SuperhistogramS

— or —

abstract algebra for fun and profit

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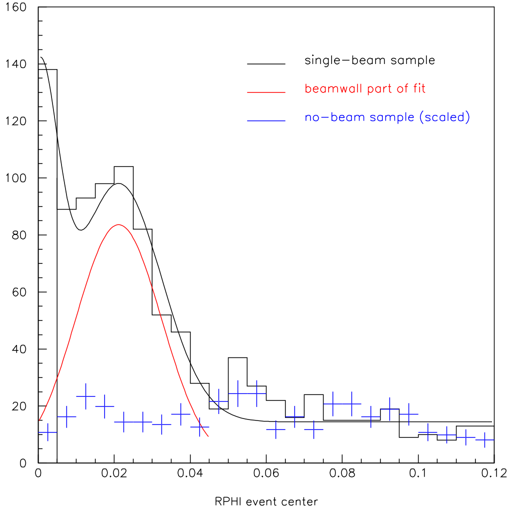
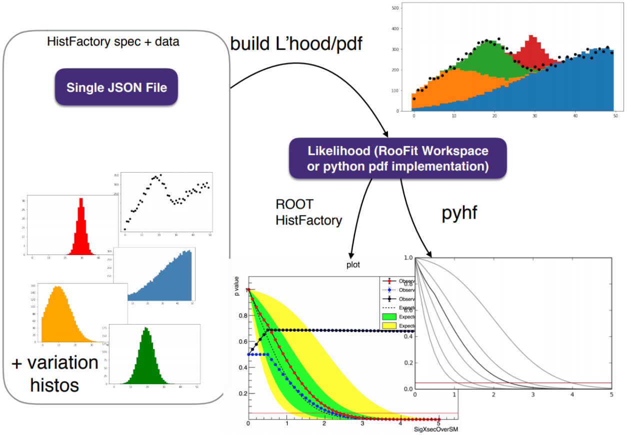
Vague statement of the problem

Long, long ago, histograms were individual objects that were managed individually.



Vague statement of the problem

Now, histograms are more often used collectively, with thousands of histograms in a single fit.



A "superhistogram" is a large collection of histograms that are meant to be interpreted together.

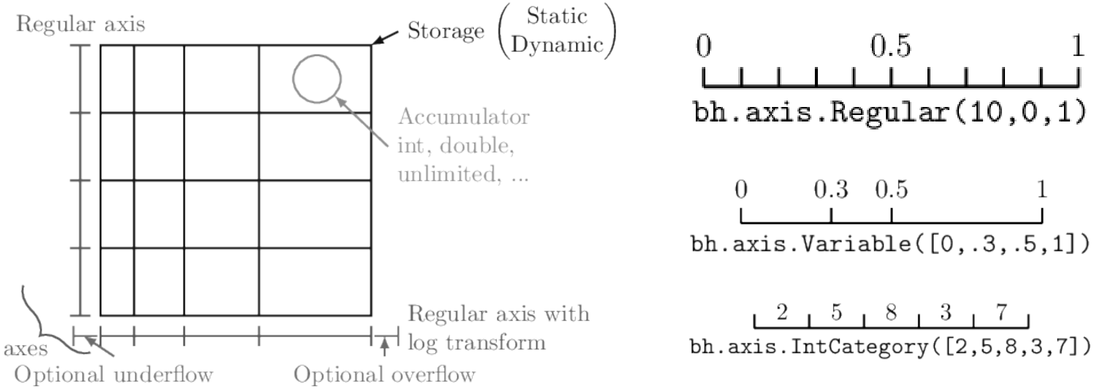
Vague statement of the problem

Representing a superhistogram with a directory of ordinary histograms is

- wasteful because much of the same metadata is copied in memory or on disk, and many small buffers of bin contents is less efficient than one big buffer.
- inconvenient because the object with a common meaning has to be managed as individual objects without an explicit connection. (Often in practice, they're only linked by naming conventions.)

Vague statement of the problem

Boost::Histogram provides a generic way to create an *n*-dimensional space with regular, variable, and categorical axes.



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- each scalar `fill` operation increments *one* bin in that space.

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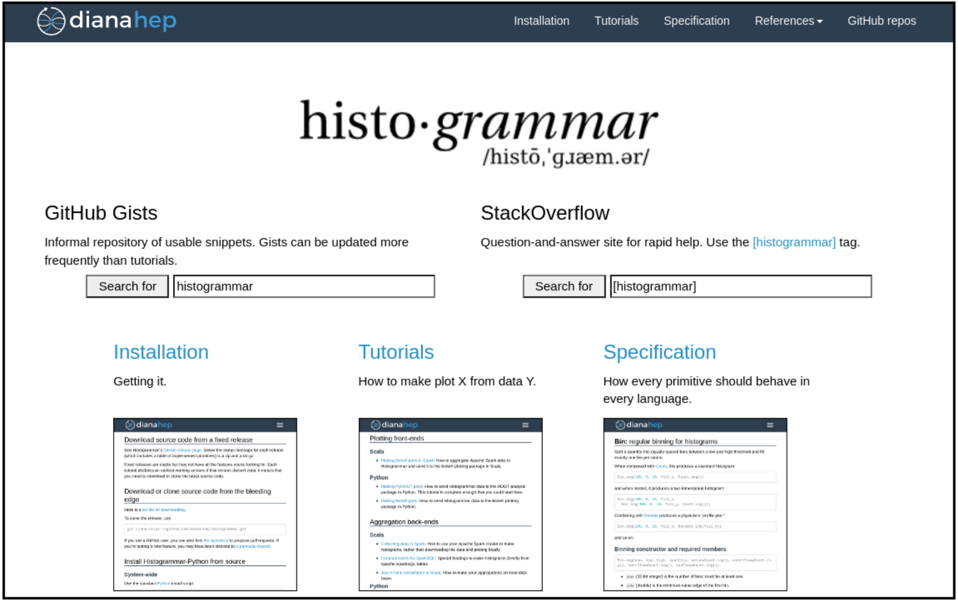
A superhistogram has multiple sources, channels, and systematics.

- Not all histograms in the collection have the same number of bins or the same dimensions, but many do.
- One `fill` of the superhistogram would increment every histogram in the collection.

Something like this

```
h = SuperHist(SuperHist(
        SuperHist(
            Hist.new.Reg(100, 0, 30, name="syst-up"),
            Hist.new.Reg(100, 0, 30, name="nominal"),
            Hist.new.Reg(100, 0, 30, name="syst-down"),
            name="pt",
        ),
        SuperHist(
            Hist.new.Reg(50, -5, 5, name="syst-up"),
            Hist.new.Reg(50, -5, 5, name="nominal"),
            Hist.new.Reg(50, -5, 5, name="syst-down"),
            name="eta",
        ),
        name="data",
    ),
    ...,      # similarly for name="mc"
)
h.fill(df)    # one DataFrame row is a scalar fill operation
```

So far, this is looking like Histogrammar



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R.I.P. Histogrammar

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We need a relationship that restricts the superhistogram to a useful subset of all possible trees.

Abstract algebra for fun (profit comes later)

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The most common is a **monoid**, which is any set S with a binary operation $x \cdot y = z$ (x, y, and z are all in S) that

is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

has an identity: there is an $e \in S$ such that $e \cdot x = x$ and $x \cdot e = x$ for all $x \in S$.

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(You may be familiar with **groups**, which are **monoids** without inverses.)

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- Strings (e.g. `"abc"`, `"def"`) under concatenation (e.g. `"abc"` $+$ `"def"` $\rightarrow$ `"abcdef"`). The identity is the empty string (`""`). (Not commutative!)

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- Boost::Histogram axes under Cartesian product: n axes, an n-dimensional space, combined with m axes, an m-dimensional space, forms an $(n + m)$-dimensional space. An axis with 1 bin could be called an identity.

For superhistograms, we need a semiring

A **semiring** is any set S with two operations, "$+$" and "$\times$", such that

- S under $+$ is a monoid; let's call its identity "0".
- S under $\times$ is a monoid; let's call its identity "1".
- $+$ is commutative: $\boxed{a + b = b + a}$.
- 0 absorbs everything under $\times$: $\boxed{a \times 0 = 0 = 0 \times a}$.
- $\times$ is distributive over $+$: $\boxed{a \times (b + c) = (a \times b) + (a \times c)}$
  and $\boxed{(b + c) \times a = (b \times a) + (c \times a)}$.

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Example: natural numbers under ordinary addition and multiplication.

Superhistograms as a semiring

To build a superhistogram, we put axes together in two ways:

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```
StrCategory(["data", "mc"], name="src") * (
    Reg(100, 0, 30, name="pt") + Reg(50, -5, 5, name="eta")
)
```

is equal to

```
StrCategory(["data", "mc"], name="src") * Reg(100, 0, 30, name="pt") +
StrCategory(["data", "mc"], name="src") * Reg(50, -5, 5, name="eta")
```

Two histograms have the same categorical axis, different regular axes.

Checking the properties

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- Superhistograms under $\times$ is a monoid: an n-dimensional space $\times$ an m-dimensional space forms an $(n + m)$-dimensional space.
- The identity of $\times$ is a one-bin axis.
- $+$ and $\times$ obey a distributive property: if a, b, and c are axes,

$$a \times (b + c) = (a \times b) + (a \times c)$$

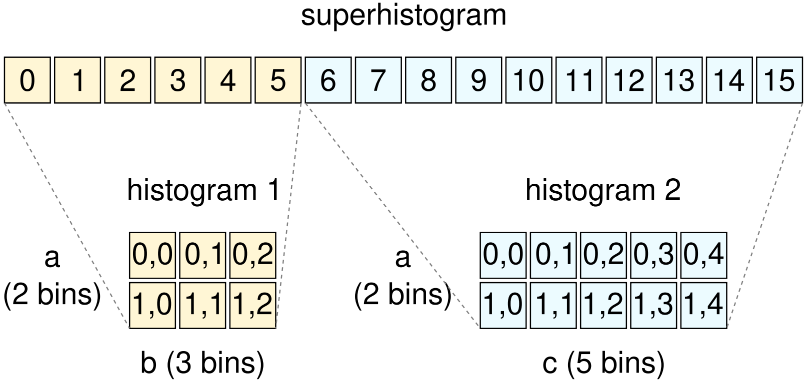
represents two 2-dimensional histograms, both with the same first axis a, differing in their second axes b or c.

A superhistogram is a jagged array

In Boost::Histogram terms, `Axis` a, b, and c in expressions like

$$a \times (b + c) = (a \times b) + (a \times c)$$

would share one `Storage`, such as `Double` (an array of floating-point values).



A superhistogram is filled by DataFrames

Each `fill` operation increments 1 bin in each multiplicative space, and every such space in an additive collection.

```
h = StrCategory(["data", "mc"], name="src") * (
    Reg(100, 0, 30, name="pt") + Reg(50, -5, 5, name="eta")
)
```

`h.fill(`

	src	pt	eta
0	data	22.1	-1.4
1	data	15.8	2.8
2	data	25.0	0.5
3	mc	19.4	-3.1
4	mc	28.5	-0.7

`)`

fills the histogram with axes `"src"` and `"pt"` 5 times, and fills the histogram with axes `"src"` and `"eta"` 5 times.

Why does it matter that superhistograms form a semiring?

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Maybe this could also simplify the interface. (Slicing across all the histograms?)

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By adding this algebraic structure, we restrict the possibilities, but in ways that have benefits to how we want to fill, store, and manipulate histograms.

To Peter's talk!