Refining EIC Beam-Beam Simulation Models

Derong Xu
Brookhaven National Laboratory

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Introduction — EIC beam-beam features

The EIC is designed to achieve a peak luminosity of up to 10^{34} cm^{-2}s^{-1}, incorporating a crossing angle of 25 mrad

- Large Beam-beam parameters: proton ~ 0.015, electron ~ 0.1, **combination never demonstrated before**
- Crab crossing collision with local crab cavities: crab cavities demonstrated at KEKB and SPS, **not used in hadron collider operation**
- Flat hadron beam with large transverse emittance ratio: experimentally demonstrated at RHIC[1], **not tested with large beam-beam parameters**

Compared with HERA, the EIC seeks to increase in luminosity by **two to three orders of magnitude**, accompanied by a **fourfold** increase in both proton and electron beam-beam parameters

[1] Luo et al., "Experimental Demonstration of a Large Transverse Emittance Ratio $11:1$ in the Relativistic Heavy Ion Collider for the Electron-Ion Collider".

Introduction — Model review in EIC BB simulation

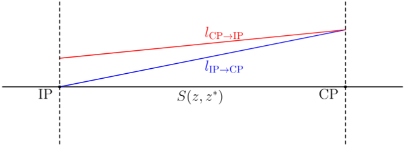
In EIC beam-beam study, we focus on **proton beam emittance growth and its long-term prediction**. Multiple models have been used

- Weak-strong (WS): Assumes a rigid Gaussian electron beam to study resonances induced by beam-beam. Over-simplified, not self-consistent
- Hybrid weak-strong and strong-strong model: Tracks both beams using strong-strong simulation initially, then fixes the electron beam distribution and continues with weak-strong simulation. Time consuming, not noise-free
- Soft-Gaussian based strong-strong (SG): Simulates both beams assuming Gaussian distributions, even if the actual distributions deviate from a perfect Gaussian. May not capture non-Gaussian features
- Particle-In-Cell based strong-strong (PIC): Fully dynamic model using a grid-based approach for precise, detailed simulations of beam-beam interactions. Time consuming, a lot of numerical noise

Enhanced Weak-strong — Problem

Weak-strong simulation is a key method for studying single-particle dynamics. Hirata's approach[2] provides a symplectic mapping for beam-beam interactions considering synchrotron motion.

- This model involves a virtual drift — beam-beam kick — virtual drift process for particle-slice interactions
- This model assumes constant longitudinal coordinates (z), ignoring z-variation.



"This makes also a tiny change of z (path length effect) during the whole beam-beam collision and thus brings a difficulty in using the concept of the longitudinal slice. What we have neglected is just this chromatic effect over the bunch length. This is, however, quite small when compared to the chromatic effect over the whole drift space."

The accumulation of $z-$variation may matter in proton tracking.

[2] Hirata, Moshammer, and Ruggiero, "A symplectic beam-beam interaction with energy change".

Enhanced Weak-strong — Chromatic drift Hamiltonian

The separation between IP and CP: $S(z, z^*) = (z - z^*)/2,\ S' = 1/2$

Left: Original Hirata's virtual drift, Right: proposed modification

Hamiltonian: $h_0 = \frac{p_x^2+p_y^2}{2}$

Lie operation:

$$\mathcal{D}_0 = \exp\left[- : Sh_0 :\right]$$

leads to transformation

$$\mathcal{D}_0 x = x + p_x S,\ \mathcal{D}_0 p_x = p_x$$

$$\mathcal{D}_0 y = y + p_y S,\ \mathcal{D}_0 p_y = p_y$$

$$\mathcal{D}_0 z = z,\ \mathcal{D}_0 p_z = p_z - \frac{p_x^2 + p_y^2}{4}$$

Hamiltonian: $h_1 = \frac{p_x^2+p_y^2}{2(1+p_z)}$

Lie operation $\mathcal{D}_1 = \exp\left[- : Sh_1 :\right]$

$$\Phi(p_x, p_y, p_z) = \sqrt{1 - S' \cdot \frac{p_x^2 + p_y^2}{(1+p_z)^2}} - 1$$

$$\mathcal{D}_1 x = x + \frac{Sp_x}{1+p_z}, \qquad \mathcal{D}_1 p_x = p_x$$

$$\mathcal{D}_1 y = y + \frac{Sp_y}{1+p_z}, \qquad \mathcal{D}_1 p_y = p_y$$

$$\mathcal{D}_1 z = z + \frac{S}{S'}\Phi(p_x, p_y, p_z)$$

$$\mathcal{D}_1 p_z = p_z + (1+p_z)\,\Phi(p_x, p_y, p_z)$$

(1) Both energy change and $z-$variation are included in the proposed transformation; (2) The non-zero Δz alters the particle's arrival time, preventing it from colliding at the pre-calculated CP, though this is unlikely to pose major issues as $\Delta z \ll \sigma/\gamma^*$

Enhanced Weak-strong — Exact drift Hamiltonian

The transformation for the Lie operator with the exact drift Hamiltonian is computationally challenging. We use generating function instead.

- The distance between IP and CP remains $S = (z - z^*)/2$, but the relation $z = z(z_0, p_{x,0}, p_{y,0}, p_{z,0})$ remains unclear.
- Determine the spatial coordinate transformation using the real drift Hamiltonian and the expression $S(z, z^*)$

$$x = x_0 + \left(\frac{p_{x,0}}{p_{s,0}}\right) S(z, z^*), \quad y = y_0 + \left(\frac{p_{y,0}}{p_{s,0}}\right) S(z, z^*), \quad z = z_0 - \left(\frac{H_0}{p_{s,0}}\right) S(z, z^*)$$

- Obtain the momentum transformation from the generating function:

$$H_0 = 1 + p_{z,0} - \sqrt{(1 + p_{z,0})^2 - p_{x,0}^2 - p_{y,0}^2}$$

$$G_3 = -x p_{x,0} - y p_{y,0} - z p_{z,0} + H_0 S(z, z^*)$$

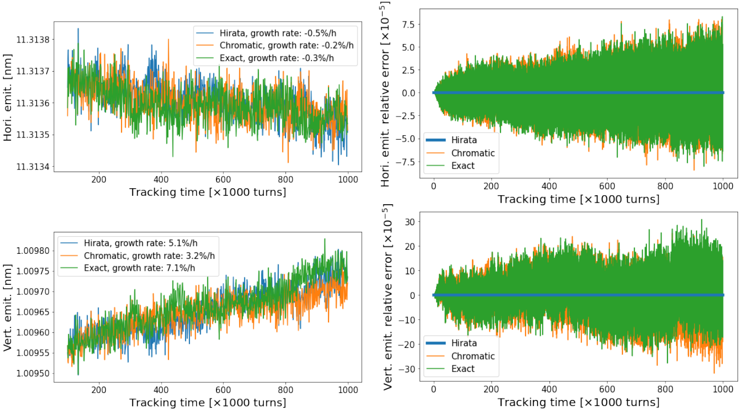
For relativistic case, the transformation from the IP to the CP is:

$$S = \frac{(z_0 - z^*)\, p_{s,0}}{2p_{s,0} + H_0}, \qquad z = z^* + 2S, \qquad p_z = p_{z,0} - \frac{H_0}{2},$$

$$x = x_0 + \left(\frac{p_{x,0}}{p_{s,0}}\right) S, \quad p_x = p_{x,0}, \quad y = y_0 + \left(\frac{p_{y,0}}{p_{s,0}}\right) S, \quad p_y = p_{y,0}$$

Enhanced Weak-strong — Simulation results

Comparison of three different models with EIC design parameters. Left: emittance evolution, Right: relative error normalized by Hirata's model



- No significant resonance. The relative error grows linearly.
- Relative error of 10^{-5} per turn may not be considered negligible when compared to the IBS diffusion $\sim 10^{-7}$ per turn

Enhanced Weak-strong — Conclusion

For more details, see the publication[3]

- Following Hirata's spirit, two transformations are proposed to incorporate longitudinal variation during beam-beam interactions. The symplecticity of overall map, drift - kick -drift, is guaranteed.
- The enhanced models along with Hirata's original model should be regarded as approximations. These models presuppose that the transformations between the IP and the CP adhere to symplectic principles.
- An alternate approach involves the real drift map between IP and CP, and modify the beam-beam kick accordingly[4,5]. This approach mirrors the real physical process but the symplecticity is not guaranteed unless every step chooses exact formula.

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[3] Derong Xu et al., "Enhanced beam-beam modeling to include longitudinal variation during weak-strong simulation".

[4] Sagan, *The Bmad Reference Manual*.

[5] Shatilov and Zobov, "Beam-beam collisions with an arbitrary crossing angle: analytical tune shifts, tracking algorithm without Lorentz boost, crab-crossing".

Extended Soft-Gaussian — Motivation

The soft-Gaussian model is also used in the EIC beam-beam simulation

- **Reduced Numerical Noise**: The soft-Gaussian model produces significantly less numerical noise than PIC-based strong-strong simulations, making it a reliable tool for cross-verifying PIC results.
- **Incorporation of Physical Effects**: It allows the consideration of additional physical phenomena without being shadowed by numerical noise, such as the electron pinch effect, offering a more realistic simulation framework
- **Lower Computational Cost**: The soft-Gaussian model requires less computational power, making it possible for longer simulations.

However, the basic soft-Gaussian model is over-simplified.

Proposed enhancements in our model:

- Considers the correlation between horizontal and vertical coordinates of macro-particles
- Includes 3rd-order moments for better accuracy in electromagnetic field calculations.

Extended Soft-Gaussian — Including beam tilt

A general 2D Gaussian distribution can be described by its Σ matrix

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}, \qquad \rho_g\left(x, y\right) = \frac{1}{2\pi\sqrt{\det\,\Sigma}} \exp\left[-\frac{1}{2}\left(x, y\right)\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

Applying a rotation to diagonalize Σ matrix on (x, y)

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \begin{bmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} = A^{\mathrm{T}} \begin{bmatrix} \overline{\sigma}_{yy} & 0 \\ 0 & \overline{\sigma}_{xx} \end{bmatrix} A$$

A possible solution

$$\cos 2\theta = \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2}}, \quad \sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2}}$$

$$\overline{\sigma}_{xx} \text{ or } \overline{\sigma}_{yy} = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) \pm \frac{1}{2}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2}$$

In the rotated frame, the beam-beam kick can be computed using the Bassetti-Erskine formula. This rotation technique is widely utilized and documented in many beam-beam simulation codes.

Extended Soft-Gaussian — Third-order moments

A general beam distribution can be expanded with Hermite polynomials[6]:

$$\rho\left(x,y\right)=a_{ij}H_i\left(\frac{x}{\sigma_x}\right)H_j\left(\frac{y}{\sigma_y}\right)\rho_g(x,y),\ \text{where } H_n(x)=(-1)^n\exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(-\frac{x^2}{2}\right)$$

The coefficients a_{mn} are determined by

$$a_{mn}=\frac{1}{m!n!}\int H_m\left(\frac{x}{\sigma_x}\right)H_n\left(\frac{y}{\sigma_y}\right)\rho\left(x,y\right)\ \mathrm{d}x\mathrm{d}y=\frac{1}{m!n!}\left\langle H_m\left(\frac{x}{\sigma_x}\right)H_n\left(\frac{y}{\sigma_y}\right)\right\rangle$$

In the rotated frame, the first two orders are corrected zero

$$a_{00}=1,\quad a_{10}=a_{01}=a_{20}=a_{11}=a_{02}=0$$

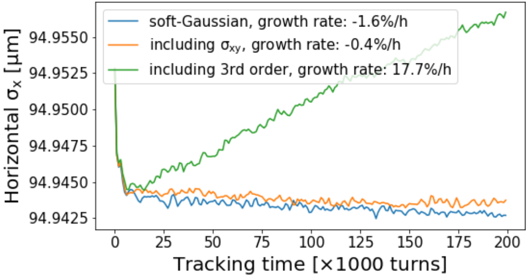
$$a_{30}=\frac{1}{6}\left\langle\frac{x^3}{\sigma_x^3}\right\rangle,\ \ a_{21}=\frac{1}{2}\left\langle\frac{x^2y}{\sigma_x^2\sigma_y}\right\rangle,\ \ a_{12}=\frac{1}{2}\left\langle\frac{xy^2}{\sigma_x\sigma_y^2}\right\rangle,\ \ a_{03}=\frac{1}{6}\left\langle\frac{y^3}{\sigma_y^3}\right\rangle$$

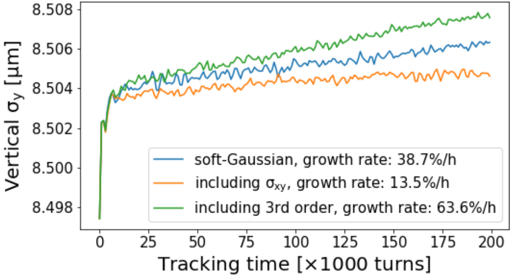
Upto 3rd order, the beam-beam potential is

$$U=U_g-a_{30}\sigma_x^3U_{xxx}-a_{21}\sigma_x^2\sigma_yU_{xxy}-a_{12}\sigma_x\sigma_y^2U_{xyy}-a_{03}\sigma_y^3U_{yyy}$$

[6] Yokoya, “Limitation of the Gaussian approximation in beam-beam simulations”.

Extended Soft-Gaussian — Simulation results





Both horizontal and vertical emittance growth rate become larger, which is a different feature compared with our resonance study

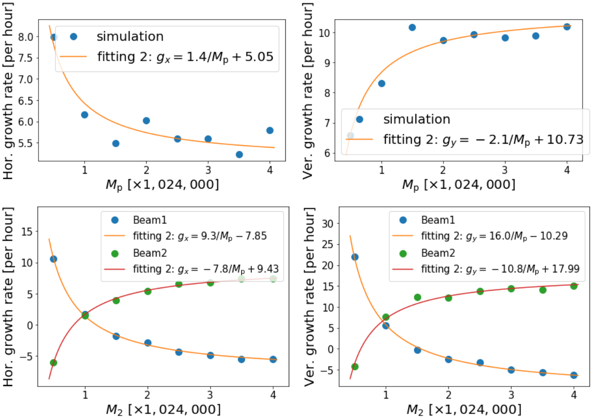
- **Physically**, Non-Gaussian beam distributions enhance nonlinear effects and resonances, leading to emittance growth.
- **Numerically**, Monte Carlo noise impacts higher-order moment calculation, increasing statistical errors in beam-beam simulations.

More details are available in[a]

[a] D. Xu, Hao, et al., "Extended Soft-Gaussian Code for Beam-Beam Simulations".

Model parameters in PIC — Proton macro particles

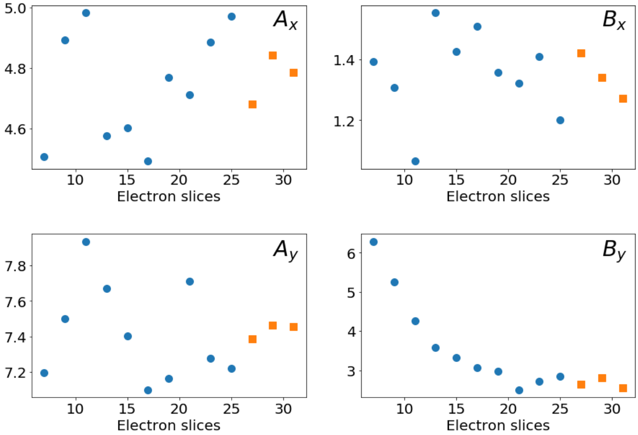
Proton growth rate with different number of proton macro particles M_p
top: e-p with radiation on, bottom: p¯-p with radiation off



There happens an emittance exchange which relates to the number of macroparticles. The synchrotron radiation prevents the exchange.

Model parameters in PIC — Electron longitudinal slices

Fitting $A_{x,y}$ and $B_{x,y}$ for different number of slices $g_{x,y} = \frac{A_{x,y}}{M_\mathrm{e}} + B_{x,y}$

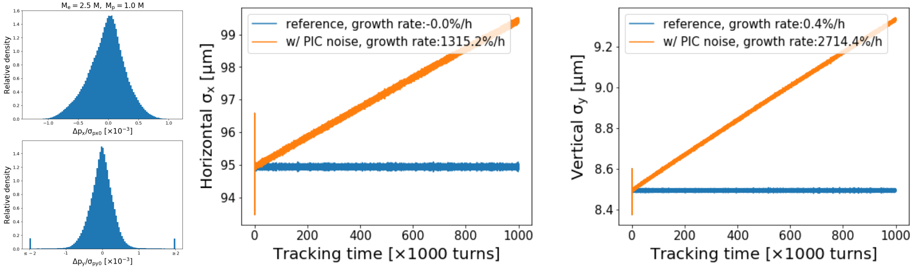


B_y converges when the number of electron slices ≥ 21. There is no clear pattern for $A_{x,y}$ and B_x. With more slices, $B_x \sim 140\%/\mathrm{h}$, $B_y \sim 300\%/\mathrm{h}$

Model parameters in PIC — Import PIC noise in WS

Regenerating PIC noise by Markov chain Monte Carlo sampling

- Obtain Δp_x and Δp_y from PIC solver and analytic formula
- Get the approximate distribution from the histogram of $\Delta p_{x,y}$
- Use the Hastings-Metropolis algorithm to get the equilibrium distribution
- Apply the PIC noise to particles in W-S simulation in every turn



We regenerate proton emittance growth in WS by importing PIC noise, which turns out the large emittance growth in SS is dominated by PIC noise. The large PIC noise may shadow real physical mechanism.

Model parameters in PIC — Conclusion

- PIC-based strong-strong simulation is not ideal for studying resonance effects and other physical factors on proton long-term behavior.
- Proton emittance growth is highly sensitive to the number of electron slices; a large number is required.
- Emittance exchange depends on the number of macroparticles in strong-strong simulations. Its origin (numerical or physical) is unclear, so we recommend matching the macroparticle ratio to the real particle ratio to minimize its impact.

For more discussion on other model parameters in SS simulation, please refer to[7][8][9]

[7] D. Xu, Hao, et al., "Numerical Noise Study in EIC Beam-Beam Simulations".
[8] D. Xu, Hao, et al., "Model Parameters Determination in EIC S-S Simulation".
[9] D. Xu, "Convergence of S-S beam-beam simulations for EIC".

Summary

Different models are used in EIC BB simulation.

- **Enhanced Weak-Strong Model**:
  - Introduced modifications to account for longitudinal variations.
  - Ensured symplecticity in the drift-kick-drift approach for better accuracy.
- **Extended Soft-Gaussian Model**:
  - Provides reduced numerical noise compared to PIC-based simulations.
  - Enhanced with third-order moments and consideration of beam tilt for more realistic beam dynamics.
- **Model Parameters in PIC**:
  - Proton emittance growth is highly sensitive to the number of electron slices; a larger number improves accuracy.
  - Emittance exchange is influenced by the number of macroparticles; matching macroparticle ratios to real particle ratios is recommended.

More study related to EIC BB models in this workshop

- @J. Qiang **Overview of the numerical tools and related challenges for the modeling of beam-beam effects**
- @Y. Luo **Beam-Beam simulation models and numerical noises for flat beam collisions in the Electron-Ion Collider**

Thank you!