Introductory Lectures on Resurgence

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Continuum Foundations of Lattice Gauge Theories

Basic Introduction to Resurgence

1. - Stokes Phenomenon and Trans-series
   - Borel Summation basics
   - Recovering Non-perturbative Connection Formulas
2. - Nonlinear Stokes Phenomenon
   - Parametric Resurgence & Phase Transitions
   - Gross-Witten-Wadia unitary matrix model
3. - QFT: Euler-Heisenberg and Effective Field Theory
   - Resurgence analysis
   - Inhomogeneous fields
4. - Resurgent Extrapolation
   - The Physics of Padé Approximation
   - Probing the Borel Plane Numerically

Generic Bender-Wu-Lipatov "Factorial/Power" Large Order Behavior of Perturbation Theory

- consider a series that appears to be asymptotic, with generic leading large order behavior of the coefficients

$$c_n \sim \mathcal{S} \, \frac{\Gamma(a\, n + b)}{A^n} \left(1 + \frac{c}{n} + \dots\right) + \dots \qquad , \quad n \to \infty$$

- $a \to$ appropriate expansion variable
- $A \to$ location of the leading Borel singularity
- $b \to$ nature of the leading Borel singularity
- the Stokes constant $\mathcal{S} \to$ normalization
- c subleading power-law correction

There are systematic ways to probe the "data" (i.e. coefficients) to extract these important parameters ...

Richardson Acceleration (very useful for studying large-order growth)

- Suppose some series coefficients a_n have large-order behavior

$$a_n \sim b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \dots$$

- basic idea (*cf* "improved actions" in lattice theory)

$$(n+1)\, a_{n+1} - n\, a_n \sim b_0 + b_2 \left(\frac{1}{n+1} - \frac{1}{n}\right) + O\left(\frac{1}{n^3}\right) \sim b_0 - \frac{b_2}{n^2} + O\left(\frac{1}{n^3}\right)$$

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$$(n+1)\, a_{n+1} - n\, a_n \sim b_0 + b_2 \left(\frac{1}{n+1} - \frac{1}{n} \right) + O\left(\frac{1}{n^3}\right) \sim b_0 - \frac{b_2}{n^2} + O\left(\frac{1}{n^3}\right)$$

The pattern continues. At each order, multiply the a_{n+k} row by $(n+k)^k$ and form the following combination:

$$A_n := \sum_{k=0}^{N} \frac{(-1)^{k+N}\,(n+k)^N\, a_{n+k}}{k!(N-k)!}$$

Then

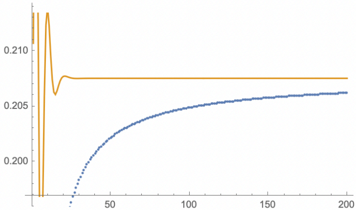
$$A_n \sim b_0 + O\left(\frac{1}{n^{N+1}}\right)$$

N is called the "order" of the Richardson acceleration.

- expansion coefficients: anomalous dimension in ϕ^3 theory in 6 dimensions

```
rat3[m_] := G[m] / 2 ^ m / Gamma[m + 1 / 2]

ListPlot[{Table[rat3[m], {m, 1, 200}], Table[Rich[rat3, 10, m], {m, 1, 200}]}, Joined → {False, True}]
```




```
Table[SetPrecision[Rich[rat3, 100, m], 50], {m, 180, 200}]

{0.20755374871029735167013412472066887260369024789112,
 0.20755374871029735167013412472066872179104163577257, 0.20755374871029735167013412472066863696493766490979,
 0.20755374871029735167013412472066860373635721060921, 0.20755374871029735167013412472066860368281499274783,
 0.20755374871029735167013412472066862038135988916965, 0.20755374871029735167013412472066864197661110094938,
 0.20755374871029735167013412472066866146294219978366, 0.20755374871029735167013412472066867577031981668954,
 0.20755374871029735167013412472066868445637006512487, 0.20755374871029735167013412472066868848788576865959,
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 0.20755374871029735167013412472066868238023213702442, 0.20755374871029735167013412472066868219487969160940,
 0.20755374871029735167013412472066868219897105755438, 0.20755374871029735167013412472066868229771959168840}
```

Richardson Acceleration (very useful for studying large-order growth)

- what is this number?

$$0.2075537487102973516701341247206686 82\ldots$$

```
WolframAlpha["0.20755374871"]
```

Input interpretation:

0.20755374871

Number line:

Rational form:

$$\frac{20\,755\,374\,871}{100\,000\,000\,000}$$

Number name:

zero point two zero seven five five three seven four eight seven one

Continued fraction: Fraction form

[0; 4, 1, 4, 2, 53, 1, 1, 1, 14]

Possible closed forms: More

$$\frac{1}{e\sqrt{\pi}} \approx 0.20755374871\,02973$$

$$\frac{4}{11}\pi \sinh^{-1}\left(\frac{839}{1910}\right)^2 \approx 0.20755374871\,70097$$

$$\frac{11\,\Phi}{900} + \frac{1}{5} \approx 0.20755374875\,1387$$

sinh^(−1)(x) is the inverse hyperbolic sine function »

Φ is the golden ratio conjugate »

WolframAlpha

Exercise 4.1: In resurgence it is convenient to re-write a factorial large order growth expression, with power-law corrections, as an expansion in "diminishing" factorials:

$$b_n \sim \Gamma(n+a) \sum_{m=0}^{\infty} \frac{d_m}{n^m} \qquad , \quad n \to +\infty$$

can be written as

$$b_n \sim \sum_{k=0}^{\infty} c_k \, \Gamma(n+a-k) \qquad , \quad n \to +\infty$$

where the coefficients c_k are expressed in terms of the d_m via the Stirling numbers of the first kind (hint: dlmf.26.8.ii):

$$c_k = \sum_{l=0}^{k} S^{(1)}(k,l) \sum_{j=0}^{l} (-a)^l \binom{j-l}{j} d_{l-j}$$

Verify this with some examples.

Exercise 4.2: Given the explicit formula for c_n in terms of gamma functions, it is easy to derive the large-order behavior analytically. But in the absence of an explicit formula this can still be done numerically. Try this for the Airy function:

1. Generate the first 100 c_n coefficients, and use ratio tests and Richardson acceleration to deduce *numerically* the leading large order asymptotics shown on the previous slide.
2. Extract *numerically* the first two subleading power-law corrections to the large order growth.

Exercise 4.3: The perturbative expansion $C(x) \sim \sum_{n=1}^{\infty} c_n\, x^n$ determines the anomalous dimension in the Hopf algebraic renormalization of 4 dimensional Yukawa theory. The coefficients c_n are positive integers, enumerating combinatorial objects known as "connected chord diagrams". This sequence is listed on the OEIS as https://oeis.org/A000699.

1. Generate 100 terms using the recursion formula listed on the OEIS and then analyze them using Richardson acceleration to show that
$$c_n \sim \frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{e\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}+O\left(\frac{1}{n^3}\right)\right)$$
2. $C(x)$ satisfies a nonlinear ODE: $C(x)\left(1-2x\frac{d}{dx}\right)C(x)=x-C(x)$. Show that the first non-perturbative correction term $C_{\rm np}(x)$ satisfies a linear ODE $\frac{d}{dx}\ln\left(C(x)C_{\rm np}(x)/x\right)=\frac{1}{2xC(x)}$. Therefore $C_{\rm np}(x)$ is immediately solved in terms of $C(x)$.
3. Hence expand $C_{\rm np}(x)$ at small x and compare with part 1.

Darboux Theorem

- Darboux theorem

$$f(t) \sim \phi(t)\left(1-\frac{t}{t_0}\right)^{-\beta} + \psi(t) \qquad , \quad t \to t_0$$

- large-order growth of Taylor coefficients of $f(t)$ at origin:

$$b_n \sim \frac{\binom{n+\beta-1}{n}}{t_0^n}\left[\phi(t_0) - \frac{(\beta-1)\,t_0\,\phi'(t_0)}{(n+\beta-1)} + \frac{(\beta-1)(\beta-2)\,t_0^2\,\phi''(t_0)}{2!(n+\beta-1)(n+\beta-2)} - \ldots\right.$$

- log branch cut $\Rightarrow$

$$b_n \sim \frac{1}{t_0^n}\cdot\frac{1}{n}\left[\phi(t_0) - \frac{t_0\,\phi'(t_0)}{(n-1)} + \frac{t_0^2\,\phi''(t_0)}{(n-1)(n-2)} - \ldots\right]$$

- in Borel plane $\Rightarrow$ large-order/low-order resurgence relations

- large-order growth encodes details of Borel singularities

Exercise 4.4:

1. investigate Darboux's theorem numerically for the hypergeometric function, which has a branch point at $t = 1$

$$ {}_2F_1\left(a, b, c; t\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)n!} t^n $$

   Experiment with various choices for the parameters a, b, c.

2. Compare with the exact expansion of the hypergeometric function about $t = 1$ (see dlmf.15.8)

3. What happens if $a + b - c =$ integer?

• conclusion: it can make a BIG difference how we sum

• processing the same perturbative input data in different ways can lead to vastly different levels of precision

• the basic toolkit: ratio tests, series acceleration methods (*e.g.* Richardson), Padé approximants, orthogonal polynomials, Szegö asymptotics, continued fractions, conformal maps, uniformization maps, ...

• the good news: many of these are actually very easy to implement → a simple set of *exploratory* procedures

• the quality of the extrapolation of an asymptotic series is governed by the quality of the analytic continuation of the Borel transform

$$f(x) = \int_0^\infty dt\, e^{-x\,t} \mathcal{B}[f](t)$$

• lesson 1: it is better to work in the Borel plane

Padé-Borel Method

- given a FINITE number of terms of an asymptotic series

$$\sum_{n=0}^{2N} \frac{c_n}{x^{n+1}} = \int_0^\infty dt\, e^{-xt} \sum_{n=0}^{2N} \frac{c_n}{n!}\, t^n = \int_0^\infty dt\, e^{-xt}\, \mathcal{B}_{2N}[f](t)$$

- recall that the singularities of $\mathcal{B}_{2N}[f](t)$ determine the non-perturbative physics

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- but $\mathcal{B}_{2N}[f](t)$ is a polynomial !

- as $N \to \infty$, $\mathcal{B}_{2N}[f](t)$ may develop singularities

- Padé = simple & powerful extrapolation method

- Padé is an excellent "low resolution" detector of singularity structures: "Padé-Borel" method

Basics of Padé Approximation

- simple and efficient method to analytically continue a series beyond its radius of convergence
- *rational* approximation to function $F_{2N}(t) = \sum_n^{2N} c_n \, t^n$

$$P_{[L,M]} \left\{ F_{2N} \right\} (t) = \frac{R_L(t)}{S_M(t)} = \sum_n^{2N} c_n \, t^n + \mathcal{O}\left(t^{2N+1}\right)$$

- completely algorithmic and algebraic (“built-in”)

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• completely algorithmic and algebraic ("built-in")

• near-diagonal Padé: polynomials $R_N(t)$ & $S_N(t)$ satisfy the same 3-term recursion relation

• hence a deep connection to orthogonal polynomials, and their asymptotics (Szegö ...)

$$\frac{R_L(t)}{S_M(t)} - \frac{R_{L+1}(t)}{S_{M+1}(t)} = (\#)\frac{t^{M+L+1}}{R_L(t)} S_{M+1}(t)$$

Basics of Padé Approximation

1. at very high orders Padé can be numerically unstable: a ratio of polynomials with very large coefficients. It is often more stable to convert to an 'equivalent' partial fraction

$$\sum_n^{2N} c_n t^n \quad \leftrightarrow \quad \frac{\sum_n^N a_n\, t^n}{\sum_n^N d_n\, t^n} \quad \leftrightarrow \quad \sum_n^N \frac{r_n}{t - t_n}$$

2. truncated series $\rightarrow$ continued fraction, which often converges in all of $\mathbb{C}$, minus a number of poles/cuts

$$1 + c_1\, t + \cdots + c_{2N}\, t^{2N} + \mathcal{O}\left(t^{2N+1}\right) = \cfrac{1}{1 + \cfrac{h_1\, t}{1 + \cfrac{h_2\, t}{1 + \cfrac{h_3\, t}{\ddots\, 1 + h_{2N}\, t}}}} + \mathcal{O}\left(t^{2N+1}\right)$$

3. near-diagonal Padé are related to continued fractions

Basics of Padé Approximation

- a Padé approximant only has *pole* singularities. But for practical applications we are often interested in *branch point* singularities, *e.g.* Borel plane, critical exponents, ...

- branch point $\rightarrow$ accumulation point of a line of poles

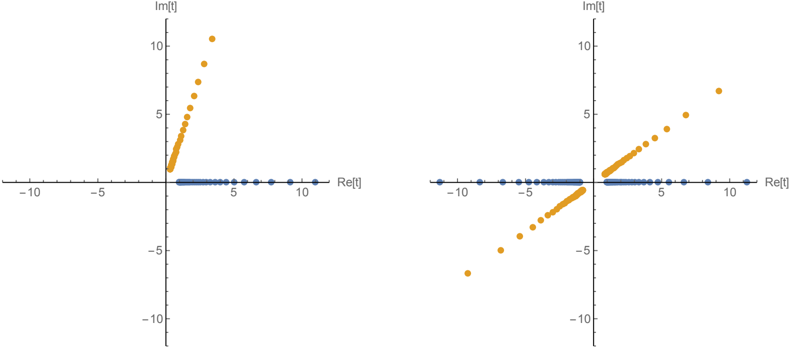


Figure: Padé poles of $\frac{1}{(1-t)^{\frac{1}{5}}}$, $\frac{1}{(e^{\frac{2\pi i}{5}}-t)^{\frac{1}{7}}}$, and of $\frac{1}{(1-t^2)^{\frac{1}{5}}}$, $\frac{1}{(e^{\frac{2\pi i}{5}}-t^2)^{\frac{1}{7}}}$.

Basics of Padé Approximation

- symmetry of singularities is important for Padé
- in applications: complex conjugate pair of singularities

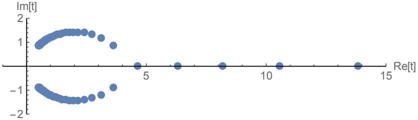


Figure: Padé poles of $1/(1-t+t^2)^{1/5}$

- poles on the real t axis !?

- in applications: complex conjugate pair of singularities

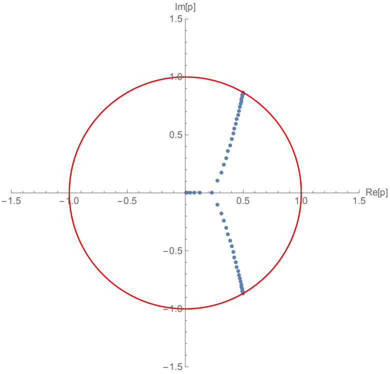


Figure: $1/t$ of Padé poles of $1/(1-t+t^2)^{1/5}$

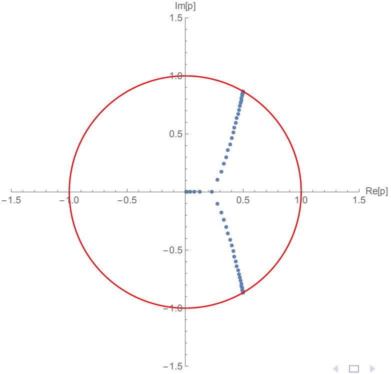
- the point at infinity is also a branch point

The Physics of Padé Approximation

• Stahl (1997): in the limit $N \to \infty$ Padé produces the "minimal capacitor"

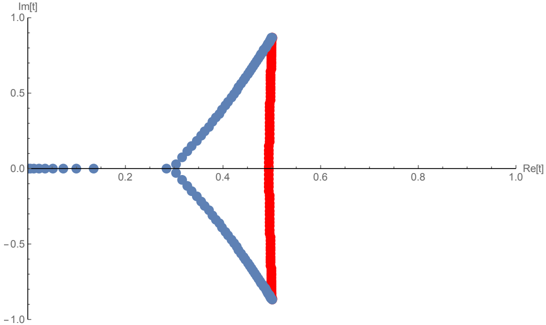
• poles = charges; in the limit they form flexible "wires", fixed at the actual singularities

• "wires" and junction points deform to minimize the capacitance



- amazing sensitivity to exponent $\frac{1}{2}$

inverse Padé poles of $\frac{1}{(1-t+t^2)^{\frac{1}{2}}}$ & $\frac{1}{(1-t+t^2)^{\frac{1}{2}-10^{-6}}}$

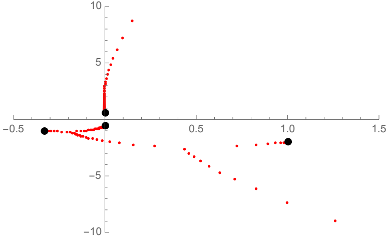


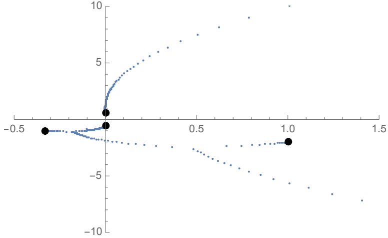
- this sensitivity can be used to our advantage

The Physics of Padé Approximants

- Padé solves a 2d electrostatics problem

- it effectively generates its own conformal map: cuts are deformable wires connecting fixed singularities, in such a way that the capacitance is minimized

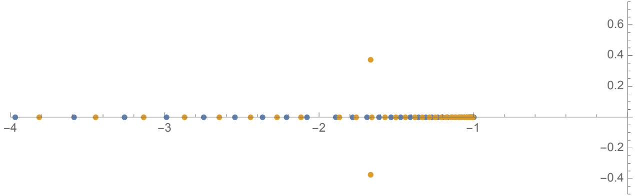




- use physical intuition to probe the true singularities

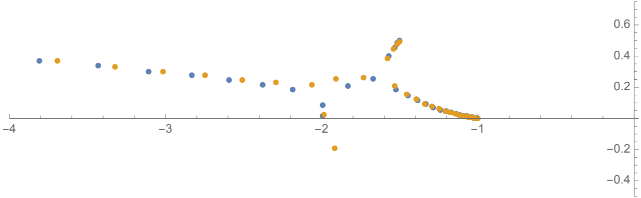
The Physics of Padé Approximation

- collinear singularities occur frequently (e.g. multi-instanton Borel singularities): e.g. $(1+t)^{-1/3} + \log(t+2)$



- insert a "probe charge": $\left(\frac{3}{2} - \frac{i}{2} + t\right)^{-1/7}$

- wires deform while genuine singularities remain fixed



Exercise 4.5: Explore the Padé pole structures for various functions with interesting singularities.

The Physics of Padé Approximation

- dramatic improvement: conformal map, then Padé

- even a conformal map based on leading singularities makes a big difference: (Costin, GD: 2003.07451, 2108.01145)

- 2 dim electrostatics $\leftrightarrow$ conformal mapping

- Padé is "trying to make a conformal map"

Problems with Padé Approximation

• Padé approximants are not very accurate near a branch point and branch cut, because of accumulating poles

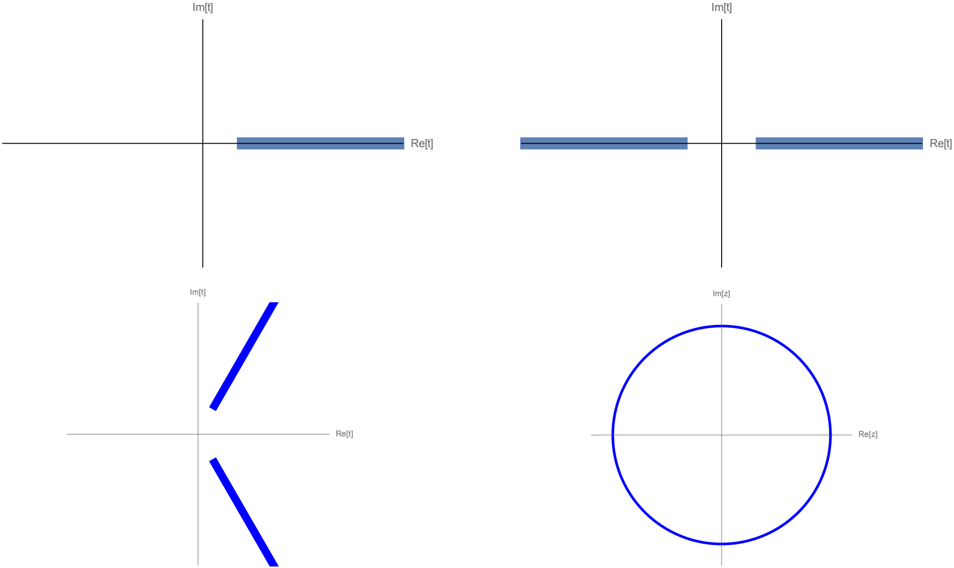
• Padé obscures repeated repeated singularities along the same line; but this is exactly what happens for resurgent Borel transforms in nonlinear problems

• Padé generically produces spurious poles and zeros ("Froissart doublets"): ultimately due to lack of a true Hilbert space for the complex orthogonal polynomials

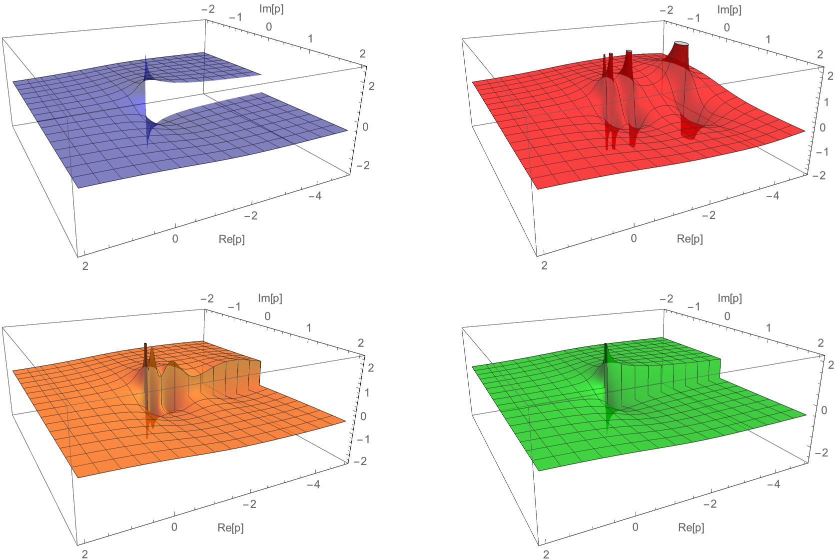
• conformal & uniformizing maps can help to overcome these problems

Conformal Mapping

- conformal map of cut domains: $t \to z$

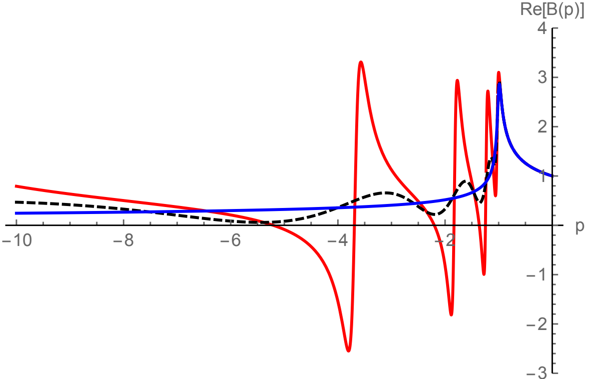


Padé-Conformal-Borel: 10-term approximation to $(1+p)^{-1/3}$



Padé-Conformal-Borel: 10-term approximation to $(1+p)^{-1/3}$

- precision of the Borel transform along the cut



blue: exact and Padé-Conformal-Borel approx. (10 terms)
red: Padé-Borel approx. (10 terms)
black-dashed: Conformal-Borel approx. (10 terms)

- Padé-Conformal-Borel is generically much more accurate near the singularity and along the cut

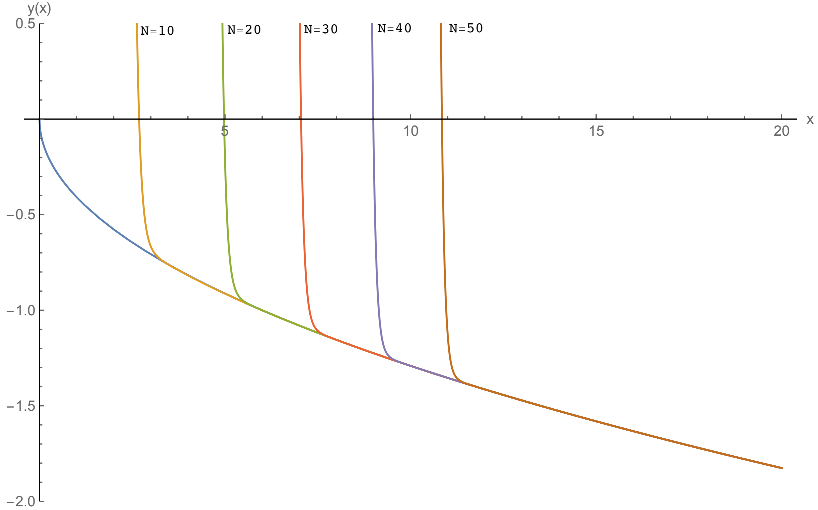
- the improved precision can be quantified

$$c_n \sim \Gamma(n+\alpha) \Rightarrow \quad \begin{array}{ll} \text{frac. error} \sim e^{-(N\,x)^{1/2}} & x\text{-Padé} \\ \text{frac. error} \sim e^{-(N^2 x)^{1/3}} & \text{Padé-Borel} \\ \text{frac. error} \sim e^{-(N^2 x)^{1/3}} & \text{Taylor-Conformal-Borel} \\ \text{frac. error} \sim e^{-(N^4 x)^{1/5}} & \text{Padé-Conformal-Borel} \end{array}$$

- for a chosen precision (*e.g.*, 1% accuracy), with exactly the same input data we can extrapolate from an N-term expansion at $x=+\infty$ down to $x_{\rm min}\to 0^+$, scaling with N as:

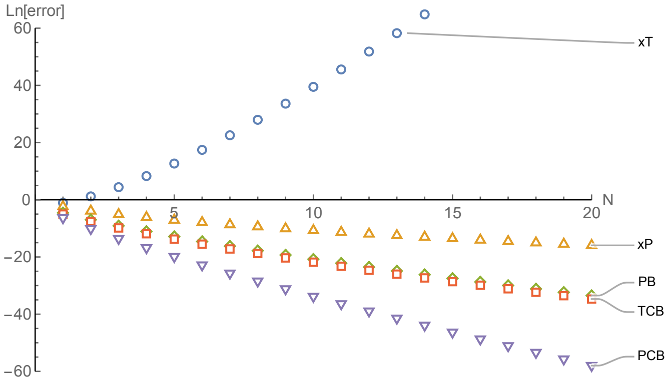
extrapolation	$x_{\rm min}$ scaling
truncated series	$x_{\rm min} \sim N$
x Padé	$x_{\rm min} \sim N^{-1}$
Padé-Borel	$x_{\rm min} \sim N^{-2}$
Conformal-Borel	$x_{\rm min} \sim N^{-2}$
Padé-Conformal-Borel	$x_{\rm min} \sim N^{-4}$

Perturbative Large x Expansion is an Asymptotic Series



- typical asymptotic series: larger N gets worse at small x

More is Better



• log of the fractional error for different approximations of $x^{-\frac{2}{3}}\, e^{x}\, \Gamma\left(\frac{2}{3}, x\right)$ at $x = 1$ as a function of truncation order

- another advantage of the conformal mapping is that it resolves repeated resurgent Borel singularities

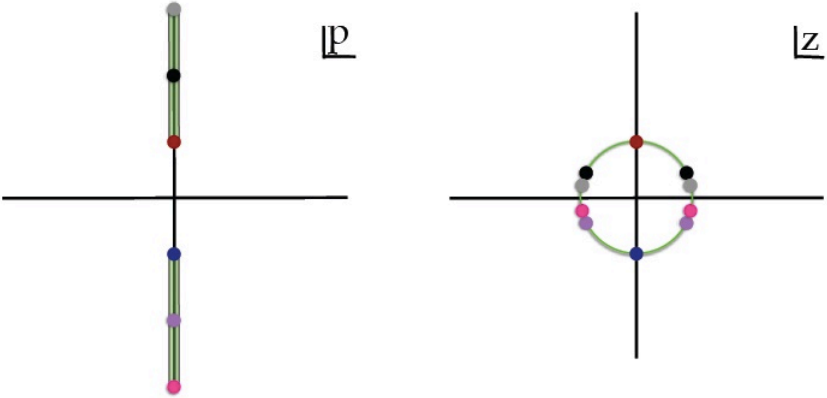
- recall that in a nonlinear problem a Borel singularity is expected to be repeated in (certain) integer multiples

- these can be obscured by Padé's attempt to represent a cut as an accumulation of poles

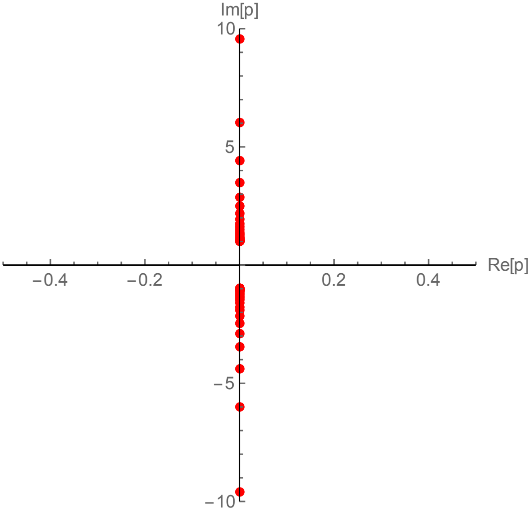
Conformal Mapping of Borel Plane

- map the doubly-cut Borel p plane to the unit disc

$$z = \frac{p}{1 + \sqrt{1 + p^2}} \qquad \longleftrightarrow \qquad p = \frac{2\,z}{1 - z^2}$$

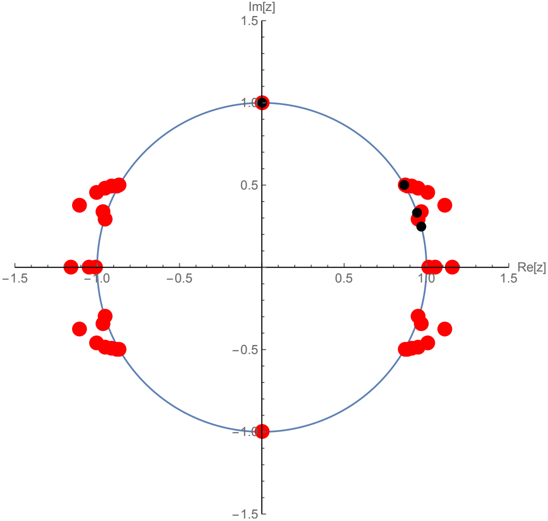


Borel plane singularities: Padé-Borel transform



- before conformal map: resurgence is hidden

After Conformal Map: Resurgent Poles in z Plane



• Conformal map reveals resurgent structure in Borel plane: Borel singularities separated into their resurgent patterns

Optimal Method: Uniformizing Map

• an even better procedure: replace the conformal map with a uniformizing map

• Optimality Theorem (O. Costin & GD, 2009.01962): given information about the Riemann surface of the Borel transform (known in many cases for resurgent functions), the optimal extrapolation procedure is to use a uniformizing map.

• super-precise exploratory tool

• singularity elimination: exponential enhancement in the vicinity of the singularities

• application: sensitive determination of the location and nature of a singularity, and its 'Stokes constant'

• permits extrapolation onto higher Riemann sheets

Ultra-Precise Probing the Neighborhood of an Isolated Singularity

- exponential distortion near a uniformized singularity
- ${}_2F_1(\frac{1}{6}, \frac{5}{6}, 1; \omega)$ with elliptic nome function (20 terms)

$$z = \exp\left[-\pi \frac{\mathbb{K}(1-\omega)}{\mathbb{K}(\omega)}\right] \quad \longleftrightarrow \quad \omega = \varphi(z) = 16z - 128z^2 + 704z^3 + \ldots$$

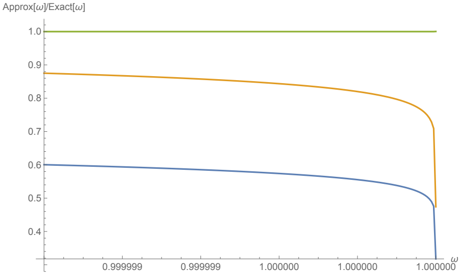
- uniformizing:

$\omega \approx 1 - 10^{-40} \longleftrightarrow z \approx 0.9$

- conformal:

$\omega \approx 1 - 10^{-40} \longleftrightarrow z \approx 1 - 10^{-20}$

$\omega \approx 1 - 10^{-3} \longleftrightarrow z \approx 0.9$



- practical applications: Stokes constant and exponent

- "Singularity elimination" method (O. Costin & GD, 2009.01962)
- extreme sensitivity to the **location** and **exponent** of the singularity
- a fractional derivative (a linear transform) can adjust the exponent to whatever you want
- change to the exponent you suspect: then make a conformal or uniformizing map
- if you were correct, the singularity is removed
- now you can expand there, and move on to another singularity

Exercise 4.6: Generate a finite amount (e.g. 50 orders) of perturbative data by expanding the function $x^{-\frac{1}{3}}\, e^{x}\, \Gamma\left(\frac{1}{3}, x\right)$ as $x \to +\infty$, and use this as input for resummation using the following methods:
(i) optimal truncation; (ii) Padé in the x plane; (iii) Padé in the Borel plane; (iv) Padé in the Borel plane after a conformal map into the unit disc; (v) Padé in the Borel plane after a uniformizing map.
Explore how things change as you change the amount of perturbative data.
Comment on the similarities and differences between the resulting reconstructions of the function.

Exercise 4.7: Generate a finite amount (e.g. 20 orders) of perturbative data by expanding the Borel transform $B(p)$ of the Airy function: $B(p) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -p\right)$, and use this as input for extrapolation using the following methods. Probe specifically the vicinity of the branch point at $p = -1$ and cut $p \in (-\infty, -1]$

1. Padé
2. Padé after a conformal map into the unit disc

$$p = \frac{4z}{(1-z)^2} \quad \longleftrightarrow \quad z = \frac{\sqrt{1+p}-1}{\sqrt{1+p}+1}$$

3. Padé after a uniformizing map via the elliptic nome function

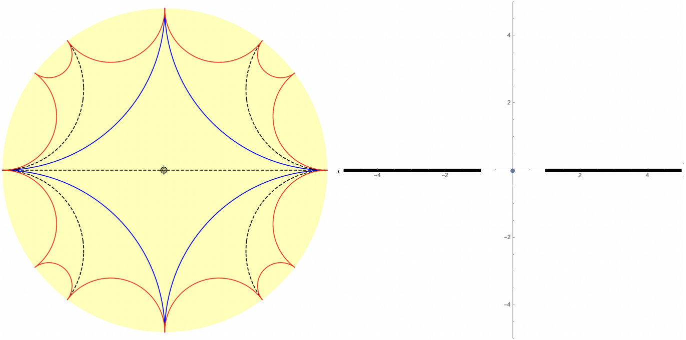
$$p = -\varphi(z) = -16z + 128z^2 - 704z^3 + \ldots \longleftrightarrow z = \exp\left[-\pi \frac{\mathbb{K}(1+p)}{\mathbb{K}(-p)}\right]$$

where $\varphi(z) = \text{InverseEllipticNomeQ}[z]$ in Mathematica.

Exploring Different Riemann Sheets

- uniformization of $\hat{\mathbb{C}} \setminus \{-1, 1, \infty\}$: modular λ function

$$w(z) = -1 + 2\lambda\left(i\left(\frac{1+i\,z}{1-i\,z}\right)\right)$$



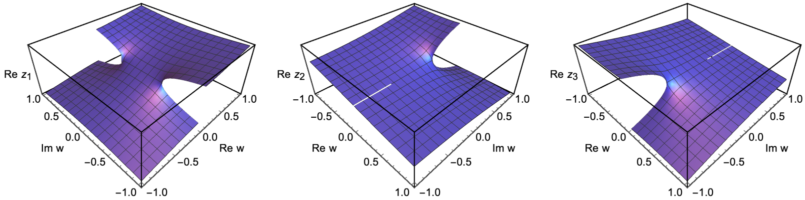
- interactive Mathematica file for uniformization
- physics : crossing Riemann sheets near critical points

Uniformizing Lee-Yang Zeros (Basar, GD, Yin 2112.14269)

- chiral random matrix model for QCD (Halasz et al 1998 ...)
- 3d Ising universality class

$$\text{effective potential:} \quad \Omega = -h\, M + \frac{r}{2} M^2 + \frac{1}{4} M^4$$

- near critical point: scaling $w := h r^{-\beta\delta}$, $z := M r^{-\beta}$
- mean field ($\beta = \frac{1}{2}$, $\delta = 3$): $\frac{\partial \Omega}{\partial M} = 0 \Rightarrow \boxed{w = z + z^3}$
- three sheets: $z_1(w)$ = high T sheet; $z_2(w)$ = low T sheet; $[z_3(w) = -z_2(-w)]$

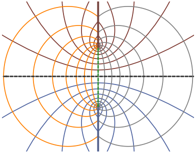


High T Equation of State: Extrapolation on First Riemann Sheet

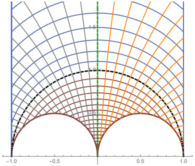
- high T expansion: $z_1(w) = w - w^3 + 3w^5 - 12w^7 + \dots$
- uniformization map: $\lambda(\tau)$= modular lambda function

$$w(\tau) = i(-1 + 2\lambda(\tau)) \qquad ; \qquad \tau(\zeta) = i\left(\frac{1 + i\,\zeta}{1 - i\,\zeta}\right)$$

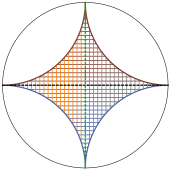
high T: w plane



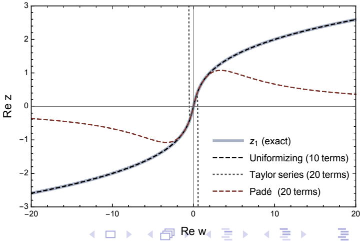
high T: τ plane



high T: ζ plane



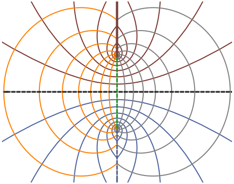
- extrapolation based on 10 terms of the high T expansion



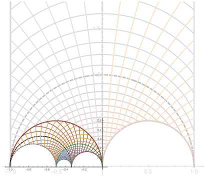
(Basar, GD, Yin 2112.14269)

Equation of State: Continuation to Low Temperature Sheet

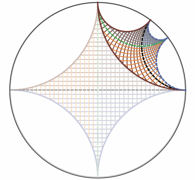
low T: w plane



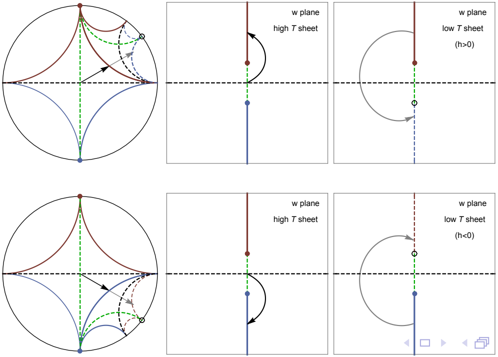
low T: τ plane



low T: ζ plane

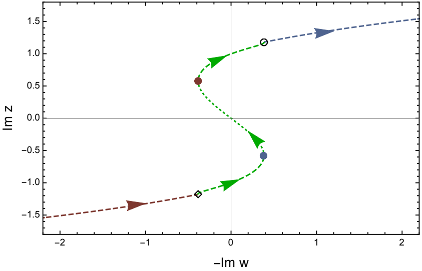


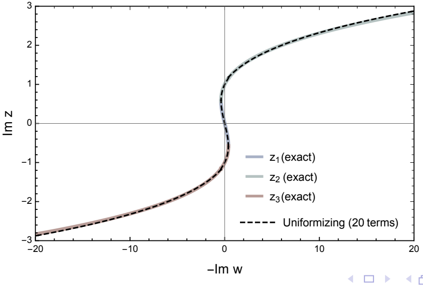
- traverse between sheets by moving in unit ζ disk



Equation of State: Continuation to Low T Riemann Sheet

- Padé in ζ plane $\to$ reconstruct function on low T sheet





Application: Heisenberg-Euler Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l}\mathfrak{E}, \mathfrak{B} \quad \text{Kraft auf das Elektron.} \\ |\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.}\end{array}\right)$$

- the first (non-perturbative) QFT computation
- paradigm of "effective field theory" (non-linear)
- compute: $\ln\det(\not{D}+m) \quad , \quad \not{D} := \not{\partial} + ie\not{A}$
- E field: $\mathrm{Im}\left[\mathcal{L}\left(\alpha, \frac{eE}{m^2}\right)\right] \sim \frac{\alpha E^2}{2\pi^2}\, e^{-\pi\, m^2/(eE)}$
- generating function for multi-leg one-loop amplitudes

Extrapolating Heisenberg-Euler

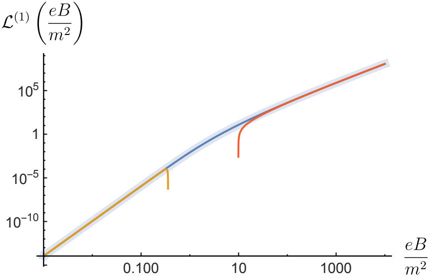
Florio 1911.03489, GD/Harris 2101.10409

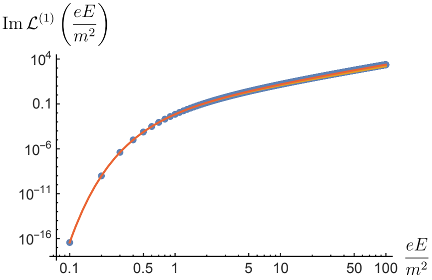
$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)}$$

$$\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}}\zeta(2n+4)\left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2$$

$$\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \dots \quad , \quad eB \gg m^2$$

- small B → large B; small B → large E (from 10 terms!)

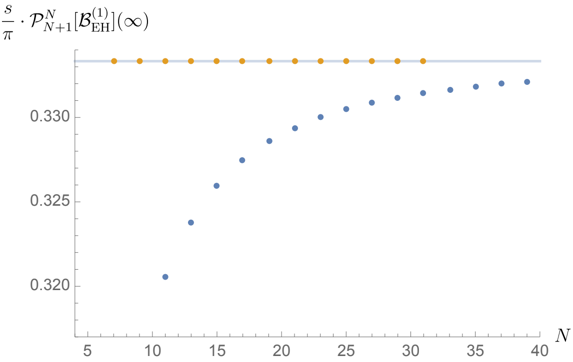




- exponentially suppressed terms are also accessible
- also at 2 loop (no Borel representation) (GD/Harris 2101.10409)

- the leading strong-field limit coefficient can be extracted from the weak field expansion

$$\mathcal{L}^{(1)} \sim \frac{B^2}{2}\,\alpha\,\beta_1\,\ln\left(\frac{eB}{m^2}\right) + \dots$$



Inhomogeneous Background Fields (GD & Z. Harris, 2212.04599)

- locally-constant-field-approximation (LCFA) is heavily used in plasma and astro intense-field simulation codes, but it is known to fail badly for very inhomogeneous fields

- precision tests for soluble cases (Narozhnyi/Nikishov, Popov, ...)

$$B(x) = B\,\text{sech}^2\,(x/\lambda) \qquad E(t) = E\,\text{sech}^2\,(t/\tau)$$

- analytic continuations: $B^2 \mapsto -E^2$, $\lambda^2 \mapsto -\tau^2$

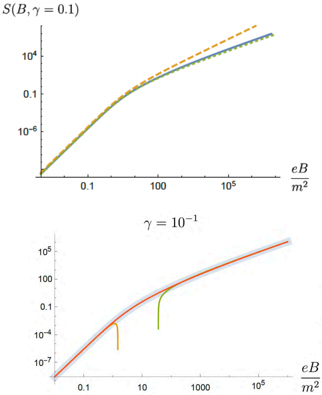
- Keldysh inhomogeneity parameter $\gamma = \frac{m}{eB\lambda} \mapsto \frac{m}{eE\tau}$

- WKB approximation: (Popov, Marinov, ...)

$$\text{Im}\,[S(E,\omega)]_{\text{WKB}} \sim L^3 \frac{m^4\tau}{8\pi^3}\left(\frac{eE}{m^2}\right)^{5/2}(1+\gamma^2)^{5/4}\exp\left[-\frac{\pi m^2}{eE}\frac{2}{\sqrt{1+\gamma^2}+1}\right]$$

Resurgent Extrapolation for Inhomogeneous Background Fields

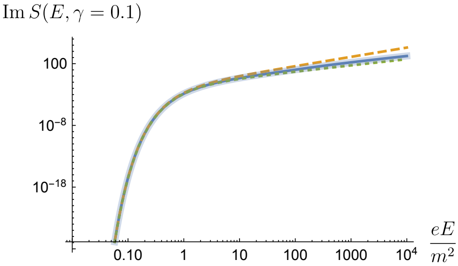
- analytic continuation: weak B field to strong B field
- with just 15 perturbative input terms

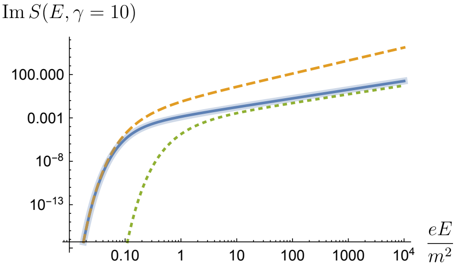


- accurate agreement over many orders of magnitude
- superior to locally-constant-field approximation or WKB

Resurgent Extrapolation for Inhomogeneous Background Fields

- analytic continuation: <u>weak B field to strong E field</u>
- with just 15 perturbative input terms





- accurate agreement over many orders of magnitude
- far superior to WKB or LCFA