A Topological Approach to the Problem of Beam-Beam Compensation

P. Bélanger, G. Sterbini

Special thanks to:
D. Kaltchev for his help and vast knowledge on non-linear maps;
T. Planche and R. Baartman for constant support and fruitful discussions on the topic;
Many others for the discussions.

Prelude: basic assumptions

- **Thin lens** are considered throughout the presentation
- **All plots** and **jargon** are given in **linearly normalized space**
  - $(\tilde{x}, \tilde{p}_x, \tilde{y}, \tilde{p}_y)$ according to Xsuite's physics manual [1] (Ref. at the end)
  - **No ellipses, only circles**
- Capital letter are for **topology**: $(\tilde{x}, \tilde{p}_x) \mapsto (\tilde{X}, \tilde{P}_x)$
- Numerical results aim to be informative about **HL-LHC** (scales should be accurate)

Introduction

- Disconnect between **linear** and **non-linear** formalism **(à la Newton vs. Relativity)**
  - Dipoles + Quadrupoles, **linear machine: forever stable**
  - Sextupoles + Octupoles + … + Beam-Beam**, non-linear machine: chaos, diffusion, stability?**

➜ **Let's try to find a natural extension to describe non-linear dynamics**

- **Practical consequences:**
  - DA studies
  - Beam losses
  - Particle distribution, emittance, etc.
  - B1 vs B2 discrepancies?
- Application on the **difficult problem of compensation**, and most of all, **Beam-Beam**

Topological description

Transport and compensation

Beam-Beam LR (academic)

Beam-Beam LR (HL-LHC)

KAM Theorem

- Henon map: the solutions are either **quasiperiodic**, or **chaotic**

Quasiperiodic

> They seem to **lie exactly on a curve**. [...] It may be interesting to remark that the successive points **rotate regularly** around the curve.
>
> — M. Henon

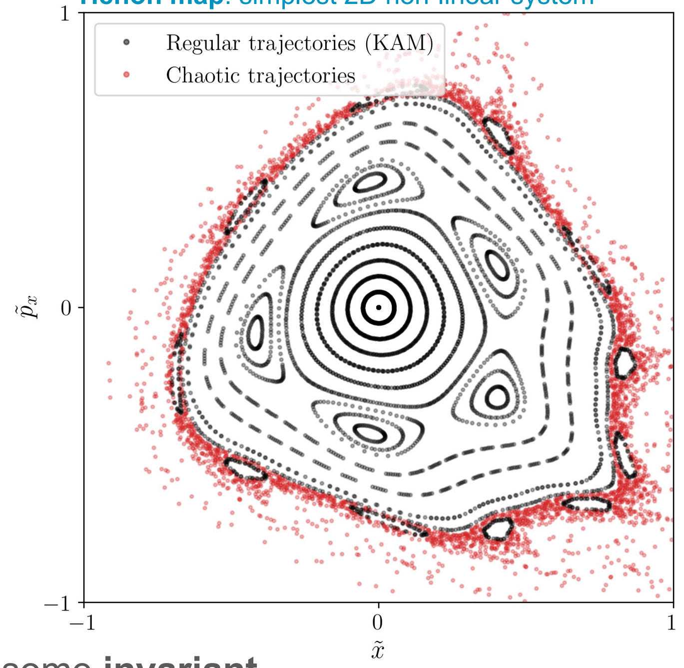
[2] (1964)

Chaotic

> They behave in a **completely different way**. It is clearly **impossible to draw any curve** through them. They seem to be **distributed at random**, in an area left free between the closed curves. Most striking is the fact that this change of behavior seems to **occur abruptly**...
>
> — M. Henon

[2] (1964)

Henon map: simplest 2D non-linear system



➟ Theorem: For **non-integrable conservative systems**, some **invariant tori (KAM tori) persist** close to integrable regions of the system.

Topological description: non-linear dynamics

- **Outside of chaos**, one can forget about the single-particle turn-by-turn picture: the points are anyhow **constrained on curves!**

- **Quasiperiodic expansion** (NAFF) can help us find the **underlying topological curve** (natural extension to the **ellipse** jargon)**.**

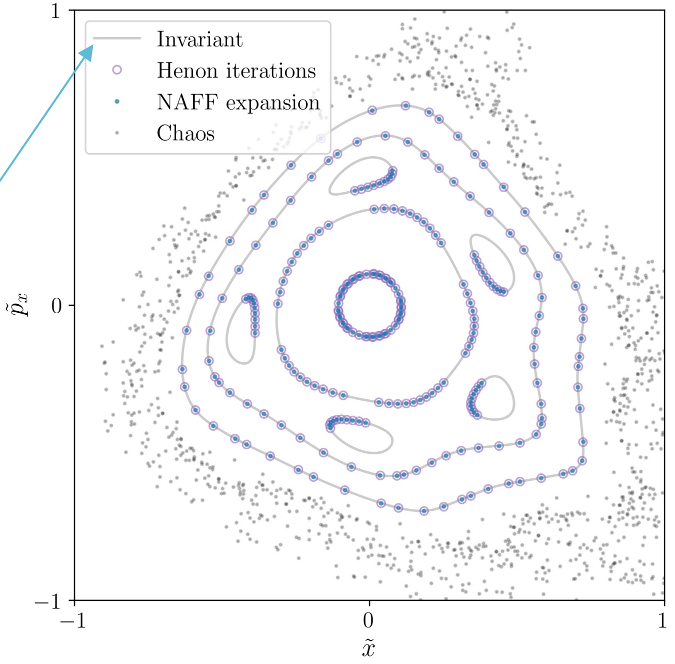
- In linearly-normalized space, linear solutions are **simple circles**:

$$\tilde{X} - i\tilde{P}_x = \sqrt{2I_x/\varepsilon_x}e^{i\Theta_x} = r_x e^{i\Theta_x}$$

- Non-linear solutions are a **natural extension**. (See *Almagest of Ptolemy,* epicycle theory)

$$\tilde{X} - i\tilde{P}_x = \sum_{k=0}^{N_h} A_k e^{i[n_k \Theta_x]}$$

- Corollary to KAM theorem: n_k **is an integer**!

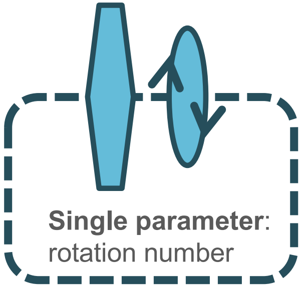


Topological description: eigensolutions

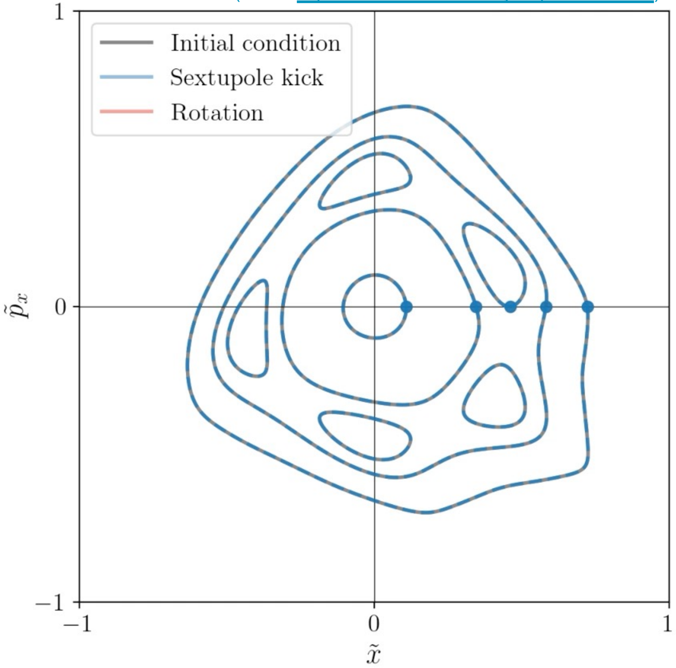
- **Why** the topological jump?
  - A fixed point is described by a single point (particle)
  - An invariant curve is described by a curve

- The **phase can advance**, but the curve (topology) is **unchanged**

- *Exempli gratia*: the Henon map, a **closed ring containing**
  1. A thin sextupole (shear)
  2. A linear segment (rotation)



$\omega = 0.2071$ (video: https://cernbox.cern.ch/s/jhxTpFLn1BrcBQr)



- In fact, the **invariant curves** are the **eigensolutions** of the transformation

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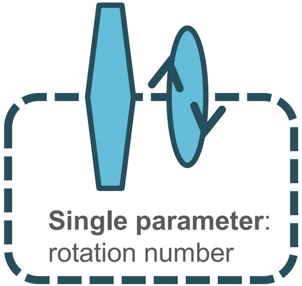
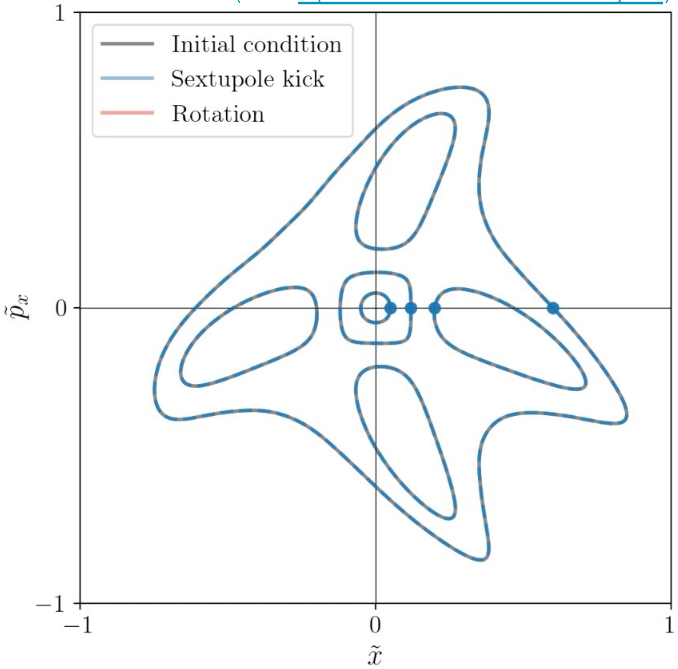


$\omega = 0.2514$ (video: https://cernbox.cern.ch/s/UZoHtQZZ76jAnke)



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Topological description

Transport and compensation

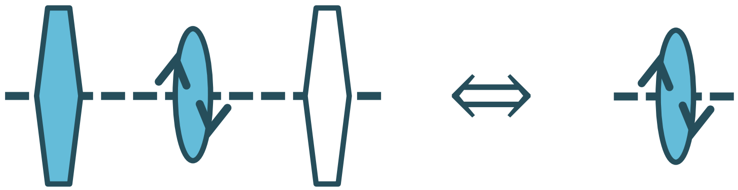
Beam-Beam LR (academic)

Beam-Beam LR (HL-LHC)

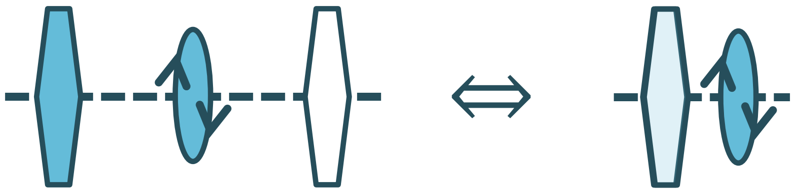
Compensation (*i.e.* linearization)

- Accelerator physics **transformations** can act on **topological objects.** They are **operators**.
- For **undesirable** non-linearities (like BB), to **compensate** is to **linearize:**
  - The **eigensolutions** of a fully compensated non-linear system are **simple circles**
  - In other words, let's find a configuration for which the system is equivalent to a simple rotation

Total compensation



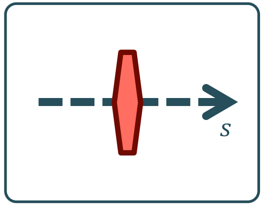
Partial compensation



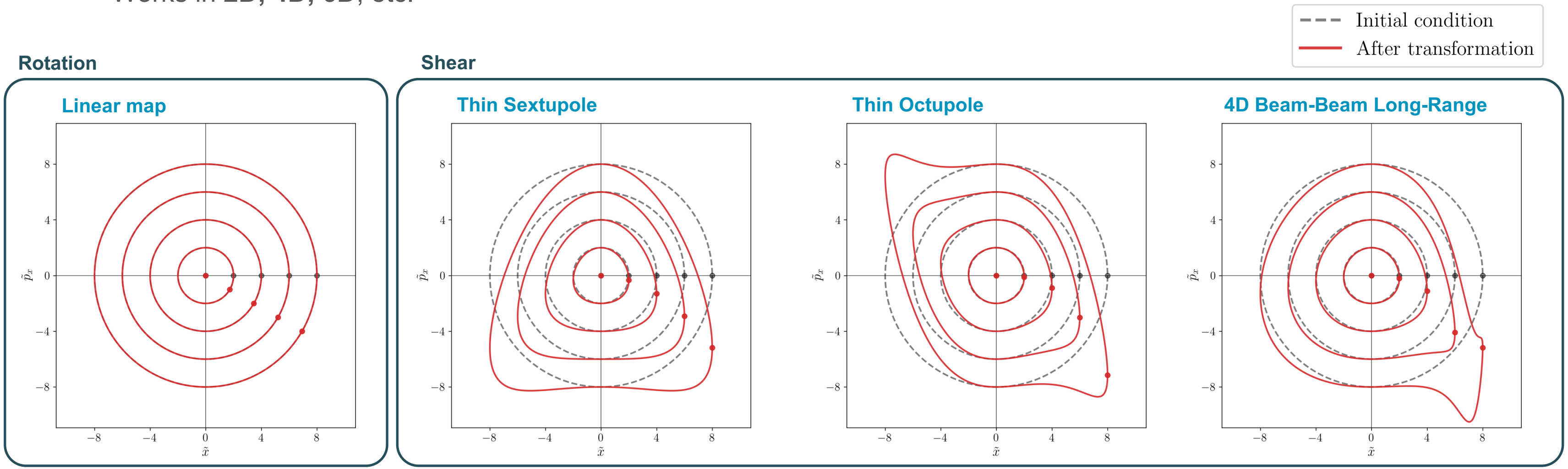
- In this framework, compensation becomes a **transport problem**!

The single-pass transport problem

- **Analytical:** with **Lie algebra**, one can find the transformed topology for common elements
- **Numerical:** with **tracking**, one can find the transformed topology for **any element**, including **beam-beam**
  - ➡ This approach is **as exact as the tracking** (no truncation)



- Works in **2D, 4D, 6D**, etc.



*** Exaggerated strength for the sake of the argument**

Example: 4D formalism

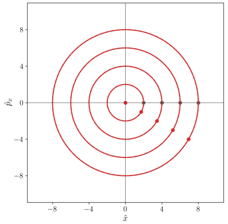
General form:

$$\begin{cases}\tilde{X} - i\tilde{P}_x = r_x e^{i\Theta_x} \\ \tilde{Y} - i\tilde{P}_y = r_y e^{i\Theta_y}\end{cases} \rightarrow \begin{cases}\tilde{X}' - i\tilde{P}'_x = \sum_{k=0}^{N_h} A_k e^{i[n_{1k}\Theta_x + n_{2k}\Theta_y]} \\ \tilde{Y}' - i\tilde{P}'_y = \sum_{k=0}^{N_h} B_k e^{i[m_{1k}\Theta_x + m_{2k}\Theta_y]}\end{cases}$$

(Coupling)

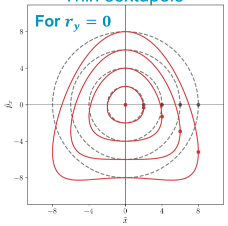
(Coupling)

Rotation



$$\begin{cases}\tilde{X} - i\tilde{P}_x = r_x e^{i\Theta_x} \\ \tilde{Y} - i\tilde{P}_y = r_y e^{i\Theta_y}\end{cases} \rightarrow \begin{cases}\tilde{X}' - i\tilde{P}'_x = e^{i\mu_x}\left(r_x e^{i[\Theta_x]}\right) \\ \tilde{Y}' - i\tilde{P}'_y = e^{i\mu_y}\left(r_y e^{i[\Theta_y]}\right)\end{cases}$$

Thin sextupole



$$\begin{cases}\tilde{X} - i\tilde{P}_x = r_x e^{i\Theta_x} \\ \tilde{Y} - i\tilde{P}_y = r_y e^{i\Theta_y}\end{cases} \rightarrow$$

$$\begin{cases}\tilde{X}' - i\tilde{P}'_x = r_x e^{i[\Theta_x]} + \frac{i\beta_x^{3/2}k_2}{8}\sqrt{\varepsilon_x}\, r_x^2 \left(2 + e^{i[2\Theta_x]} + e^{i[-2\Theta_x]}\right) - \frac{i\sqrt{\beta_x}\beta_y k_2}{8}\frac{\varepsilon_y}{\sqrt{\varepsilon_x}} r_y^2 \left(2 + e^{i[2\Theta_y]} + e^{i[-2\Theta_y]}\right) \\ \tilde{Y}' - i\tilde{P}'_y = r_y e^{i[\Theta_y]} - \frac{i\sqrt{\beta_x}\beta_y k_2}{4}\sqrt{\varepsilon_x}\, r_x r_y \left(e^{i[\Theta_x+\Theta_y]} + e^{i[\Theta_x-\Theta_y]} + e^{i[-\Theta_x+\Theta_y]} + e^{i[-\Theta_x-\Theta_y]}\right)\end{cases}$$

(Coupling)

(Coupling)

Topological description

Transport and compensation

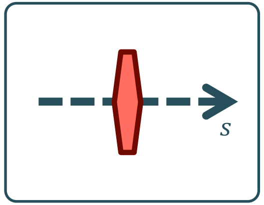
Beam-Beam LR (academic)

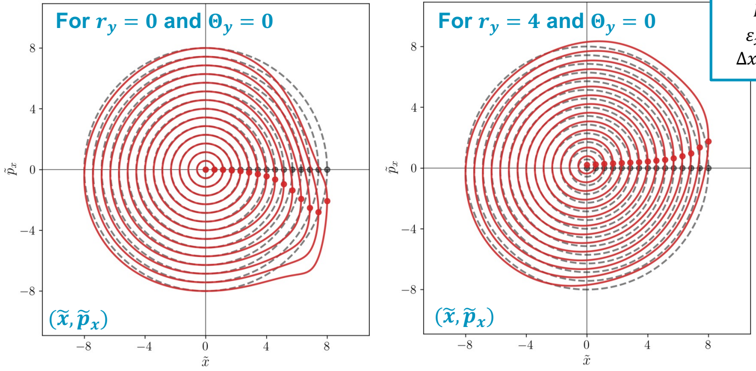
Beam-Beam LR (HL-LHC)

Beam-Beam Long-Range (BBLR)

$E = 7$ TeV
$N_{LR} = 40$
$N_b = 2.3 \times 10^{11}$
$\varepsilon_{x,y} = 2.5\ \mu\text{m}$
$\Delta x_{LR} = 9\ \sigma_x$

- **Transformation** from a **4D BBLR**:
  - Many dimensions to inspect
  - Coupling





- To assess the **non-linear residual**, we can define:

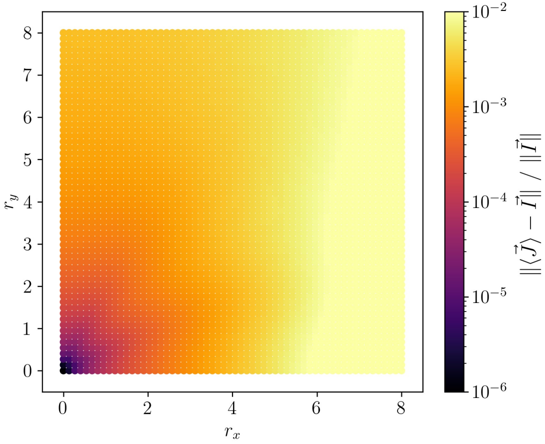
$$\vec{I} = \begin{pmatrix} I_x \\ I_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \varepsilon_x r_x^2 \\ \varepsilon_y r_y^2 \end{pmatrix} \quad \text{and} \quad \langle \vec{J} \rangle = \frac{1}{2}\begin{pmatrix} \varepsilon_x \left\langle \tilde{X}^2 + \tilde{P}_x^2 \right\rangle \\ \varepsilon_y \left\langle \tilde{Y}^2 + \tilde{P}_y^2 \right\rangle \end{pmatrix}$$

Action — C.-S. like action

- For a **linear** transformation, the **Courant-Snyder** action **is** the action:

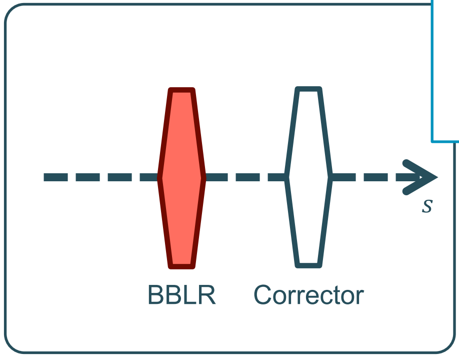
$$\|\langle \vec{J} \rangle - \vec{I}\| \ / \ \|\vec{I}\|$$

(Non-linear residual)

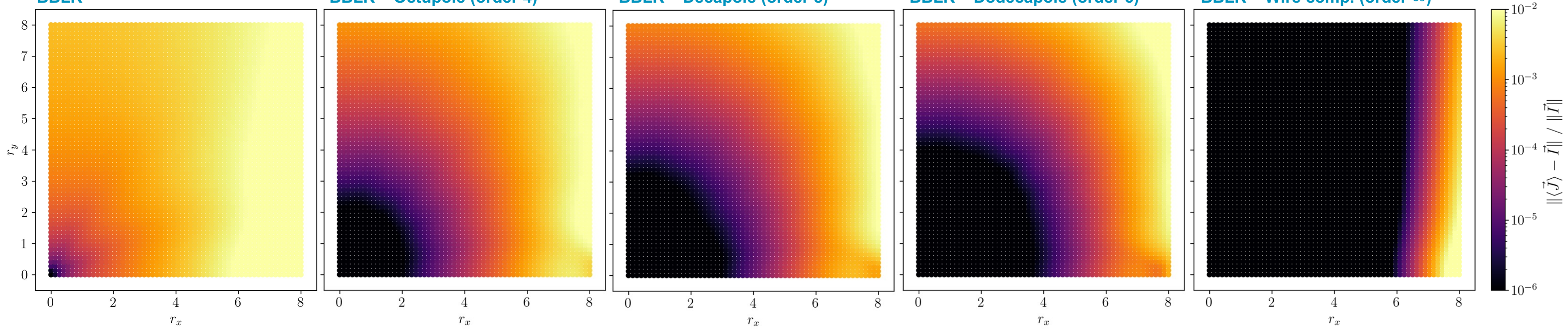


BBLR – Multipole equivalence: compensation

- Let's compensate a **single** 4D-BBLR with an ideal corrector package:
  - ➡ Assuming **round-beam** BBLR
  - ➡ **Multipoles** of increasing order
  - ➡ **Wire** compensator

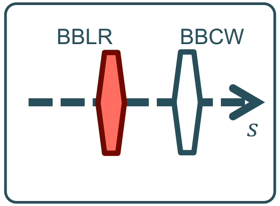


$$E = 7\ \text{TeV}$$
$$N_{LR} = 40$$
$$N_b = 2.3 \times 10^{11}$$
$$\varepsilon_{x,y} = 2.5\ \mu\text{m}$$
$$\Delta x_{LR} = 9\ \sigma_x$$
$$I_w = N_{LR} \cdot N_b \cdot (ec)$$

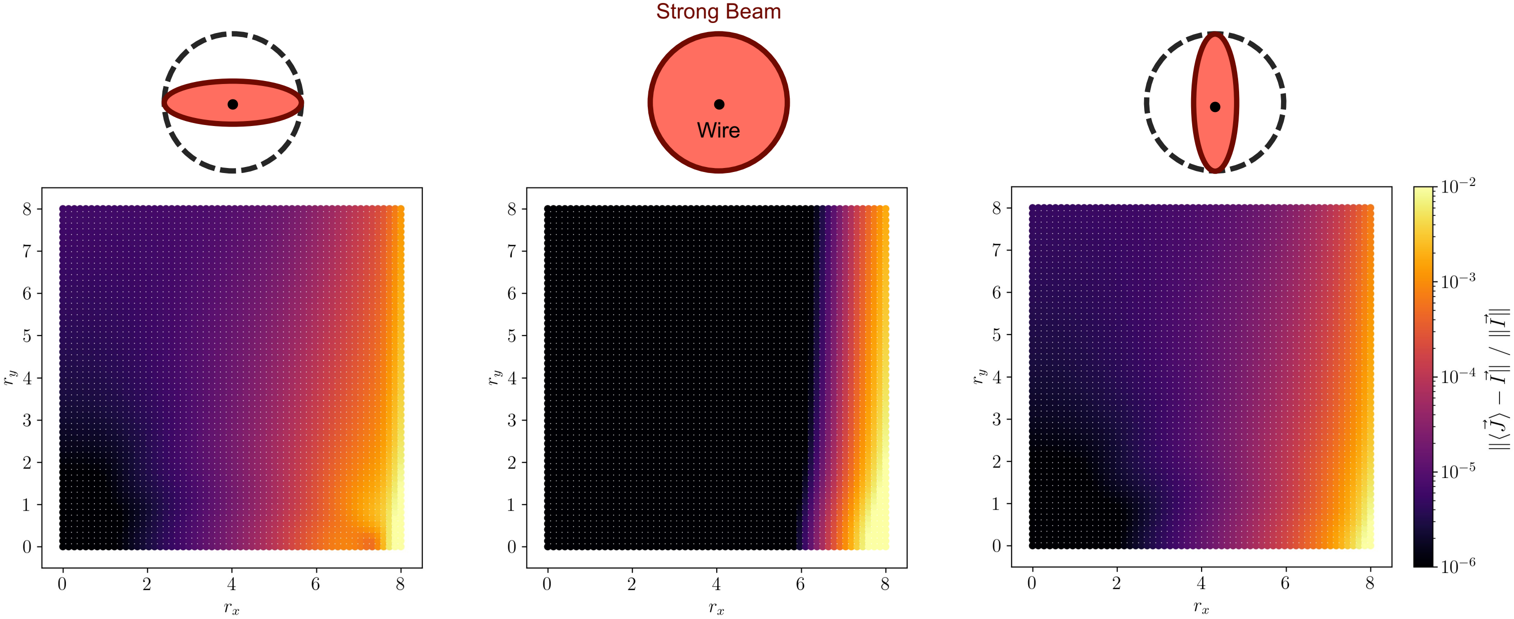


BBLR – Wire equivalence: flatness

- Let's compensate a **single** 4D-BBLR with a wire compensator:
  - ➡ **H-flat** strong beam, **Round** strong beam, **V-flat** strong beam
  - ➡ The equivalence should not be taken for granted!



$$
\begin{aligned}
E &= 7\ \text{TeV}\\
N_{LR} &= 40\\
N_b &= 2.3\times 10^{11}\\
\varepsilon_{x,y} &= 2.5\ \mu\text{m}\\
\Delta x_{LR} &= 9\ \sigma_x\\
I_w &= N_{LR}\cdot N_b\cdot (ec)
\end{aligned}
$$



Topological description

Transport and compensation

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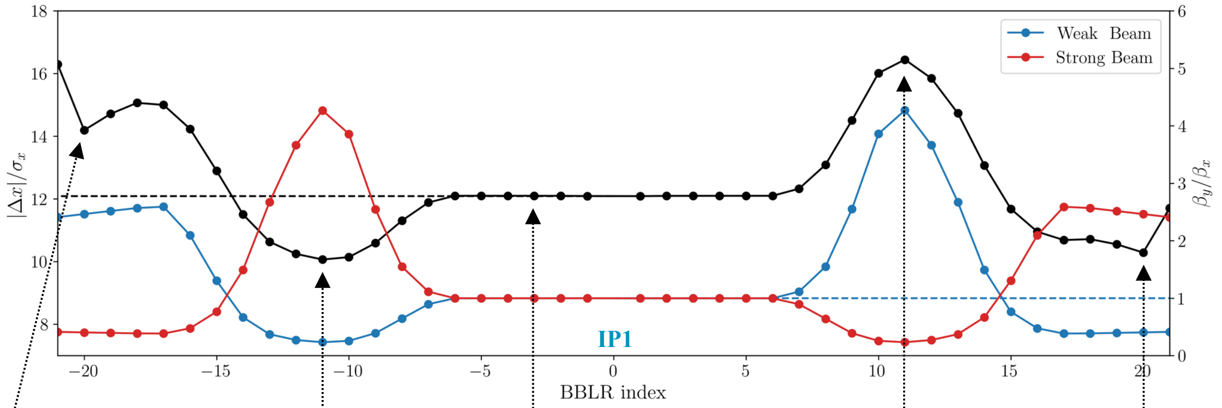
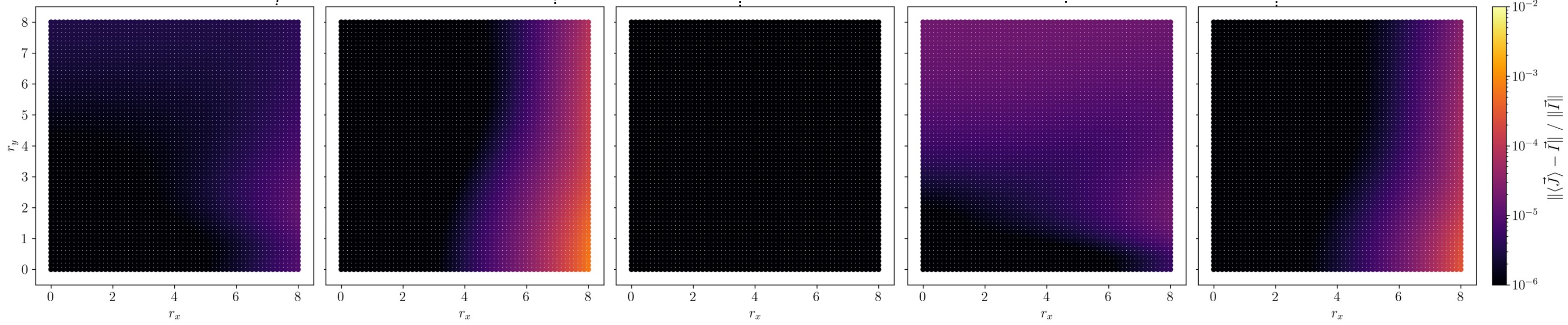
BBLR – Wire equivalence: HL-LHC





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\begin{aligned}
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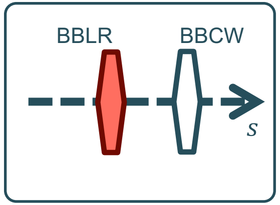
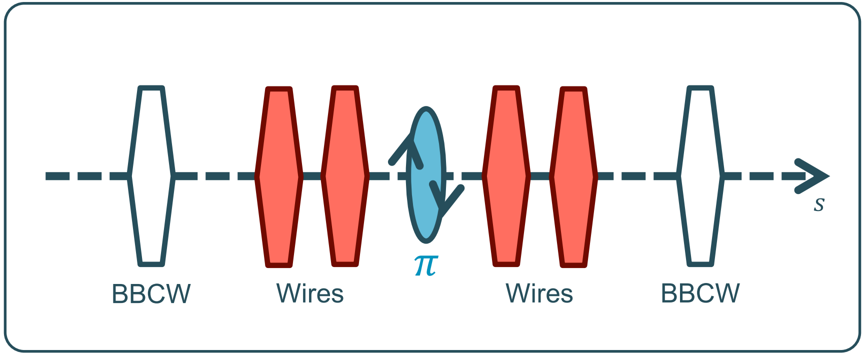
N_b &= 2.3 \times 10^{11} \\
\varepsilon_{x,y} &= 2.5\ \mu\text{m} \\
\Delta x_{LR} &= \Delta x(s) \\
\theta_c/2 &= 190\ \mu\text{rad} \\
\beta^* &= 15\ \text{cm} \\
I_w &= N_{LR} \cdot N_b \cdot (ec)
\end{aligned}
$$



*** Single BBLR considered (Parametric study)**

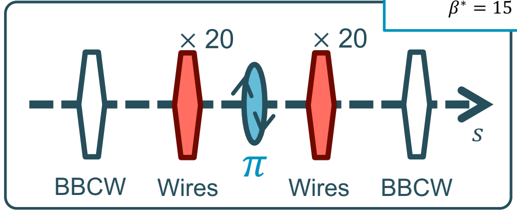
4-Wire Problem

- A full **BBLR – BBCW** compensation scheme will ultimately become an **optimization problem**
- Let's consider in the meantime the **academic** formulation of a **multi-BBLR** compensation scheme:
  - ➟ **BBLR** are replaced by pure **DC wires**
  - ➟ Practical **symmetry considerations** are kept (**antisymmetric β-functions** and **π-phase advance at the IP**)
- The simplest formulation of the problem contains **2 BBCW** compensating **4 DC wires**, placed symmetrically:
  - ➟ Case 1: **round optics**, $\beta_x = \beta_y$ and **constant** Δx (trivial case, **solution exists**)
  - ➟ Case 2: **varying optics** ratio, β_y/β_x **and/or varying** Δx (optimization case, **no perfect solution**)

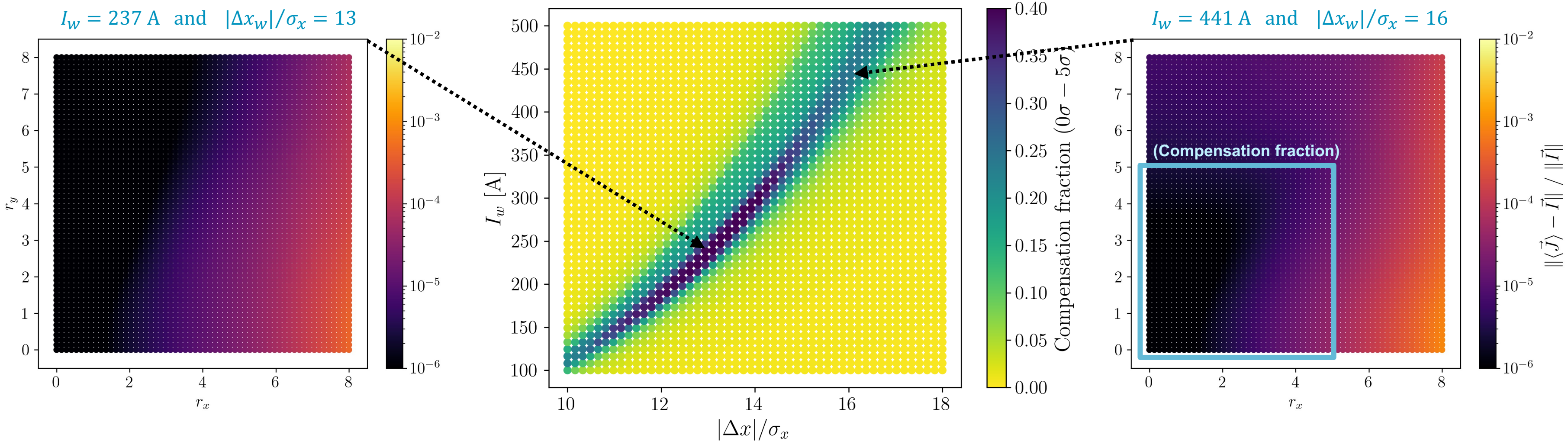


40-Wire Problem (HL-LHC case)

$E = 7$ TeV
$N_{LR} = 40$
$N_b = 2.3 \times 10^{11}$
$\varepsilon_{x,y} = 2.5\ \mu$m
$\theta_c/2 = 190\ \mu$rad
$\beta^* = 15$ cm



- Let's consider the **full optics of HL-LHC** and compensate the **BBLR (as wires) around IP1**
  - ➟ **2 antisymmetric BBCWs**, constrained to have $\frac{\beta_y^R}{\beta_x^R} = \frac{\beta_x^L}{\beta_y^L} = 1.75$.
    Optimization of I_w and $|\Delta x_w|/\sigma_x$.



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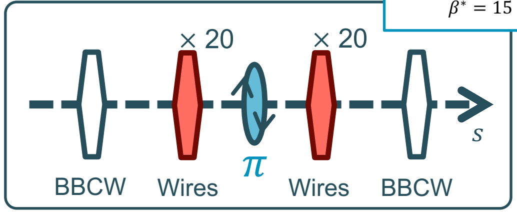
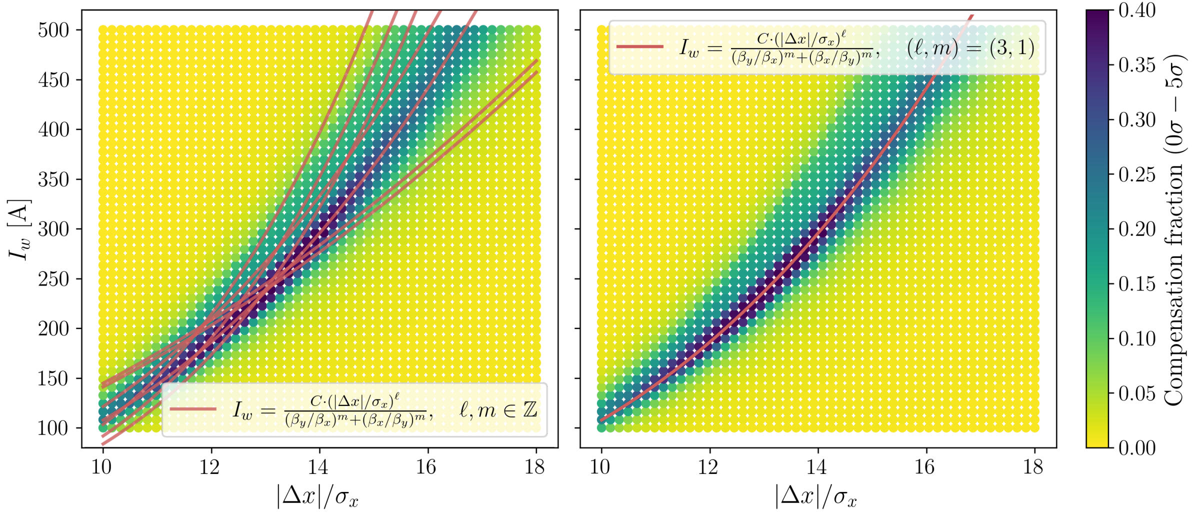


- **Optimal point** is found to correspond to **analytic predictions**:

$$\frac{I_w}{(|\Delta x|/\sigma_x)^\ell}\left[(\beta_y/\beta_x)^m + (\beta_x/\beta_y)^m\right] = \sum_{\text{BBLR}} \frac{N_b \cdot (ec)}{(|\Delta x|/\sigma_x)^\ell}(\beta_y/\beta_x)^m = C, \quad \ell, m \in \mathbb{Z}$$

Octupole-like **target**: $\frac{I_w}{(|\Delta x|/\sigma_x)^3}\left[(\beta_y/\beta_x) + (\beta_x/\beta_y)\right] = C$



*See [3]

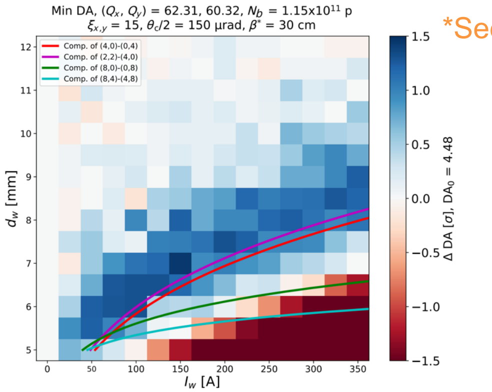


FIG. 9. DA variation (from red for a loss of DA to blue for a gain) as a function of the beam-wire distance and the wires current for the actual LHC case. The different colored lines show the configurations needed to compensate for a given RDT.

Summary and Outlook

- Presented **a novel** and **complementary description** of **non-linear beam dynamics**
  - **Topology is key**
  - **Exact** via tracking**,** no truncation needed
  - Invariants are **eigensolution** of operators
  - **Non-linear transport problem** can be described in an analogous way to **Twiss**
  - Valid in **2D, 4D, 6D**

- **Beam-Beam Long-Range compensation** was reviewed using this formalism
  - **Wire** compensation is the only **reasonable** solution to this problem
  - **HL-LHC** compensation scheme was reviewed
  - Results are **consistent** with **previous DA studies**

- **Dedicated DA studies** should be carried out to find the correlation with **“non-linear residual”**
- Connection between **transport problem** and **periodic problem** should be studied **(emergence of chaos)**

References

[1] G. Iadarola et al. (2024), “Xsuite physics manual”. https://xsuite.github.io/xsuite/docs/physics_manual/physics_man.pdf

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[3] A. Poyet et al. (2024), “First experimental evidence of a beam-beam long-range compensation using wires in the Large Hadron Collider”.

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[5] J. Laskar (1999), “Introduction to Frequency Map Analysis”. http://link.springer.com/10.1007/978-94-011-4673-9_1

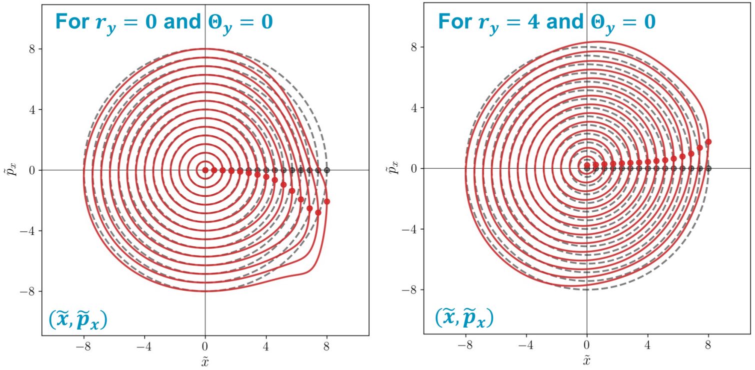
[6] S. Fartoukh (2015), “Compensation of the long-range beam-beam interactions as a path towards new configurations for the high luminosity LHC”.

HiLumi
HL-LHC PROJECT

Thank you!

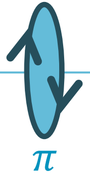
BBLR IP symmetry

$\Delta x > 0$



$E = 7$ TeV
$N_{LR} = 40$
$N_b = 2.3 \times 10^{11}$
$\varepsilon_{x,y} = 2.5\ \mu$m
$\Delta x_{LR} = +9\ \sigma_x$

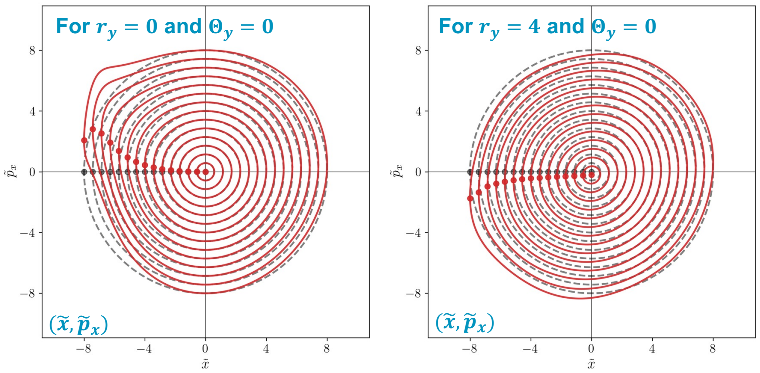
Fully equivalent!



Fully equivalent!

s

$\Delta x < 0$



$E = 7$ TeV
$N_{LR} = 40$
$N_b = 2.3 \times 10^{11}$
$\varepsilon_{x,y} = 2.5\ \mu$m
$\Delta x_{LR} = -9\ \sigma_x$

BBCW symmetry: real space vs normalized?

$E = 7$ TeV
$N_{LR} = 40$
$N_b = 2.3 \times 10^{11}$
$\varepsilon_{x,y} = 2.5\ \mu$m
$\theta_c/2 = 190\ \mu$rad
$\beta^* = 15$ cm