$\gamma^*\gamma^* \to \pi\pi$

If the particles involved have spin and/or gauge invariance plays a role, things become more complicated. We illustrate this with the help of the process $\gamma^*\gamma^* \to \pi\pi$ which is also relevant for a_μ.

If photons are off-shell this is to be viewed as a subprocess of a more complicated one, with particles emitting the virtual photons. We do not consider these and look at:

$$W^{\mu\nu}_{ab}(p_1, p_2, q_1) = i \int d^4x \, e^{-iq_1 x} \langle \pi^a(p_1)\pi^b(p_2) | T\, j^\mu_{em}(x) j^\nu_{em}(0) | 0 \rangle$$

$$\langle \pi^a(p_1)\pi^b(p_2) | \gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \rangle = i e^2 (2\pi)^4 \delta^4(P_f - P_i)\, \varepsilon^{\lambda_1}_\mu(q_1)\varepsilon^{\lambda_2}_\nu(q_2)\, W^{\mu\nu}_{ab}$$

$$\varepsilon^{\lambda_1}_\mu(q_1)\varepsilon^{\lambda_2}_\nu(q_2)\, W^{\mu\nu}_{ab} = e^{i(\lambda_1-\lambda_2)\phi}\, H^{ab}_{\lambda_1\lambda_2}$$

helicity amplitudes.

In this case the Mandelstam variables are defined as usual but the constraint reads:

$$s + t + u = q_1^2 + q_2^2 + 2M_\pi^2$$

Tensor decomposition.

In principle the $W^{\mu\nu}$ amplitude has 4×4 components, but taking into account Lorentz covariance one can decompose it as follows:

$$W^{\mu\nu} = g^{\mu\nu} W_1 + q_i^\mu q_j^\nu W_2^{ij}$$

with $q_i = \{q_1, q_2, p_2 - p_1\}$ $\Rightarrow$ 10 amplitudes.

<u>Gauge invariance</u>:

$$q_1^\mu W_{\mu\nu} = q_2^\nu W_{\mu\nu} = 0$$

$$q_1^\nu W_1 + q_1 \cdot q_i \, q_j^\nu W_2^{ij} = 0$$

$$q_2^\mu W_1 + q_i^\mu q_2 \cdot q_j W_2^{ij} = 0$$

$$q_1^\mu \left(W_1 + q_1 \cdot q_i \, W_2^{i1} \right) + q_2^\mu \, q_1 \cdot q_i \, W_2^{i2} + q_3^\mu \, q_1 \cdot q_i W_2^{i3} = 0$$

$$q_2^\mu \left(W_1 + q_2 \cdot q_j \, W_2^{2j} \right) + q_1^\mu \, q_2 \cdot q_j \, W_2^{1j} + q_3^\mu \, q_2 \cdot q_j \, W^{3j} = 0$$

6 relations in principle, but 5 linearly independent ones.

In order to satisfy them, follow Bardeen and Tung's approach:

$$I^{\mu\nu} \equiv g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}$$

$$q_{1\mu} I^{\mu\nu} = q_{2\nu} I^{\mu\nu} = 0$$

$$I_\mu{}^\lambda W_{\lambda\nu} = W_{\mu\lambda} I^\lambda{}_\nu = W_{\mu\nu}$$

$$\Rightarrow \quad W_{\mu\nu} = I_{\mu\mu'} I_{\nu\nu'} W^{\mu'\nu'} = \sum_{i=1}^{5} T^i_{\mu\nu} B_i$$

$$\bar{T}_1^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \qquad \bar{B}_1 = W_1,$$
$$\bar{T}_2^{\mu\nu} = q_1^\mu q_2^\nu - \frac{q_1^2 q_2^\mu q_2^\nu}{q_1 \cdot q_2} - \frac{q_2^2 q_1^\mu q_1^\nu}{q_1 \cdot q_2} + \frac{q_1^2 q_2^2 q_2^\mu q_1^\nu}{(q_1 \cdot q_2)^2}, \qquad \bar{B}_2 = W_2^{12},$$
$$\bar{T}_3^{\mu\nu} = q_1^\mu q_3^\nu - \frac{q_1^2 q_2^\mu q_3^\nu}{q_1 \cdot q_2} - \frac{q_2 \cdot q_3 q_1^\mu q_1^\nu}{q_1 \cdot q_2} + \frac{q_1^2 q_2 \cdot q_3 q_2^\mu q_1^\nu}{(q_1 \cdot q_2)^2}, \qquad \bar{B}_3 = W_2^{13},$$
$$\bar{T}_4^{\mu\nu} = q_3^\mu q_2^\nu - \frac{q_2^2 q_3^\mu q_1^\nu}{q_1 \cdot q_2} - \frac{q_1 \cdot q_3 q_2^\mu q_2^\nu}{q_1 \cdot q_2} + \frac{q_2^2 q_1 \cdot q_3 q_2^\mu q_1^\nu}{(q_1 \cdot q_2)^2}, \qquad \bar{B}_4 = W_2^{32},$$
$$\bar{T}_5^{\mu\nu} = q_3^\mu q_3^\nu - \frac{q_1 \cdot q_3 q_2^\mu q_3^\nu}{q_1 \cdot q_2} - \frac{q_2 \cdot q_3 q_3^\mu q_1^\nu}{q_1 \cdot q_2} + \frac{q_1 \cdot q_3 q_2 \cdot q_3 q_2^\mu q_1^\nu}{(q_1 \cdot q_2)^2}, \qquad \bar{B}_5 = W_2^{33}.$$

The $\bar{B}_i$ now contain zeros to compensate the singularities in the Lorentz structures. One can redefine the latter to remove the singularities:

$$W_{\mu\nu} = \sum_{i=1}^{5} \tilde{T}^i_{\mu\nu} \tilde{B}_i$$

$$\begin{aligned}
\tilde{T}_1^{\mu\nu} &= q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu, \\
\tilde{T}_2^{\mu\nu} &= q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu, \\
\tilde{T}_3^{\mu\nu} &= q_1^2 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_3^\nu - q_1^2 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_1^\mu q_1^\nu, \\
\tilde{T}_4^{\mu\nu} &= q_2^2 q_1 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_2^\nu - q_2^2 q_3^\mu q_1^\nu - q_1 \cdot q_3 q_2^\mu q_2^\nu, \\
\tilde{T}_5^{\mu\nu} &= q_1 \cdot q_3 q_2 \cdot q_3 g^{\mu\nu} + q_1 \cdot q_2 q_3^\mu q_3^\nu - q_1 \cdot q_3 q_2^\mu q_3^\nu - q_2 \cdot q_3 q_3^\mu q_1^\nu,
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_1 &= \frac{1}{q_1 \cdot q_2} W_1 - \frac{q_1^2 q_2^2}{(q_1 \cdot q_2)^2} W_2^{12} - \frac{q_1^2 q_2 \cdot q_3}{(q_1 \cdot q_2)^2} W_2^{13} - \frac{q_2^2 q_1 \cdot q_3}{(q_1 \cdot q_2)^2} W_2^{32} - \frac{q_1 \cdot q_3 q_2 \cdot q_3}{(q_1 \cdot q_2)^2} W_2^{33}, \\
\tilde{B}_2 &= \frac{1}{q_1 \cdot q_2} W_2^{12}, \\
\tilde{B}_3 &= \frac{1}{q_1 \cdot q_2} W_2^{13}, \\
\tilde{B}_4 &= \frac{1}{q_1 \cdot q_2} W_2^{32}, \\
\tilde{B}_5 &= \frac{1}{q_1 \cdot q_2} W_2^{33}.
\end{aligned} \tag{2.16}$$

The $\tilde{T}_i$ structures form a basis for $q_1 \cdot q_2 \neq 0$, but for $q_1 \cdot q_2 = 0$ become degenerate (Tarrach). One needs to add one more structure to resolve this degeneracy:

$$\tilde{T}_6^{\mu\nu} = \left(q_1^2 q_3^\mu - q_1 \cdot q_3 \, q_1^\mu\right)\left(q_2^2 q_3^\nu - q_2 \cdot q_3 \, q_2^\nu\right)$$

Taking into account crossing symmetry one can, however, get rid again of one structure, ending up with:

$$T_i^{\mu\nu} = \tilde{T}_i^{\mu\nu} \qquad i = 1, 2$$

$$T_3^{\mu\nu} = (t-u)\left(\tilde{T}_3^{\mu\nu} - \tilde{T}_4^{\mu\nu}\right)$$

$$T_i^{\mu\nu} = \tilde{T}_{i+1}^{\mu\nu} \qquad i = 4, 5$$

The corresponding amplitudes A_i:

$$W^{\mu\nu} = \sum_{i=1}^{5} T_i^{\mu\nu} A_i$$

are free from kinematic singularities and zeros.

$\Rightarrow$ they are amenable to a dispersive treatment.

Partial-wave expansion.

$$H_{\lambda_1\lambda_2}(s,t,u) = \sum_J (2J+1)\, d^J_{m0}(z)\, h_{J,\lambda_1,\lambda_2}(s)$$

with d^J_{m0} the Wigner d-function, $m = |\lambda_1 - \lambda_2|$.

Only even J's are allowed. For $m=0 \Rightarrow H_{++}$ and H_{00} the expansion starts at $J=0$, otherwise at $J=2$.

It is possible to derive Roy-like equations for this partial wave: when the particles involved in the scattering are not all of the same kind, the s-, t- and u-channel are, of course, not identical, so that crossing does not represent a symmetry. In this case the equations are called Roy-Steiner (Steiner was the first to analyze πK scattering with this approach):

$$h_{J,i}(s) = \sum_{J',j} \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \, K_{JJ'}^{ij}(s,s') \, \mathrm{Im}\, h_{J',j}(s') + (\text{crossed channels})$$

For this process the lowest-lying singularity is in fact in the t- and u-channel and is the exchange of a single pion. The corresponding contribution to the imaginary part of the amplitude reads (one simply has to insert a complete set of states between the two $\gamma\pi$ states and then single out the pion):

$$\mathrm{Im}_\pi^t W^{\mu\nu} = \frac{1}{2} \int d\tilde{k} \, (2\pi)^4 \delta^4(q_1 - p_1 - k)(k-p_1)^\mu (k+p_2)^\nu F_\pi^V(q_1^2) F_\pi^V(q_2^2)$$

$$\downarrow$$

$$\mathrm{Im}_\pi^t W^{\mu\nu} = F_\pi^V(q_1^2) F_\pi^V(q_2^2) \, \pi \, \delta(t - M_\pi^2) \left(q_3^\mu q_1^\nu - q_2^\mu q_3^\nu - q_2^\mu q_1^\nu + q_3^\mu q_3^\nu \right)$$

$$\mathrm{Im}_\pi^u W^{\mu\nu} = F_\pi^V(q_1^2) F_\pi^V(q_2^2) \, \pi \, \delta(u - M_\pi^2) \left(q_2^\mu q_3^\nu - q_3^\mu q_1^\nu - q_2^\mu q_1^\nu + q_3^\mu q_3^\nu \right)$$

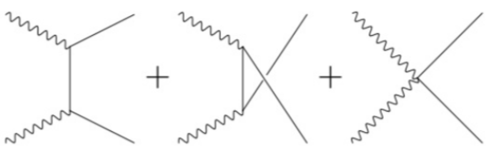
which imply for the scalar amplitudes

$$
\begin{aligned}
\mathrm{Im}^t_\pi A_1 &= F^V_\pi(q_1^2)F^V_\pi(q_2^2)\,\pi\delta(t-M_\pi^2),\\
\mathrm{Im}^t_\pi A_4 &= \frac{2}{s-q_1^2-q_2^2}F^V_\pi(q_1^2)F^V_\pi(q_2^2)\,\pi\delta(t-M_\pi^2),\\
\mathrm{Im}^u_\pi A_1 &= F^V_\pi(q_1^2)F^V_\pi(q_2^2)\,\pi\delta(u-M_\pi^2),\\
\mathrm{Im}^u_\pi A_4 &= \frac{2}{s-q_1^2-q_2^2}F^V_\pi(q_1^2)F^V_\pi(q_2^2)\,\pi\delta(u-M_\pi^2),
\end{aligned}
$$

$$
\begin{aligned}
A_1^\pi &= -F^V_\pi(q_1^2)F^V_\pi(q_2^2)\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right),\\
A_4^\pi &= -F^V_\pi(q_1^2)F^V_\pi(q_2^2)\frac{2}{s-q_1^2-q_2^2}\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right),\\
A_2^\pi &= A_3^\pi = A_5^\pi = 0.
\end{aligned}
$$

It's interesting to compare this to what one gets in sQED:

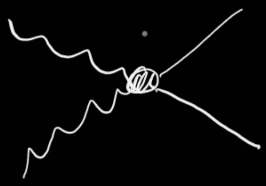
$ie^2 W^{\mu\nu}_{\mathrm{Born}} =$



$$= ie^2(2p_1^\mu - q_1^\mu)(2p_2^\nu - q_2^\nu)\frac{1}{t-M_\pi^2} + ie^2(2p_2^\mu - q_1^\mu)(2p_1^\nu - q_2^\nu)\frac{1}{u-M_\pi^2} + 2ie^2 g^{\mu\nu},$$

$$
\begin{aligned}
A_1^{\mathrm{Born}} &= -\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right),\\
A_4^{\mathrm{Born}} &= -\frac{2}{s-q_1^2-q_2^2}\left(\frac{1}{t-M_\pi^2}+\frac{1}{u-M_\pi^2}\right),\\
A_2^{\mathrm{Born}} &= A_3^{\mathrm{Born}} = A_5^{\mathrm{Born}} = 0.
\end{aligned}
$$

Notice the different meaning of the Feynman and the unitarity diagrams. In the Feynman language



seem to indicate the presence of a local interaction, without any pole, but once the sQED amplitude is projected onto the right basis one realizes that the sQED only contains the pion-pole contribution and with a constant vector form factor. The so-called seagull term is only there to ensure gauge invariance.