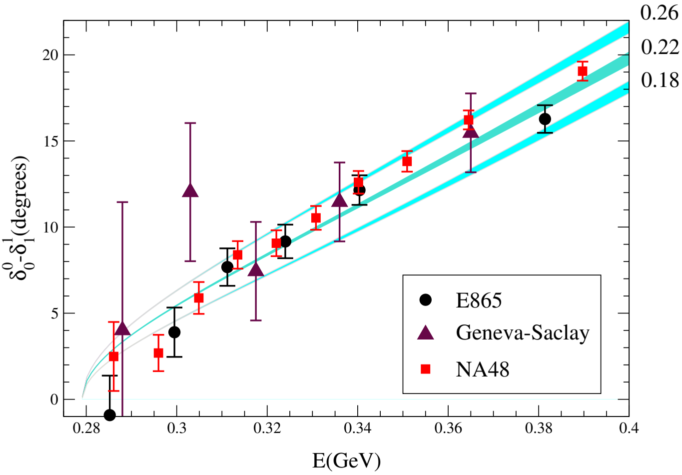
$$M_{\text{GMOR}} > 94\% M_\pi$$

Situation after new data?

Experimental tests



Experimental tests



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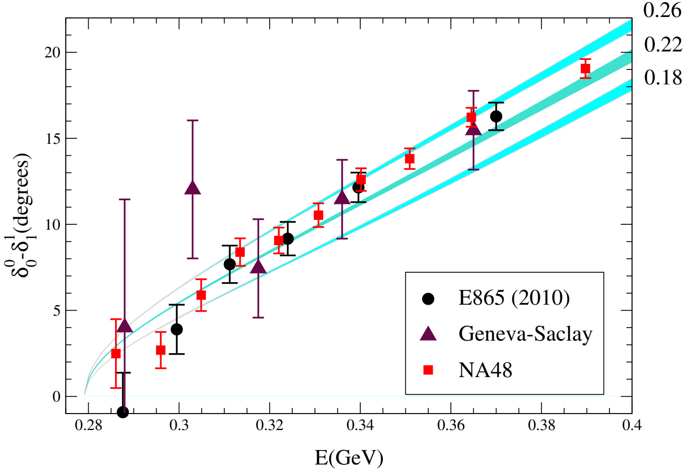
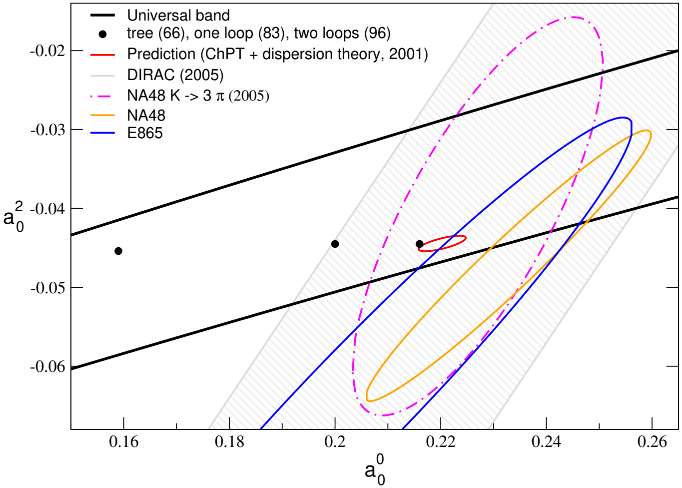


Recent update: E865 corrected their data

Experimental tests



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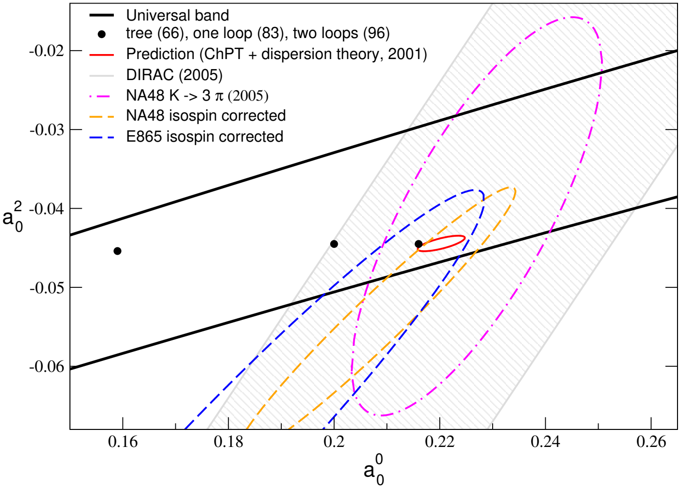
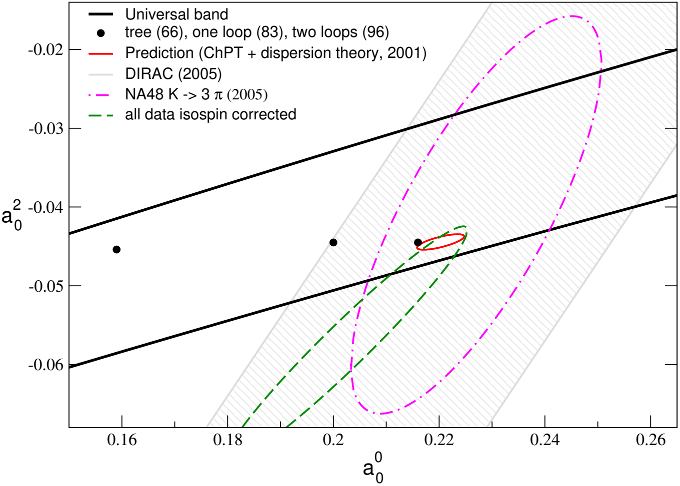
Experimental tests



isospin breaking corrections recently calculated for K_{e4} are essential at this level of precision GC, Gasser, Rusetsky (09)

Experimental tests



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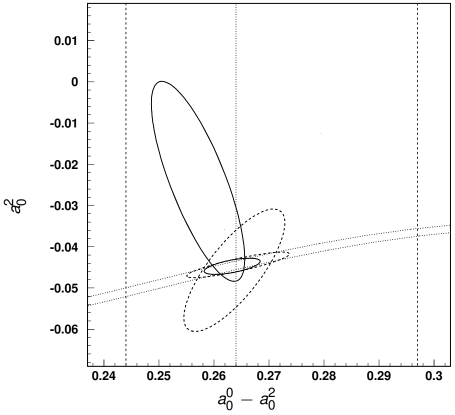

Figure from NA48/2 Eur.Phys.J.C64:589,2009

S-wave scattering lengths: current status

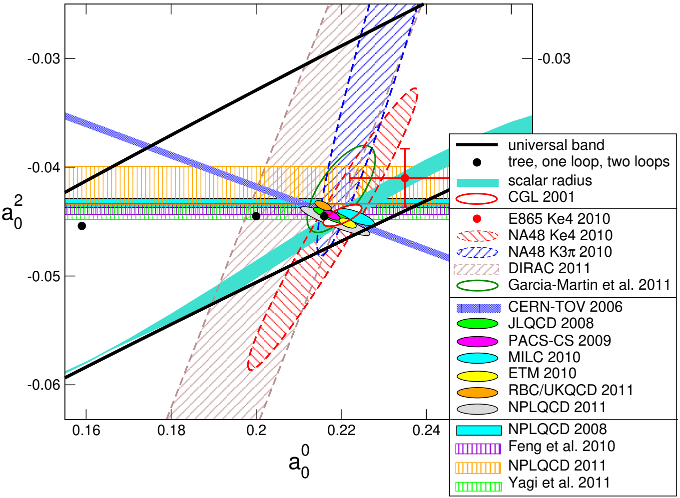

Figure courtesy of H. Leutwyler

S-wave scattering lengths: current status

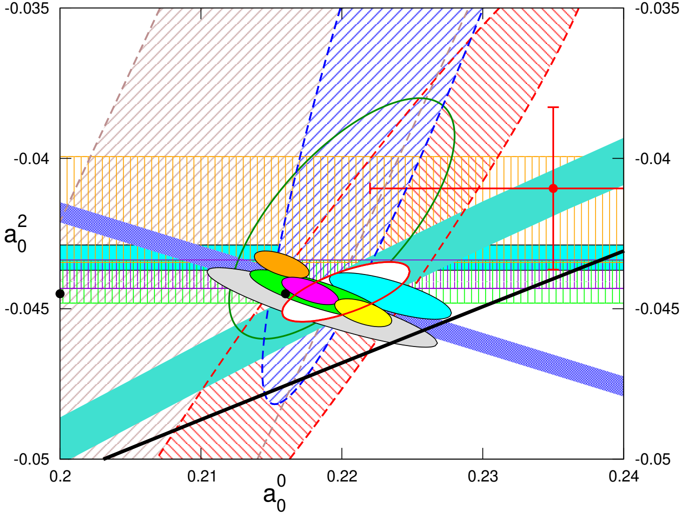

Figure courtesy of H. Leutwyler

$\delta_0^0(0.8\mathrm{GeV})$: current status

$\delta_0^0(0.8\mathrm{GeV})$	analysis
$82.3^\circ \pm 3.4^\circ$	ACGL (2000)
$91.9^\circ \pm 2.6^\circ$	PY (2003)
$82.3^{\circ}{}^{+10^\circ}_{-4^\circ}$	CCL (2006)
$85.7^\circ \pm 1.6^\circ$	GMKPY (2011)
$82.9^\circ \pm 1.7^\circ$	(1)Moussallam (2011)
$80.9^\circ \pm 1.4^\circ$	(2)Moussallam (2011)

(1) shallow dip in η_0^0
(2) deep dip in η_0^0

$\Rightarrow$ our earlier narrower range now confirmed

Outline

Roy equations

- Meaning of $\bar{\ell}_3$ and $\bar{\ell}_4$
- Current determination of a_0^0, a_0^2 and $\delta_0^0(0.8\mathrm{GeV})$

Model-independent σ-pole determination

The σ in $\gamma\gamma \to \pi\pi$

The sigma resonance

Citation: S. Eidelman *et al.* (Particle Data Group), Phys. Lett. B **592**, 1 (2004) and 2005 partial update for edition 2006 (URL: http://pdg.lbl.gov)

$f_0(600)$ or σ

$I^G(J^{PC}) = 0^+(0^{++})$

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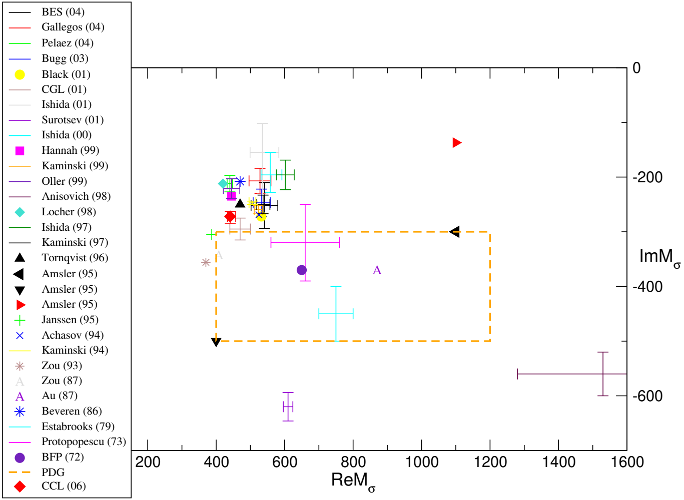
$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\text{pole}}})$.

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
(400–1200)−i(300–500) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(541 \pm 39)-i(252 \pm 42)$	[1] ABLIKIM	04A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528 \pm 32)-i(207 \pm 23)$	[2] GALLEGOS	04	RVUE	Compilation
$(440 \pm 8)-i(212 \pm 15)$	[3] PELAEZ	04A	RVUE	$\pi\pi \to \pi\pi$
$(533 \pm 25)-i(247 \pm 25)$	[4] BUGG	03	RVUE	
$532 - i272$	BLACK	01	RVUE	$\pi^0\pi^0 \to \pi^0\pi^0$
$(470 \pm 30)-i(295 \pm 20)$	[5] COLANGELO	01	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-36})-i(155^{+76}_{-53})$	[6] ISHIDA	01		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610 \pm 14 - i620 \pm 26$	[7] SUROVTSEV	01	RVUE	$\pi\pi \to \pi\pi,\ K\overline{K}$
$(558^{+34}_{-27})-i(196^{+32}_{-41})$	ISHIDA	00B		$p\overline{p} \to \pi^0\pi^0\pi^0$
$445 - i235$	HANNAH	99	RVUE	π scalar form factor
$(523 \pm 12)-i(259 \pm 7)$	KAMINSKI	99	RVUE	$\pi\pi \to \pi\pi,\ K\overline{K},\ \sigma\sigma$
$442 - i\,227$	OLLER	99	RVUE	$\pi\pi \to \pi\pi,\ K\overline{K}$
$469 - i203$	OLLER	99B	RVUE	$\pi\pi \to \pi\pi,\ K\overline{K}$
$445 - i221$	OLLER	99C	RVUE	$\pi\pi \to \pi\pi,\ K\overline{K},\ \eta\eta$
$(1530^{+90}_{[illegible]})-i(560 \pm 40)$	ANISOVICH	98B	RVUE	Compilation

The sigma resonance



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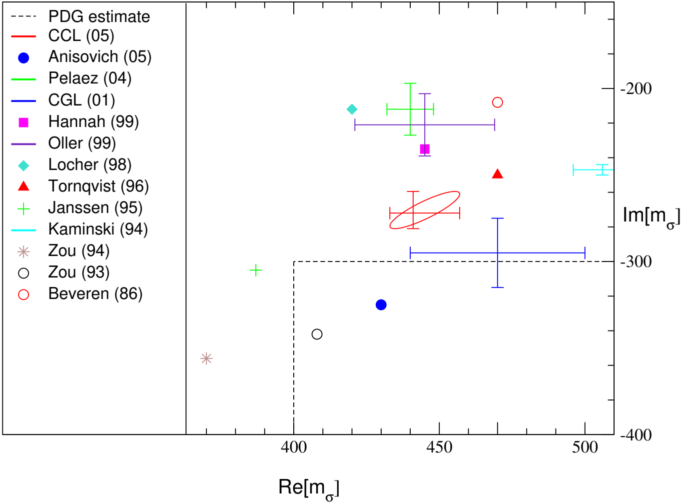


The sigma resonance

Message n. 1 of this talk:

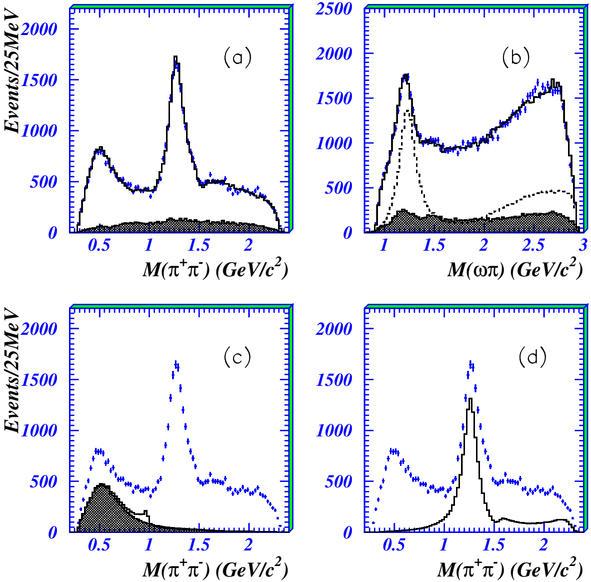
In $\pi\pi$ scattering two *S*-wave scattering lengths are the *essential parameters* at low energy

Their knowledge fixes the σ pole position to a *remarkable* level of precision

$$M_\sigma = 441 \, ^{+16}_{-8} \text{ MeV}\,, \quad \Gamma_\sigma = 544 \, ^{+18}_{-25} \text{ MeV}$$

I. Caprini, GC, H. Leutwyler, PRL 06

The σ in the data – BES (04), $J/\psi \to \omega\pi^+\pi^-$



How is the σ pole determined?

The relevant question is:

Where does the amplitude have a pole on the second Riemann sheet of the complex s plane?

The answer ought to be model- and parametrization-independent

How is the σ pole determined?

What is usually done is instead the following:
Fit the data with a parametrization, e.g.

$$\begin{aligned} f &= \frac{G_\sigma}{M^2 - s - iM\Gamma_{\rm tot}(s)} \\ \Gamma_{\rm tot}(s) &= g_1 \frac{\rho_{\pi\pi}(s)}{\rho_{\pi\pi}(M^2)} + g_2 \frac{\rho_{4\pi}(s)}{\rho_{4\pi}(M^2)} \end{aligned}$$

where $g_{1,2}$ can also be functions of s

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The fit to the data determines the σ parameters, M and $\Gamma_{\rm tot}$

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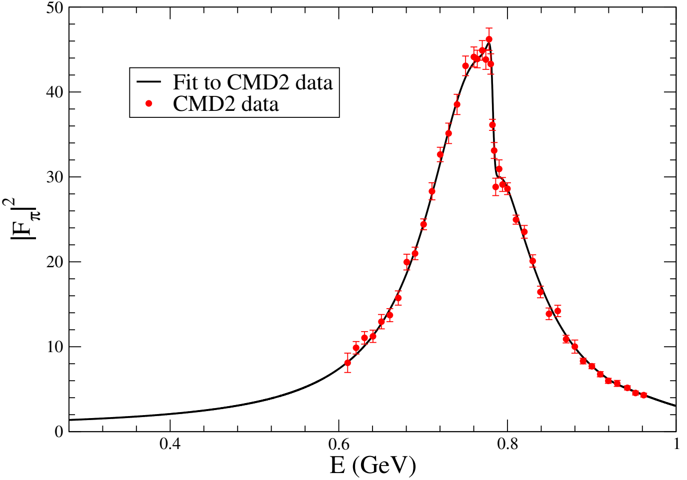
where $g_{1,2}$ can also be functions of s

The fit to the data determines the σ parameters, M and Γ_{tot}

The outcome is parametrization-dependent
Moreover, a shortcoming of many of the parametrizations used to fit data is the neglect of the left-hand cut

Compare to the ρ in $e^+e^- \to \pi^+\pi^-$



Roy representation of t_0^0

Double-subtracted, crossing symmetric dispersion relation for t_0^0

$$
\begin{aligned}
t_0^0(s) \;=\;& a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' \left\{ K_0(s,s')\, \mathrm{Im}\, t_0^0(s') \right. \\
& \left. + K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s') \right\} + d_0^0(s)
\end{aligned}
$$

$$a = a_0^0\,, \quad b = (2\,a_0^0 - 5\,a_0^2)/(12 M_\pi^2)$$

$$K_0(s,s') = \frac{1}{\pi(s'-s)} + \frac{2\ln((s+s'-4M_\pi^2)/s')}{3\pi(s-4M_\pi^2)} - \frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}$$

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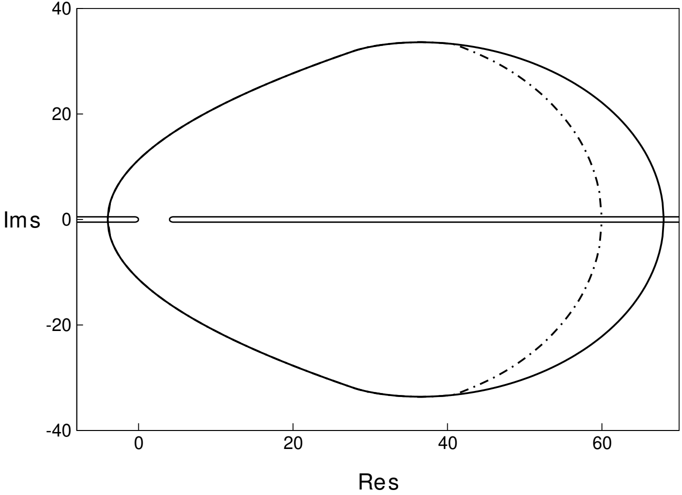
$$a = a_0^0\,, \quad b = (2\,a_0^0 - 5\,a_0^2)/(12 M_\pi^2)$$

This representation allows one to evaluate t_0^0 in the complex plane – in its domain of validity on the first sheet.

Roy representation of t_0^0



Caprini, GC, Leutwyler (05)

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\end{aligned}
$$

$$
a = a_0^0\,, \quad b = (2\,a_0^0 - 5\,a_0^2)/(12 M_\pi^2)
$$

This representation allows one to evaluate t_0^0 in the complex plane – in its domain of validity on the first sheet.

Poles, however, are to be found on the second sheet

Roy representation of S_0^0

$$S_0^0(s) = 1 - 2\sqrt{\frac{4M_\pi^2}{s} - 1}\, t_0^0(s)\,, \qquad 0 \le s \le 4M_\pi^2$$

Roy representation of S_0^0

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By analytic continuation, it is then true everywhere that

$$S_0^{0\;II}(s) = \left[S_0^{0\;I}(s)\right]^{-1}$$

Poles on the *II* sheet correspond to zeros on the *I* sheet!

Summary: method to determine the pole position

- Roy equations provide an explicit representation of t_0^0 on the first sheet, in terms of the imaginary parts of the partial waves on the real axis and two subtraction constants:

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' K_0(s,s')\, \mathrm{Im}\, t_0^0(s') + \ldots$$

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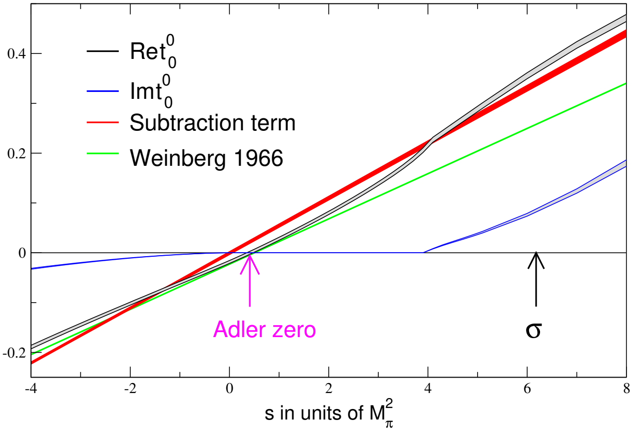
$$S_0^{0\ II}(s) = \left[S_0^{0\ I}(s)\right]^{-1}$$

- Using as input the imaginary parts of the partial waves and the two *S*-wave scattering lengths one can determine the position of the poles of the *S*-matrix on the second sheet

Importance of the scattering lengths



Zeros of S_0^0 (and S_1^1)

Input: the imaginary parts from Roy solutions below 1.15 GeV and the central values of the two scattering lengths (CHPT) we find two pairs of zeros

$$m_\sigma^2 = (6.2 \pm i\,12.3)\,M_\pi^2 \qquad m_{f_0}^2 = (51.4 \pm i\,1.4)\,M_\pi^2$$

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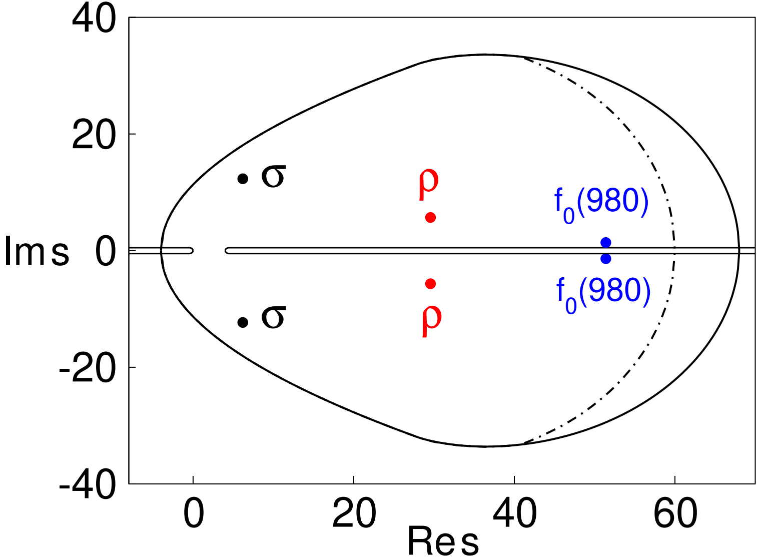

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Error analysis: [at fixed a_0^0, a_0^2 and $\delta_A \equiv \delta_0^0(0.8\text{GeV})$]

$$m_\sigma = 441 \pm 4 - i(272 \pm 6)\text{ MeV}$$

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$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005}$$

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Error analysis:

$$\begin{aligned} m_\sigma &= 441 \pm 4 - i(272 \pm 6)\text{ MeV} + (-2.4 + i3.8)\Delta a_0^0 \\ &+(0.8 - i4.0)\Delta a_0^2 \end{aligned}$$

$$\Delta a_0^0 = \frac{a_0^0 - 0.220}{0.005} \qquad \Delta a_0^2 = \frac{a_0^0 + 0.0444}{0.001}$$

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$$M_\sigma = 441\,^{+16}_{-8}\text{ MeV}\,, \quad \Gamma_\sigma = 544\,^{+18}_{-25}\text{ MeV}$$

Caprini, GC and Leutwyler, PRL (06)

Other recent σ-pole determinations

Other recent σ-pole determinations based on the same approach agree very well

	$\sqrt{s_\sigma}$ (MeV)
Caprini, GC, Leutwyler (06)	$441^{+16}_{-8} - i272^{+9}_{-12.5}$
Garcia-Martin, Pelaez, Yndurain (08)	$474 \pm 6 - i254 \pm 4$
Garcia-Martin, Kaminsky, Pelaez, Ruiz de Elvira (10)	$457^{+14}_{-13} - i279^{+11}_{-7}$
Moussallam (11)	$442^{+5}_{-8} - i274^{+6}_{-5}$

The PDG does listen...

$f_0(500)$ $I^G(J^{PC}) = 0^+(0^{++})$

also known as σ; was $f_0(600)$, $f_0(400\text{–}1200)$

See the review on "Scalar Mesons below 1 GeV."

Mass (T-Matrix Pole $\sqrt{s}$) = (400–550)$-i$(200–350) MeV
Mass (Breit-Wigner) = 400 to 800 MeV
Full width (Breit-Wigner) = 100 to 800 MeV

$f_0(500)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	–
$\gamma\gamma$	seen	–

Outline

Roy equations
- Meaning of $\bar{\ell}_3$ and $\bar{\ell}_4$
- Current determination of a_0^0, a_0^2 and $\delta_0^0(0.8\mathrm{GeV})$

Model-independent σ-pole determination

The σ in $\gamma\gamma \to \pi\pi$

The σ in other processes

- $\pi\pi$ scattering is the simplest scattering process involving two pions: the simplest where the σ can show up
- two pions in the final state with $I = J = 0$
  $\Rightarrow$ the σ is an intermediate state
- the broad bump due to the σ may look different
  [energy-dependence of the σ-coupling to the initial state?]
- BUT: (Message n.2 of this talk)
  the σ-pole does not move!
- main interest: determine the σ-coupling to the initial state
- rest of the talk: illustrate this in $\gamma\gamma \to \pi\pi$

$\gamma\gamma \to \pi\pi$: definitions

(see, e.g. Hoferichter, Phillips, Schat (11))

$$\langle \pi(p_1)\pi(p_2)|S|\gamma_{\lambda_1}(q_1)\gamma_{\lambda_2}(q_2)\rangle = ie^2\delta^4(Q-P)H_{\lambda_1\lambda_2}(s,t)e^{i(\lambda_1-\lambda_2)\varphi}$$

Partial-wave expansion

$d^J_{mm'}$ are the Wigner's functions

$$H_{++}(s,t) = \sum_J (2J+1)h_{J,-}(s), d^J_{20}(\theta)$$
$$H_{+-}(s,t) = \sum_J (2J+1)h_{J,+}(s), d^J_{00}(\theta)$$

Polarizabilities

$$\frac{2\alpha}{M_\pi s}H_{++}(s,M_\pi^2) = \alpha_1 - \beta_1 + \frac{s}{12}(\alpha 2 - \beta_2) + O(s^2)$$
$$-\frac{2\alpha}{M_\pi s}H_{+-}(s,M_\pi^2) = \alpha_1 + \beta_1 + \frac{s}{12}(\alpha 2 + \beta_2) + O(s^2)$$

$\gamma\gamma \to \pi\pi$: definitions

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The σ shows up in the partial wave

$$h_{0,+}(s)$$

The σ-pole in $\gamma\gamma \to \pi\pi$

Elastic unitarity:

$$\mathrm{Im} h_{J,\pm}(s) = \rho(s) h_{J,\pm}(s) t_J(s)^*$$

where $\rho(s) = \sqrt{1 - 4M_\pi^2/s}$
and $t_J(s)$ is the partial wave of $\pi\pi$ scattering

Analyticity:
$h_{J,\pm}(s)$ are analytic on the cut s-plane $(-\infty, 0]$ $[4M_\pi^2, \infty)$
poles are on the second Riemann sheet

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Oller, Roca, Schat PLB (08)

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Oller, Roca, Schat PLB (08)

A pole in $t_{II}(s)$ shows up also in $h_{II}(s)$:

$$t_{II}(s) \overset{s \sim s_\sigma}{\simeq} \frac{g^2_{\sigma\pi\pi}}{s_\sigma - s} \quad \Rightarrow \quad h_{II}(s) \overset{s \sim s_\sigma}{\simeq} \frac{g_{\sigma\gamma\gamma} g_{\sigma\pi\pi}}{s_\sigma - s}$$

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$g_{\sigma\gamma\gamma}$ can be calculated from $h_I(s_\sigma)$

Dispersion relations for $\gamma\gamma \to \pi\pi$

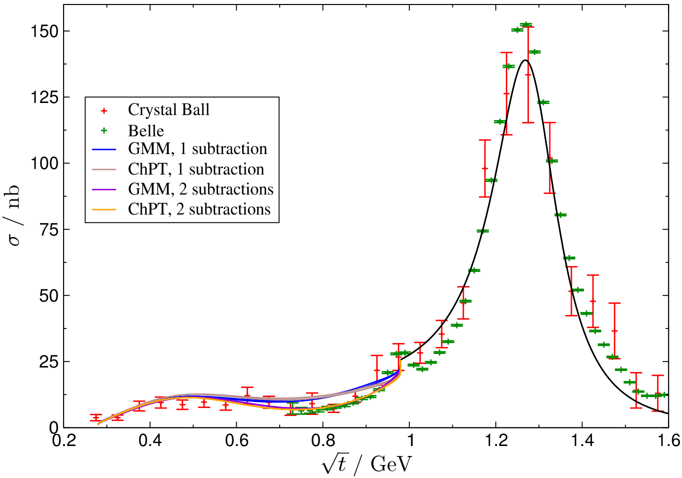
- the calculation of $h_I(s)$ for complex s is unambiguous from a dispersive representation
- recent dispersive analyses of $\gamma\gamma \to \pi\pi$:
  - Pennington (06)
  - Fil'kov Kashevarov (06)
  - Pennington, Mori, Uehara, Watanabe (08)
  - Bernabéu, Prades (08)
  - Oller, Roca, Schat (08)
  - Mennessier, Narison et al. (08,10)
  - García-Martín, Moussallam (10)
  - Hoferichter, Phillips, Schat (11)

Dispersion relations for $\gamma\gamma \to \pi\pi$

- the calculation of $h_I(s)$ for complex s is unambiguous from a dispersive representation

- following Hoferichter, Phillips, Schat (11):
- given the high-energy part, the low-energy behaviour is dictated by the subtr. constants=polarizabilities
- correlation between polarizabilities and $g_{\sigma\gamma\gamma}$
- χPT prediction for the polarizabilities $\Rightarrow$ prediction for $g_{\sigma\gamma\gamma}$
- a measurement of $\gamma\gamma \to \pi\pi$ at low energy provides a thorough test of these predictions

Results of dispersive analyses of $\gamma\gamma \to \pi\pi$



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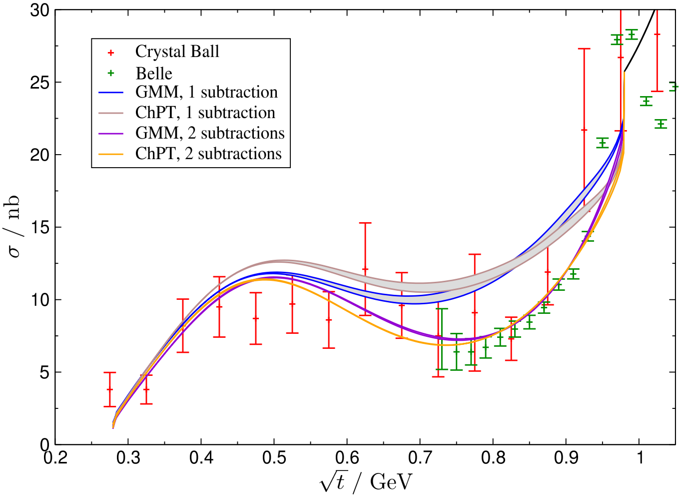

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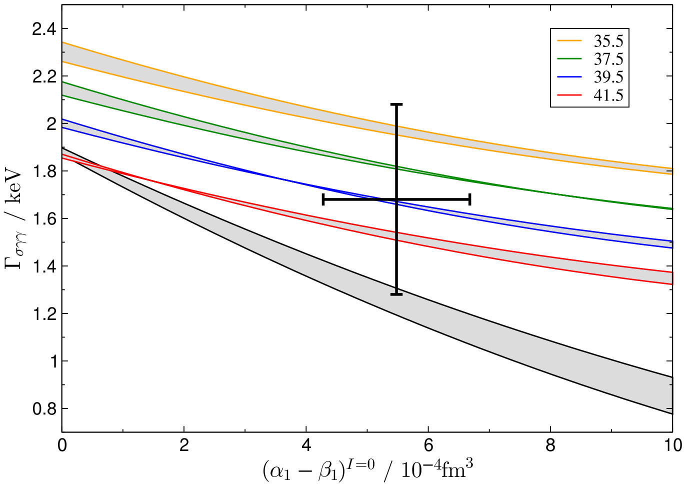

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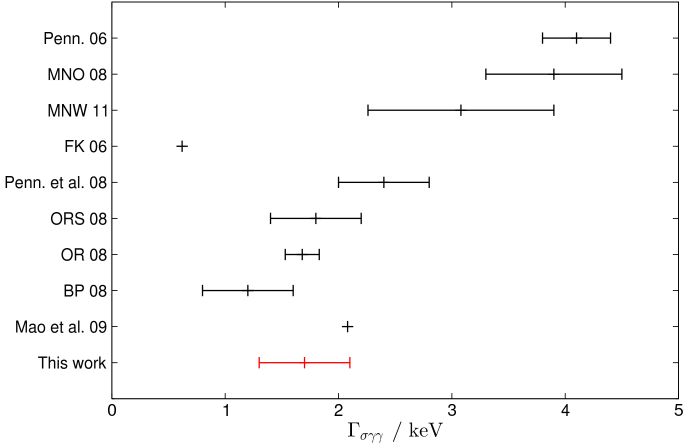

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