# Dispersive approach to the hadronic light-by-light contribution to $(g-2)_\mu$

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FOR FUNDAMENTAL PHYSICS

School on continuum foundations, CERN, 26.7.24

Based on:

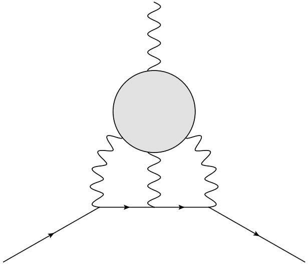
JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 ........................................................ +B. Kubis

# Outline

Setting up the stage: Master Formula

A dispersion relation for HLbL
- Pion-pole contribution
- Pion-box contribution
- Pion rescattering contribution

# Hadronic light-by-light: source of theory uncertainty



- 4-point fct. of em currents in QCD
- "*it cannot be expressed in terms of measurable quantities*"
- $<$ 2014: only model calculations
- $>$ 2014: dispersive approach
- lattice QCD is now also a player

# Different analytic evaluations of HLbL

Jegerlehner-Nyffeler 2009

| Contribution | BPaP(96) | HKS(96) | KnN(02) | MV(04) | BP(07) | PdRV(09) | N/JN(09) |
|---|---|---|---|---|---|---|---|
| $\pi^0, \eta, \eta'$ | 85±13 | 82.7±6.4 | 83±12 | 114±10 | – | 114±13 | 99±16 |
| $\pi$, $K$ loops | −19±13 | −4.5±8.1 | – | – | – | −19±19 | −19±13 |
| " " + subl. in $N_c$ | – | – | – | 0±10 | – | – | – |
| axial vectors | 2.5±1.0 | 1.7±1.7 | – | 22± 5 | – | 15±10 | 22± 5 |
| scalars | −6.8±2.0 | – | – | – | – | −7± 7 | −7± 2 |
| quark loops | 21± 3 | 9.7±11.1 | – | – | – | 2.3 | 21± 3 |
| total | 83±32 | 89.6±15.4 | 80±40 | 136±25 | 110±40 | 105±26 | 116±39 |

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*$K$s are subdominant*, see below)
- heavier single-particle poles decreasingly important

# Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible
  (in principle)
- if data not available: use theoretical calculations of subamplitudes, short-distance constraints etc.

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- First attempts:
  GC, Hoferichter, Procura, Stoffer (14)
  Pauk, Vanderhaeghen (14)
- why hasn't this been adopted before?

# The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

# Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

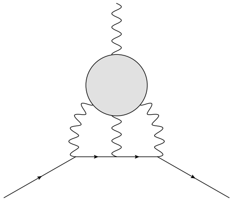
- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

# Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the $\Pi_i$
- the $\Pi_i$ are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques



GC, Hoferichter, Procura, Stoffer (2015)

# Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where $Q_i^\mu$ are the Wick-rotated four-momenta and $\tau$ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

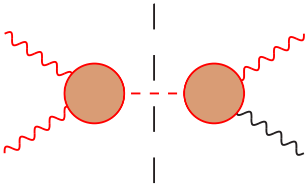
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

# Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = $\delta$-functions

Projection on the BTT basis: easy ✓

Our master formula=explicit expressions in the literature ✓

Input: pion transition form factor Hoferichter et al. (18)

First results of direct lattice calculations Gerardin, Meyer, Nyffeler (16)

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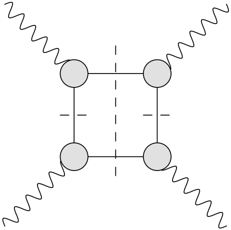
$\pi$-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole

– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

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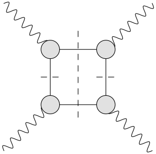
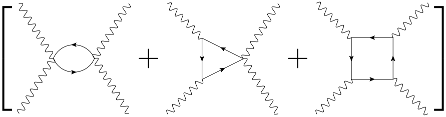


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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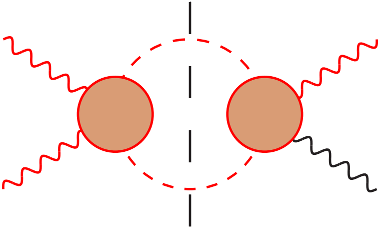

The "rest" with $2\pi$ intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

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E.g. $\gamma^*\gamma^* \to \pi\pi$ $S$-wave contributions

$$\hat{\Pi}^S_4 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s' \,\mathrm{Im} h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}^S_5 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t' \,\mathrm{Im} h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}^S_6 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u' \,\mathrm{Im} h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}^S_{11} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}^S_{16} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}^S_{17} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

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We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states are neglected in first approximation

of course, the $\eta$, $\eta'$ and other pseudoscalars pole contribution, or the kaon-box/rescattering contribution can be calculated within the same formalism

# Pion-pole contribution

- Expression of this contribution in terms of the pion transition form factor already known Knecht-Nyffeler (01)
- Both transition form factors (TFF) must be included:

$$\bar{\Pi}_1 = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2}$$

[dropping one bc short-distance not correct Melnikov-Vainshtein (04) ]

- data on singly-virtual TFF available CELLO, CLEO, BaBar, Belle, BESIII
- several calculations of the transition form factors in the literature Masjuan & Sanchez-Puertas (17), Eichmann et al. (17), Guevara et al. (18)
- dispersive approach works here too Hoferichter et al. (18)
- quantity where lattice calculations can have a significant impact Gerardin, Meyer, Nyffeler (16)

# Pion-pole contribution

Latest complete analyses:

- Dispersive calculation of the pion TFF  Hoferichter et al. (18)

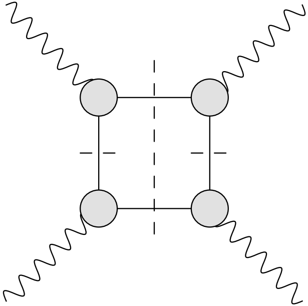
$$10^{11} a_\mu^{\pi^0} = 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\left({}^{2.2}_{1.4}\right)_{\text{BL}}(0.5)_{\text{asym}} = 62.6^{+3.0}_{-2.5}$$

- Padé-Canterbury approximants  Masjuan & Sanchez-Puertas (17)

$$10^{11} a_\mu^{\pi^0} = 63.6(1.3)_{\text{stat}}(0.6)_{a_{P;1,1}}(2.3)_{\text{sys}} = 63.6(2.7)$$

# Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



# Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

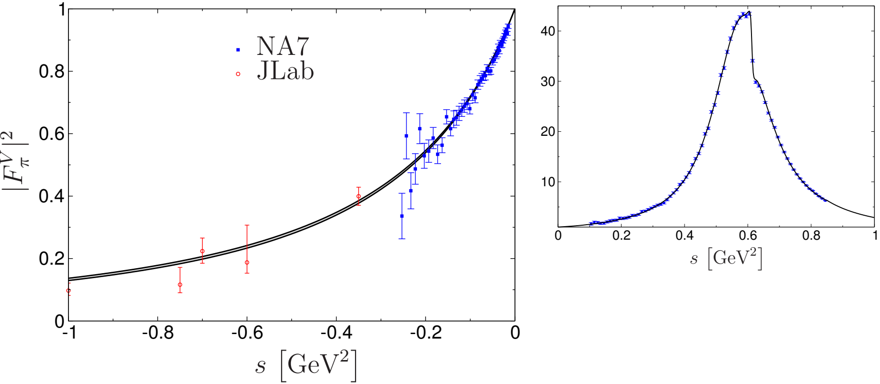
where

$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

# Pion-box contribution




Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

# Pion-box contribution

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| axial vectors | 2.5±1.0 | 1.7±1.7 | – | 22± 5 | – | 15±10 | 22± 5 |
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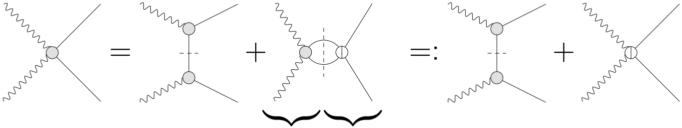
# First evaluation of *S*- wave $2\pi$-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:




Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to $\infty$]
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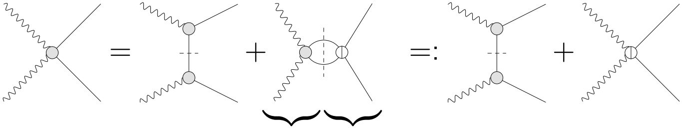


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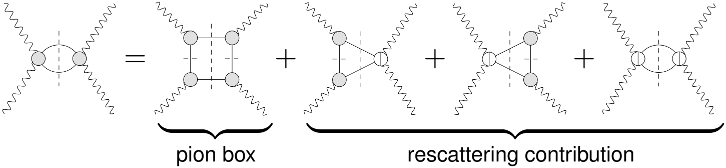
*S*-wave contributions : $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

# Two-pion contribution to $(g-2)_\mu$ from HLbL

Two-pion contributions to HLbL:




$$a_\mu^{\pi-\mathrm{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \cdot 10^{-11}$$

# $\gamma^*\gamma^* \to \pi\pi$ contribution from other partial waves

- formulae get significantly more involved with several subtleties in the calculation
- in particular sum rules which link different partial waves must be satisfied by different resonances in the narrow width approximation Danilkin, Pascalutsa, Pauk, Vanderhaeghen (12,14,17)
- data and dispersive treatments available for on-shell photons e.g. Dai & Pennington (14,16,17)
- dispersive treatment for the singly-virtual case and check with forthcoming data is very important $\to$ talks by Prencipe, & Redmer

# Improvements obtained with the dispersive approach

| Contribution | PdRV(09) *Glasgow consensus* | N/JN(09) | J(17) | WP(20) |
|---|---|---|---|---|
| $\pi^0, \eta, \eta'$-poles | 114(13) | 99(16) | 95.45(12.40) | 93.8(4.0) |
| $\pi, K$-loops/boxes | −19(19) | −19(13) | −20(5) | −16.4(2) |
| $S$-wave $\pi\pi$ rescattering | −7(7) | −7(2) | −5.98(1.20) | −8(1) |
| subtotal | 88(24) | 73(21) | 69.5(13.4) | 69.4(4.1) |
| scalars | – | – | – | } − 1(3) |
| tensors | – | – | 1.1(1) | } |
| axial vectors | 15(10) | 22(5) | 7.55(2.71) | 6(6) |
| $u, d, s$-loops / short-distance | – | 21(3) | 20(4) | 15(10) |
| $c$-loop | 2.3 | – | 2.3(2) | 3(1) |
| total | 105(26) | 116(39) | 100.4(28.2) | 92(19) |

- significant reduction of uncertainties in the first three rows

  CHPS (17), Masjuan, Sánchez-Puertas (17) Hoferichter, Hoid et al. (18), Gerardin, Meyer, Nyffeler (19)

- resonances and short-distance constraints need to be improved

  Danilkin, Hoferichter, Stoffer (21), Lüdtke, Procura, Stoffer (23), Melnikov, Vainshtein (04), Nyffeler (09), Bijnens et al. (20,21), Cappiello et al. (20), Leutgeb, Rebhan (19,21)

# Situation for HLbL