Space charge effects on beam-beam mode coupling instability

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Beam-beam Effects in Circular Colliders, BB24

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Thanks to Y. Zhang

Space charge tune shift

- Tune shift

$$-\Delta\nu_{y,sc} = \xi_{sc} = \frac{Nr_e}{(2\pi)^{3/2}\sigma_z\beta^2\gamma^3}\oint ds \frac{\beta_y}{(\sigma_x+\sigma_y)\sigma_y}$$

Beam-beam tune shift

$$\Delta\nu_{y,bb} = \xi_{bb}^{(\pm)} = \frac{N^{(\mp)}r_e}{2\pi\gamma^{\pm}}\frac{\beta_y^*}{\sigma_y^*\sigma_x^*\sqrt{1+\theta_P^2}}$$

Space charge force for dipole moment distribution in the longitudinal phase space

- Momentum change in the longitudinal phase space

$$\Delta p_y(z,\delta) = k_{sc}\rho(z)[y(z,\delta) - y(z)]\Delta s \qquad k_{sc}(s) = \frac{\alpha N r_e}{\beta^2\gamma^3}\frac{1}{\sigma_y(\sigma_x+\sigma_y)} = \frac{(2\pi)^{3/2}\alpha\xi_{sc}}{\beta_y}\frac{\sigma_z}{L}$$

$$y(z) = \int y(z,\delta)\psi(z,\delta)d\delta/\rho(z) \qquad \rho(z) = \int \psi(z,\delta)d\delta \qquad 1 \le \alpha \le 2$$

- Following the expression using Wake

$$\Delta p_y(z) = -\left[\int W_Q(z-z')\rho(z')dz' y(z) + \int W_D(z-z')\rho(z')y(z')dz'\right]$$

- Effective transverse wake for the space charge force

$$W_{Q/D}(z) = \mp k_{sc} L\delta(z) \qquad Z_{Q/D}(\omega) = \mp i k_{sc} L/c$$

Vertical cross wake in beam-beam interaction

- The cross wake exists also in the vertical plain.

$$\Sigma = \sqrt{2(\sigma_x^2 - \sigma_y^2)}.$$

$$W_y^{(\pm)}(z_\pm - z_\mp) = -\frac{N_\mp r_e}{\gamma_\pm}\frac{\partial F_y}{\partial y}\bigg|_{x=(z_\pm - z_\mp)\theta_c}$$

$$F = F_y + iF_x = \frac{2\sqrt{\pi}}{\Sigma}[w(A) - \exp(-B)w(C)]$$

For $\sigma_x \gg \sigma_y, y = 0$,

$$W_y^{(\pm)}(z) \approx -\frac{N_\mp r_e}{\gamma_\pm}\frac{1}{\sigma_x\sigma_y}\exp\left(-\frac{z^2\theta_c^2}{4\sigma_x^2}\right) = -\frac{N_\mp r_e}{\gamma_\pm}\frac{1}{\sigma_x\sigma_y}\exp\left(-\frac{z^2\theta_P^2}{4\sigma_z^2}\right)$$

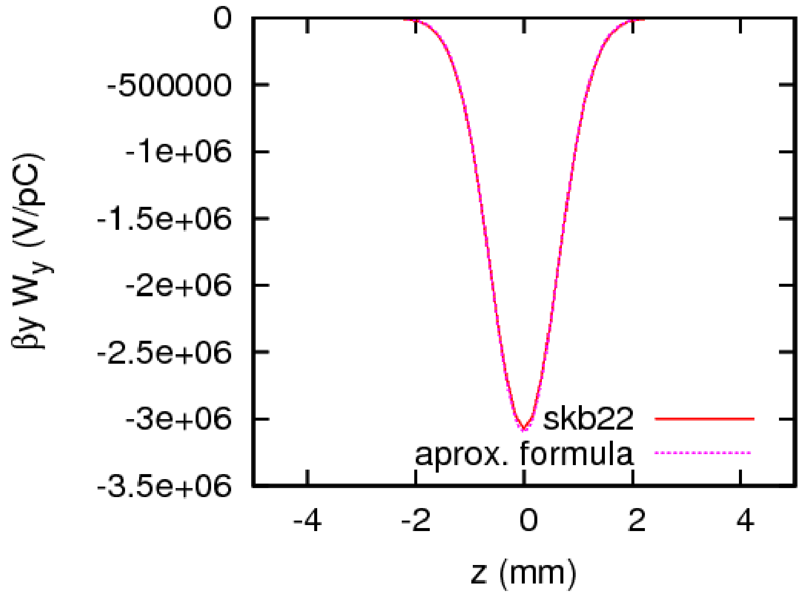
$$\beta_y^* W_{crs} = -(2\pi)^{3/2}\sqrt{2}\xi_{bb}\sigma_z\delta_{\sqrt{2}\sigma_z/\theta_P}(z)$$

$$Z(\omega) = i\int W_y(z)e^{-i\omega z/c}dz/c$$

$$= -i\frac{N^{(-)}r_e}{\gamma^{(+)}}\frac{1}{\bar{\sigma}_x\bar{\sigma}_y(s)}\frac{\sqrt{4\pi}\sigma}{c\theta_P}\exp\left(-\frac{\omega^2\sigma^2}{c^2\theta_P^2}\right)$$

$$\beta_y^* Z_{crs}(\omega) = -i(2\pi)^{3/2}\sqrt{2}\xi_{bb}\frac{\sigma_z}{c}\exp\left(-\frac{\omega^2\sigma_z^2}{c^2\theta_P^2}\right)$$

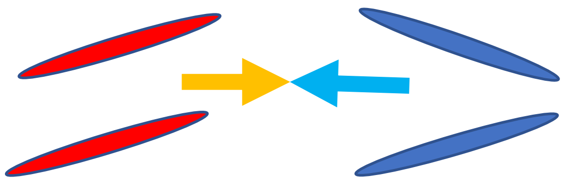
$$\Delta p_y^{(\pm)}(z) = \int_{-\infty}^{\infty} W^{(\pm)}(z-z')\rho^{(\mp)}(z')(y^{(\pm)}(z) - y^{(\mp)}(z'))dz'$$



Beam-beam force in the transparency collision

$$\frac{N^{(-)}r_e}{\gamma^{(+)}} = \frac{N^{(-)}r_e}{\gamma^{(+)}} \qquad \sigma^{(+)}_{x,y,z} = \sigma^{(-)}_{x,y,z} \qquad \beta^{*(+)}_{x,y} = \beta^{*(-)}_{x,y}$$

- σ mode $\quad y^{(+)}(z) = y^{(-)}(z') \qquad p_y^{(+)}(z) = p_y^{(-)}(z)$
- π mode $\quad y^{(+)}(z) = -y^{(-)}(z') \qquad p_y^{(+)}(z) = -p_y^{(-)}(z)$



$$\Delta p_y(z) = \int_{-\infty}^{\infty} W_{crs}(z - z')\rho(z')\big(y(z) \mp y(z')\big)\, dz'$$

$$\beta_y^* W_{crs} = -(2\pi)^{3/2}\sqrt{2}\xi_{bb}\sigma_z\delta_{\sqrt{2}\sigma_z/\theta_P}(z)$$

$$\delta_\sigma(z) \equiv \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2\sigma^2}$$

Similar formula and similar wake

Space charge force

$$\Delta p_y(z) = \int_{-\infty}^{\infty} W_{sc}(z - z')\rho(z')\big(y(z) - y(z')\big)\, dz'$$

$$\langle\beta_y\rangle W_{sc}(z) = (2\pi)^{3/2}\alpha\xi_{sc}\sigma_z\delta(z)$$

$1 \le \alpha \le 2$

TMCI/srcMC α=2

Mode analysis

- Dipole and quadrupole wake force

$$\frac{dp_y(z,\delta)}{ds} = -\beta_y(s)\int [w_Q(z-z')y(z,\delta) + w_D(z-z')y(z')]\rho(z')dz'$$

$$y(z) = \int y(z,\delta)\psi(z,\delta)d\delta/\rho(z) \qquad \rho(z) = \int \psi(z,\delta)d\delta$$

- Wake per s length
  - Dipolar $$\beta_y(s)w_D = \pm\beta_y^* W_{crs}\delta(s-s^*) + \langle\beta_y\rangle\frac{W_{sc}+W_y}{L}$$
  - Quadrupolar $$\beta_y(s)w_Q = -\beta_y^* W_{crs}\delta(s-s^*) + \langle\beta_y\rangle\frac{-W_{sc}+W_{y,Q}}{L}$$

- Radial-azimuthal mode expansion

$$\frac{y(z,\delta)}{\sqrt{\beta_y}} = \sum_{l=-\infty}^{\infty} y_l(J)e^{il\phi} \qquad \sqrt{\beta_y}p_y(z,\delta) = \sum_l p_l(J)e^{il\phi}$$

Momentum change of each mode

- Dipole and quadrupole wake force

$$\frac{dp_l(J)}{ds} = -\beta_y(s)\left[\sum_{l'} \bar{w}_{ll'}(J)y_{l'}(J) + \int \sum_{l'} w_{ll'}(J,J')\psi(J')y_{l'}(J')dJ'\right]$$

$$\bar{w}_{ll'}(J) = \frac{1}{2\pi}\int \psi(J')dJ'd\phi d\phi' w_Q(z-z')e^{-i(l-l')\phi}$$

$$w_{ll'}(J,J') = \frac{1}{2\pi}\int d\phi d\phi' w_D(z-z')e^{-il\phi+il'\phi'}$$

- Discretize for J

$$\frac{dp_l(J)}{ds} = -2\sum_{J'}\sum_{l'} m_{w,lJl'J'}y_{l'}(J')$$

$$m_{w,lJl'J'} = \frac{\beta_y}{2}\left[\bar{w}_{ll'}(J)\delta_{JJ'} + w_{ll'}(J,J')\psi(J')\Delta J\right]$$

Two types of TMCI

1. Instability caused by wakes that are considered uniformly distributed on average.
   - Modeled by uniform wake
   - The instability is betatron tune independent
   - The instability threshold is determined by the frequency difference between synchro-beta sidebands; namely synchrotron tune.

2. Additional Instability caused by locality of the wake.
   - Modeled by wake and transfer model.
   - This instability is dependent of the betatron tune.
   - Higher order mode for larger separation of the betatron tune from integer/half integer.

Uniformly distributed wake source

- Base $(y_l(J) + ip_l(J))$
- Wake $\delta(y_l(J) + ip_l(J)) = -im_{w,lJ,l'J'}\delta s(y_{l'}(J') + ip_{l'}(J'))$

  Equipartitioning
- Synchro-betatron motion $\delta(y_l(J)+ip_l(J)) = -i(\delta\phi_y + l\delta\phi_s)(y_l(J)+ip_l(J))$
- One turn matrix $e^{-i\mu}$, square matrix with the order $(2l_{max}+1)\times n_J$

$$(y_l(J) + ip_l(J))_L = \exp(-i\mu_{lJ,l'J'})(y_{l'}(J') + ip_{l'}(J'))$$

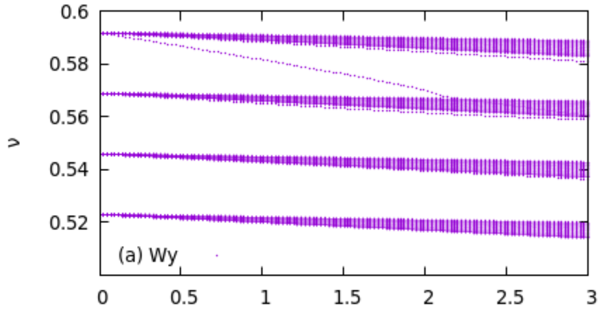
- Matrix to be solved eigenvalue problem

$$\mu = \mu_y + l\mu_s + M_W \qquad M_W = m_w L$$

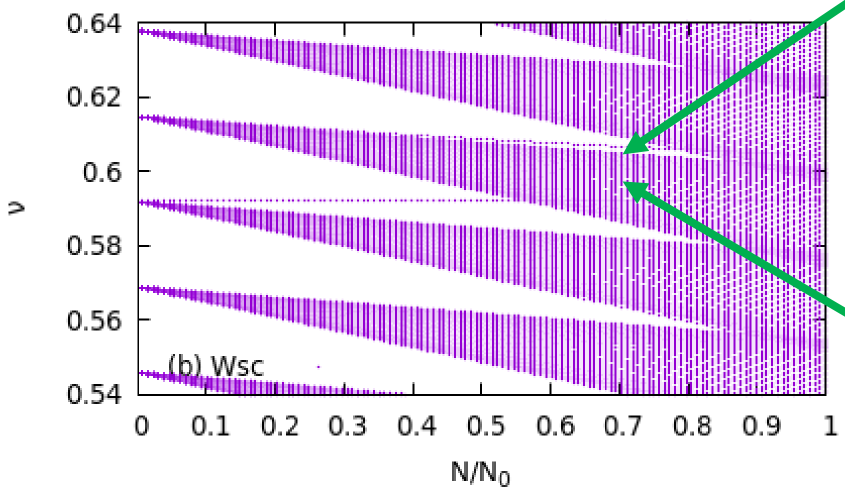
Mode spectrum for cases with one of Wy, SC or BB using uniform wake

Single e+ beam

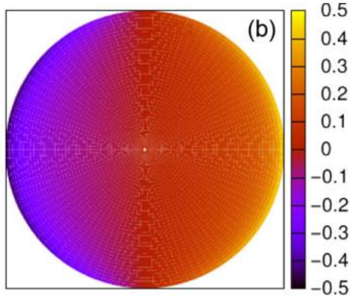
- Wy

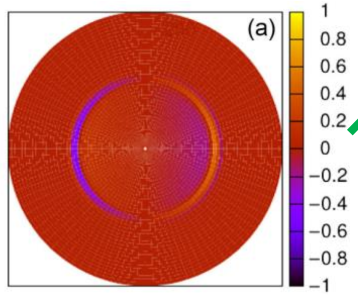


- SC



Isolated mode

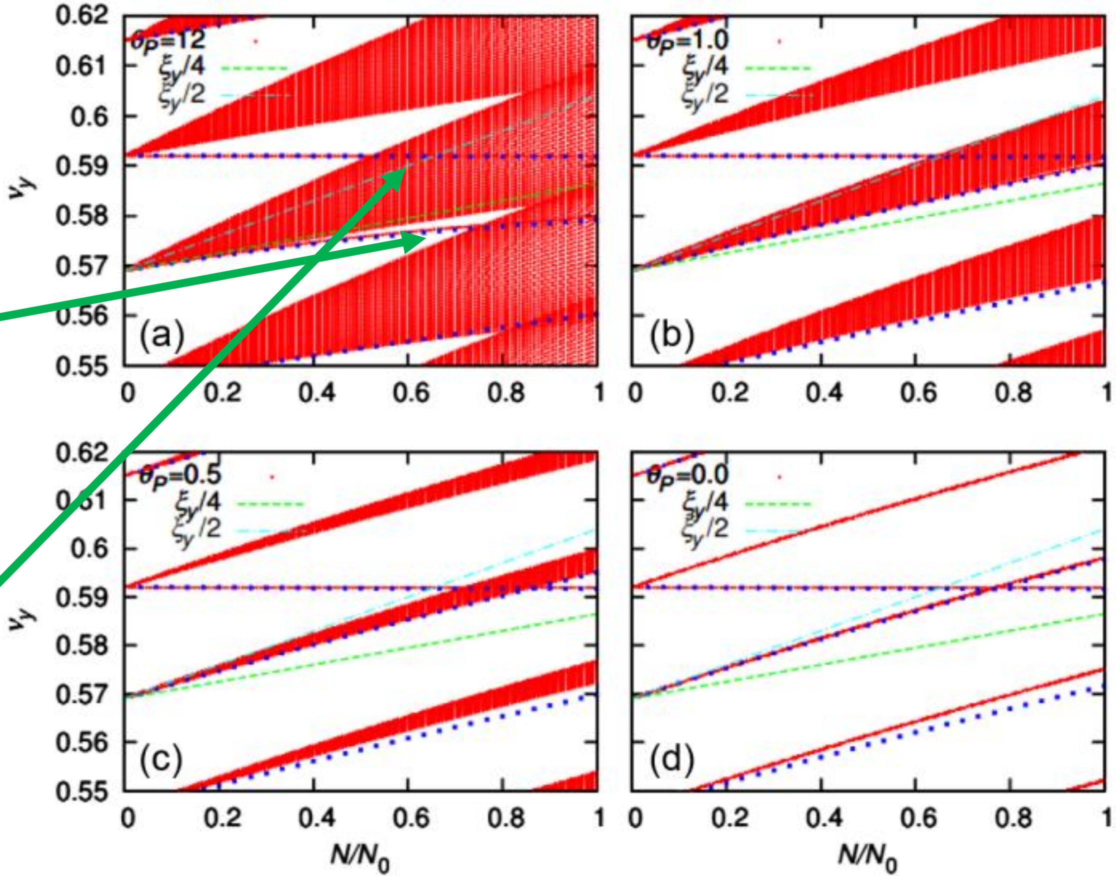




Distributed mode

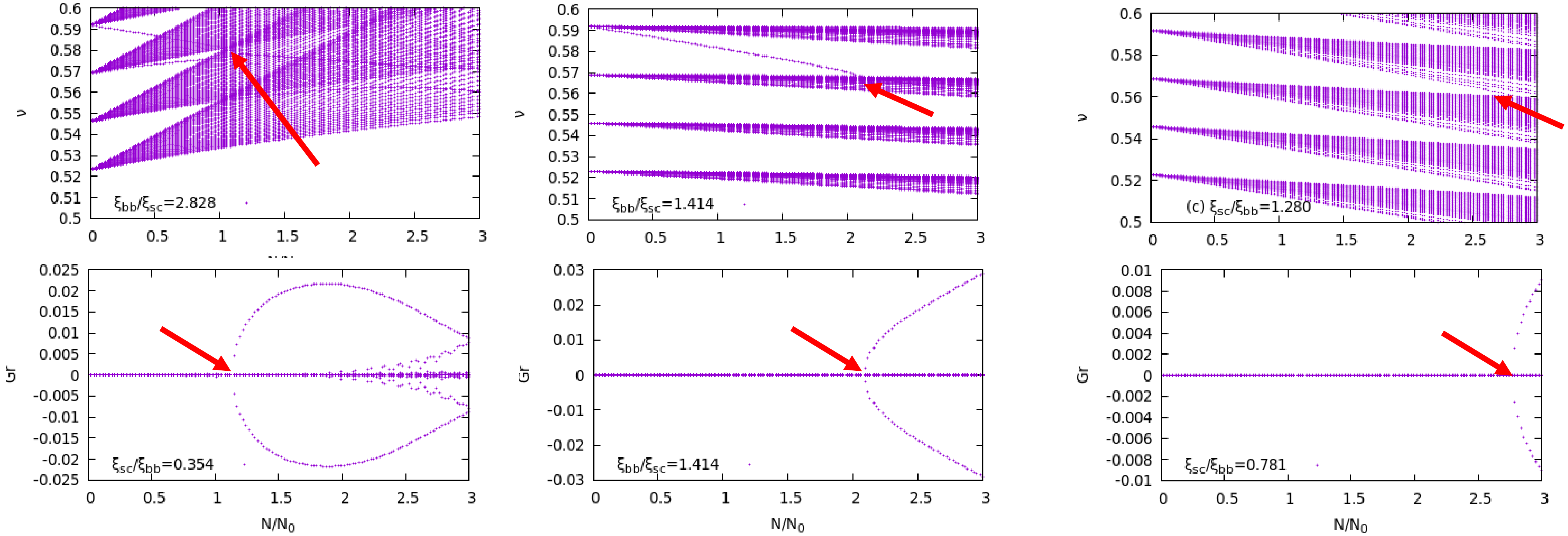
Colliding beam

BB (σ mode)



Combined Wake, BB+SC+Wy (approx. uniform)

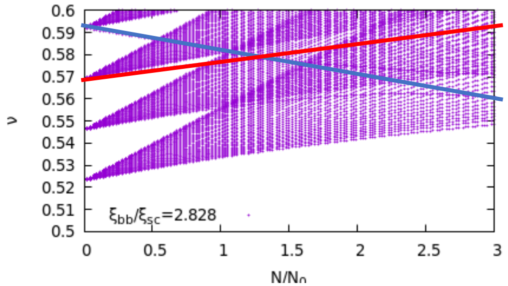
- BB dominant — perfect cancel $\xi_{bb} = \sqrt{2}\xi_{sc}$ — SC dominant

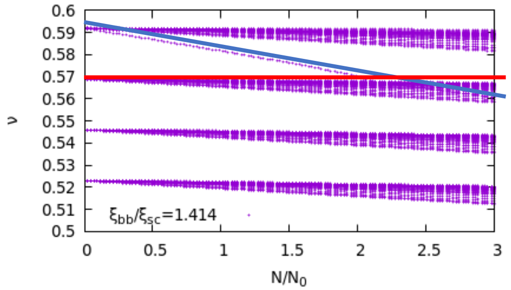


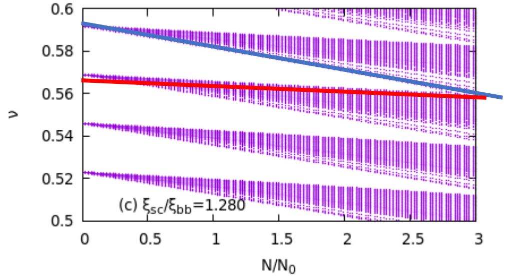
- The growth rate and threshold aree seen in the imaginary part of tune.
- The threshold of instability increases for higher space charge.
- The threshold for $\xi_{bb} = \sqrt{2}\xi_{sc}$ is the same as the case only Wy.

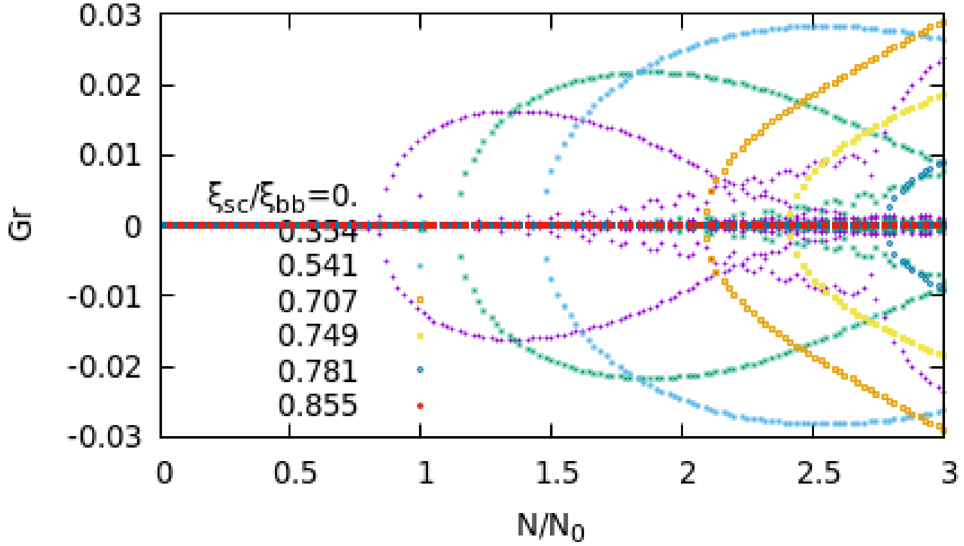
Threshold as a function of ξ_{sc}/ξ_{bb}

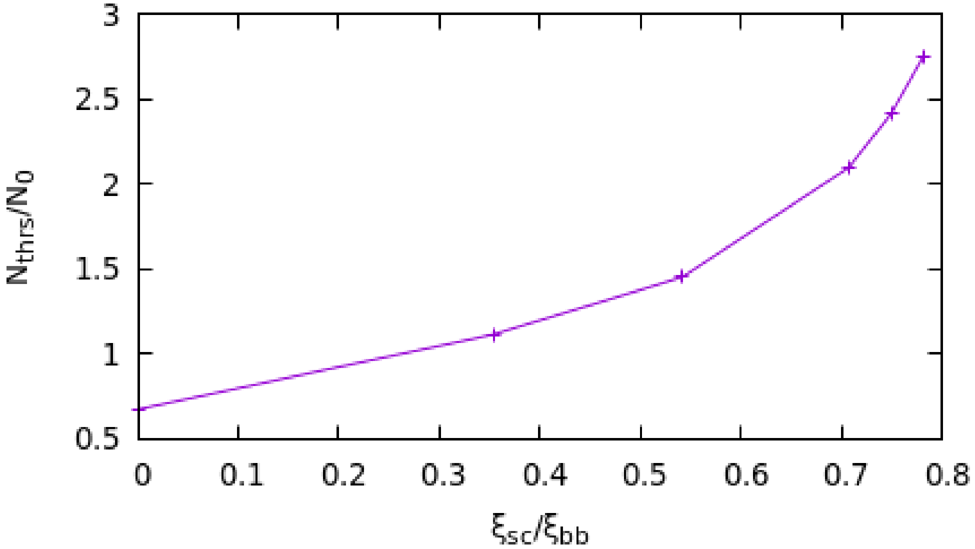
- Slope of isolated -1 mode decreases for increasing space charge force, while slope of 0 mode is determined by the impedance.
- The threshold increases for SC dominant.











Combined case of localized beam-beam and uniform space charge

Transformation for the vector (y_l, p_l) with the length $2\times(2l_{max}+1)\times n_J$

- Arc is divided by n

$$U = e^{il\phi_s}\begin{pmatrix} \cos\phi_y & \sin\phi_y \\ -\sin\phi_y & \cos\phi_y \end{pmatrix}$$

$$\phi_y = \pi\nu_y/n$$
$$\phi_s = \pi\nu_s/n$$

- Wake

$$T_{bb,sc} = \begin{pmatrix} I & 0 \\ 0 & M_{bb,sc} \end{pmatrix} \qquad M_{bb,sc} = m_{bb,sc}\delta s \qquad \delta s = L/n$$

- The revolution matrix for which the eigenvalue problem is solved

$$T = T_{bb}(UT_{sc}U)^n$$

- This formalism is easily extended to realistic space charge considering beta function variation.

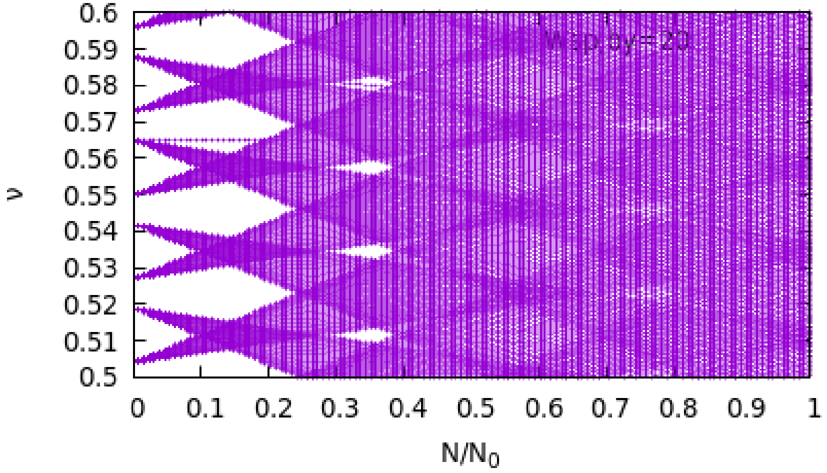
Mixture of localized beam-beam and uniform space charge forces

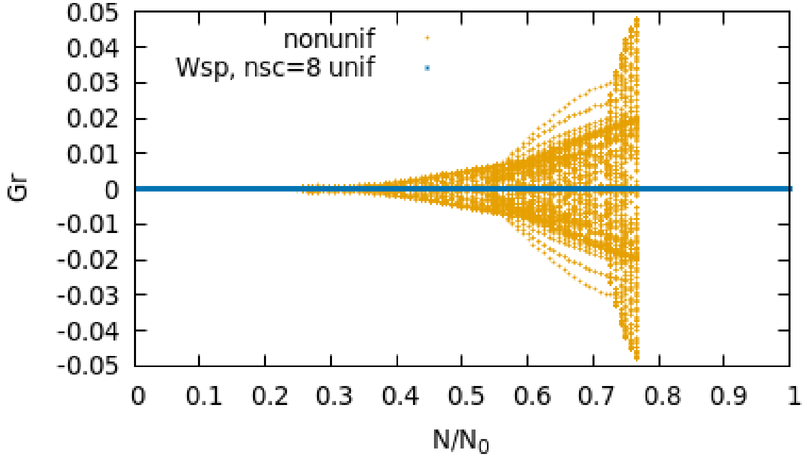
- Characteristic of the localized wake
- Mode tune is reflected (wrapped) at (half)-integer.
- Instability occurs even pure imaginary impedance coupling with a wrapped mode.
- In a large separation for (half)-integer, high order mode instability is concerned.

- In a special case satisfying $\xi_{bb} = \sqrt{2}\xi_{sc}$, no tune shift in uniform approx. However a tune shift appears in the presence of a localized wake.

How to realize uniform space charge in the matrix transformation

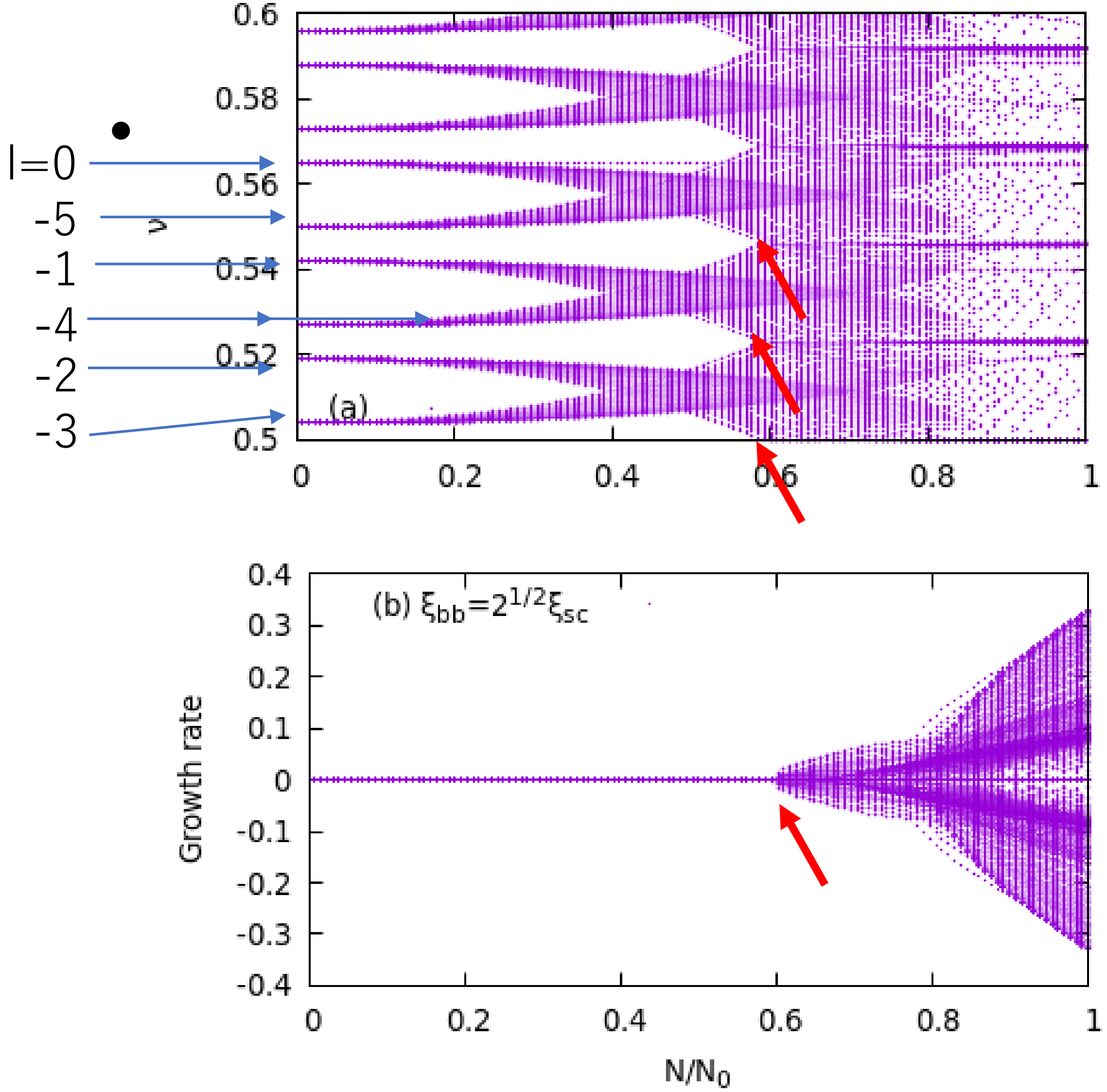
- We study linear instability. Avoid structure resonance of half integer for space charge.
- 8 points, ν_y=43.565
- Uniform $M_{sc}=M_{\nu/16} M_{sc/8} M_{\nu/16}$, $M_t= M_W (M_{sc})^8$
- Break uniformity a bit $M_{sc}=M_{sc/8} M_{\nu/9}$, $M_t= M_W M_{\nu/9} (M_{sc})^8$





- Artificial growth is avoided by space charge force applying at 8 points uniformly.
- Considering beta variation precisely, physical growth is evaluated.

Mode behavior in the case $\xi_{bb} = \sqrt{2}\xi_{sc}$



- Unstable for coupling with the same parity mode for pure imaginary impedance like beam-beam and space charge.
  - Parity l: even or odd
- Instability occurs at coupling with different parity mode, when real part of impedance is finite like ordinary wake.

PIC simulation for beam-beam, space charge and impedance

- SCTR-bb (BBSSL), GPU code developed by cuda c++.

https://kds.kek.jp/event/42931/

https://kds.kek.jp/event/43469/

https://kds.kek.jp/event/50714/

- Lattice tracking is available using SAD compatible code (Li Zhiyuan).

Simulation using a symmetric model

- $E=4\text{GeV}\times 4\text{GeV}$, $N_p=5\times 10^{10}$, $\varepsilon_x=4\text{nm}$, $\varepsilon_y=25\text{pm}$, $\beta_x^*=8\text{cm}$, $\beta_y^*=1\text{mm}$, $\sigma_z=6\text{mm}$
- Arc is tracked by the same way as the mode analysis.
  - Divided in to n=8, $T=T_{bb}(UT_{sc}U)^n$
- $(\beta_{xsc}, \beta_{ysc})=(5,20)$ $\Delta\nu_y=0.0512$
- (5,50) $\Delta\nu_{ysc}=0.0750$
- (5,100) $\Delta\nu_{ysc}=0.0980$
- (12,12) $\Delta\nu_{ysc}=0.0275$

$$\lambda_p = \frac{N_p}{\sqrt{2\pi}\,\sigma_z};$$

$$\Delta\nu_x = -\frac{\lambda_p r_e \beta_x L}{2\pi\sigma_x(\sigma_x+\sigma_y)\beta_p^2\gamma_p^3};$$

$$\Delta\nu_y = -\frac{\lambda_p r_e \beta_y L}{2\pi\sigma_y(\sigma_x+\sigma_y)\beta_p^2\gamma_p^3};$$

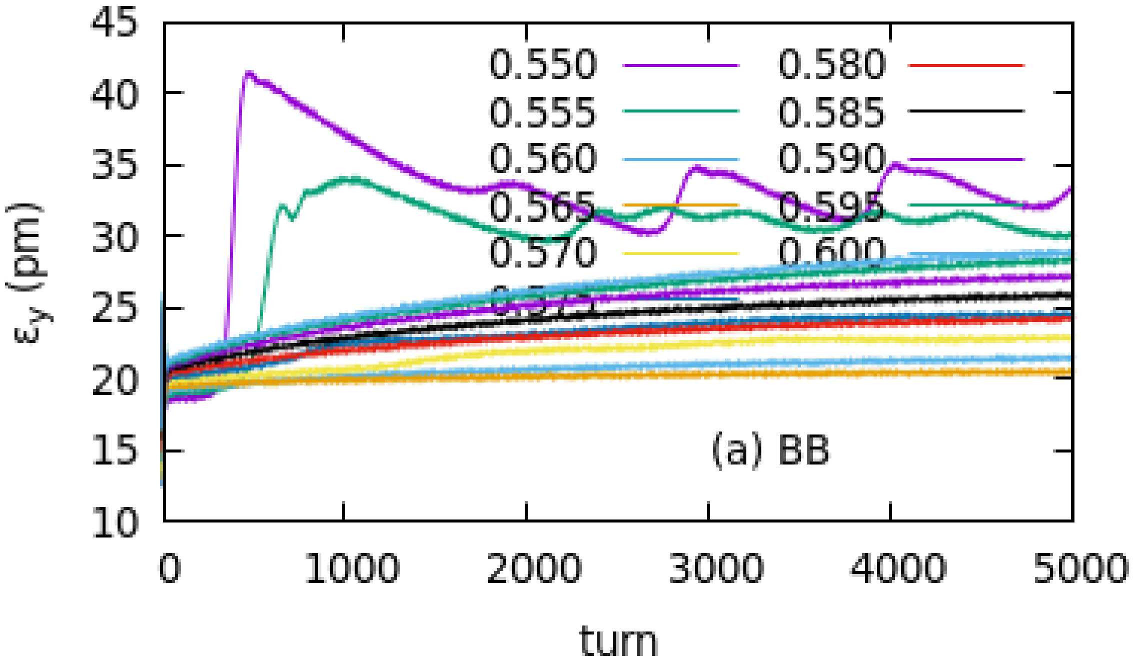
- $\Delta\nu_{ybb}=0.0726$

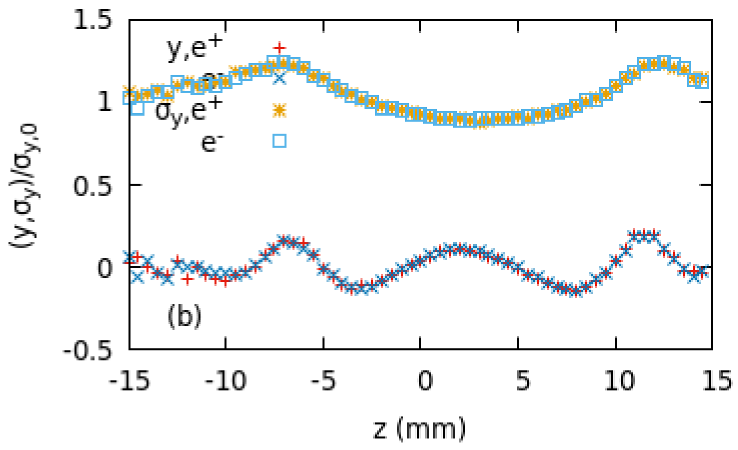
$$\Delta\nu_{yL} = \frac{N_e\ r_e}{2\pi\gamma_p}\ \frac{\beta_y}{\sigma_{xz}\sigma_y};$$

$$\sigma_{xz} = \sqrt{\epsilon_x\beta_x + (\theta_{crs}\sigma_z)^2};$$

Strong-strong simulation for only beam-beam

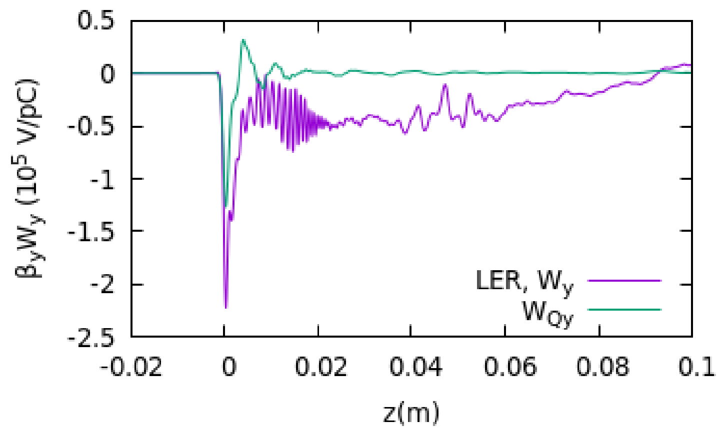
- Typical result for 2-nd type of TMCI
- The emittance growth is saturated at 2xε_y, and behaves saw-tooth.
- σy growth only at ν_y =0.550 and 0.555.
- High order (l~3) vertical beam-beam head-tail (Y-Z) instability.



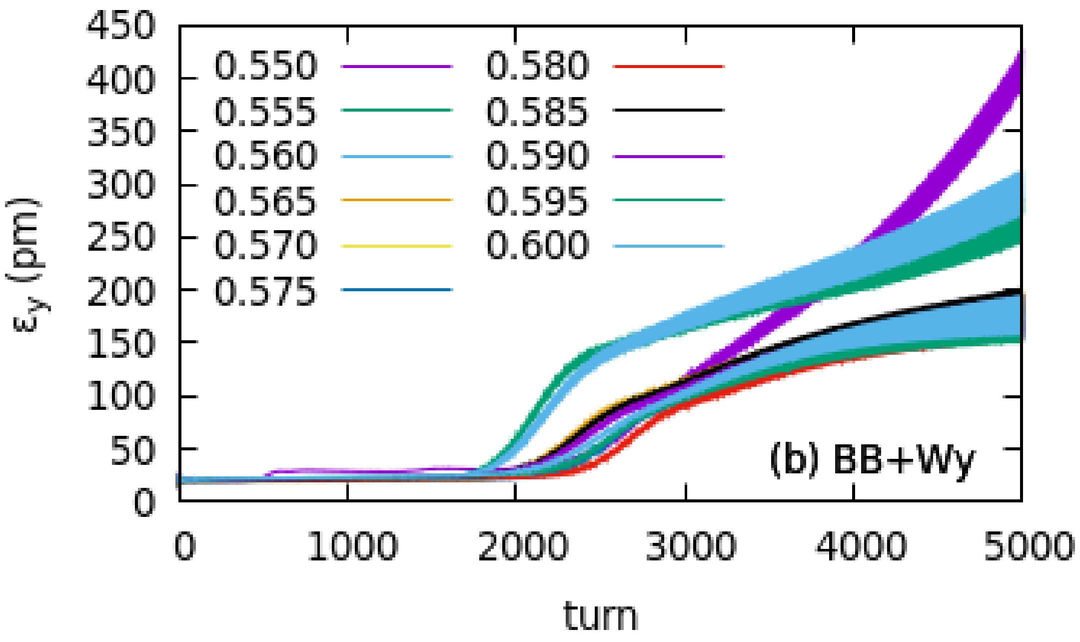


Vacuum environment impedance is added

- Impedance data by T. Ishibashi, D. Zhou, https://kds.kek.jp/event/40318/

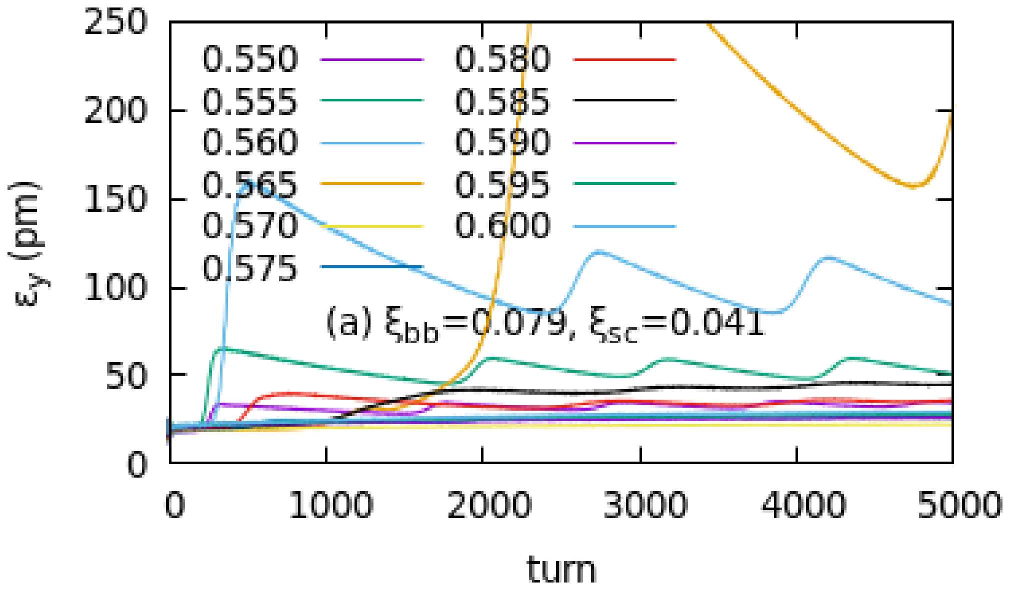


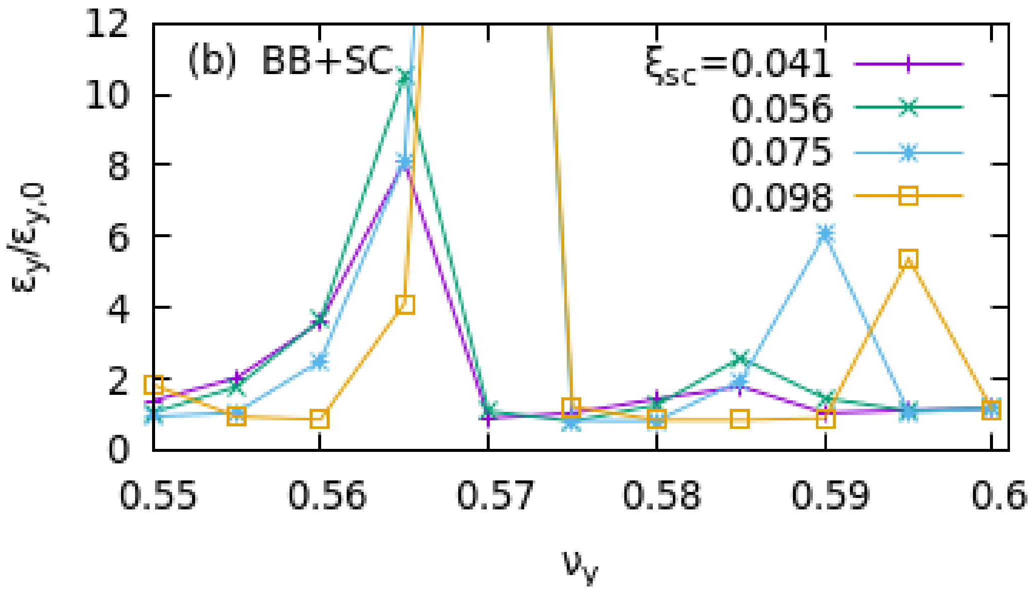
- Instability simulation considering beam-beam and the impedance
- Large emittance growth than that for only beam-beam.
- Instability occurs independent of ν_y.
- 1st type of TMCI, PRAB 26, 111001



Simulation considering beam-beam and space charge

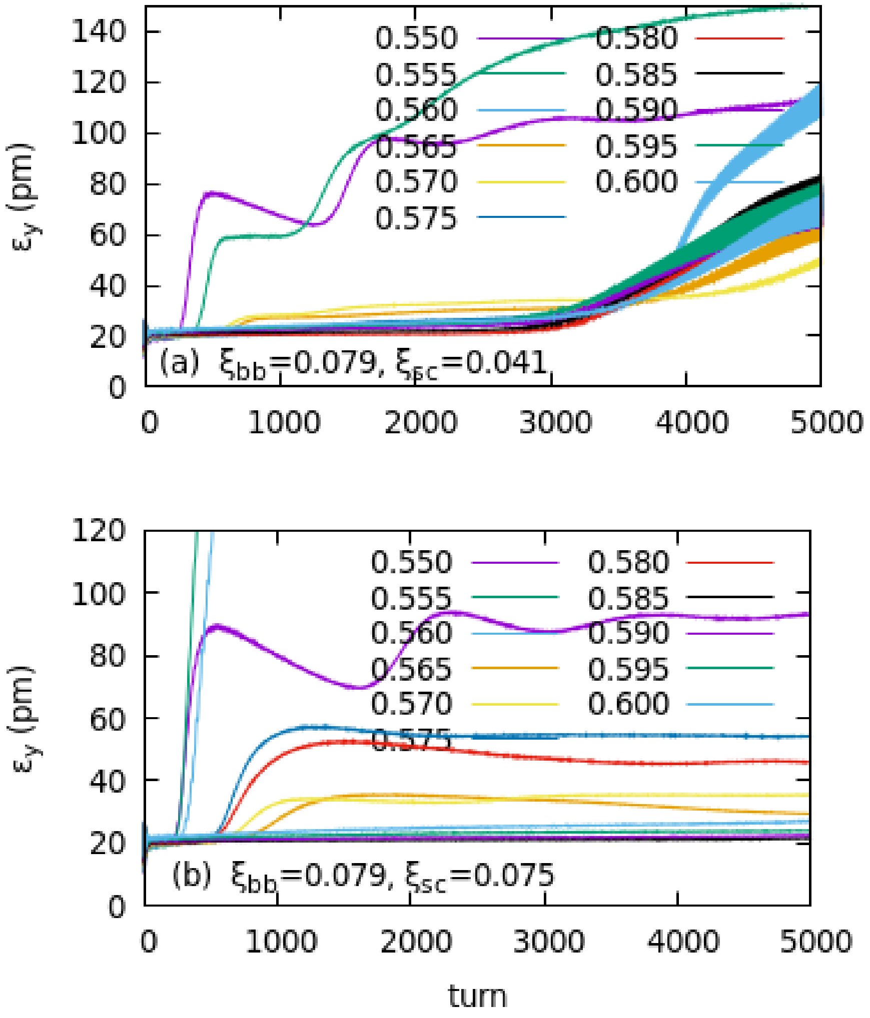
- The effective impedance is pure imaginary.
- Increasing space charge effect (tune shift), 2nd type of instability is enhanced by lack of Landau damping due to tune shift(spread) cancellation





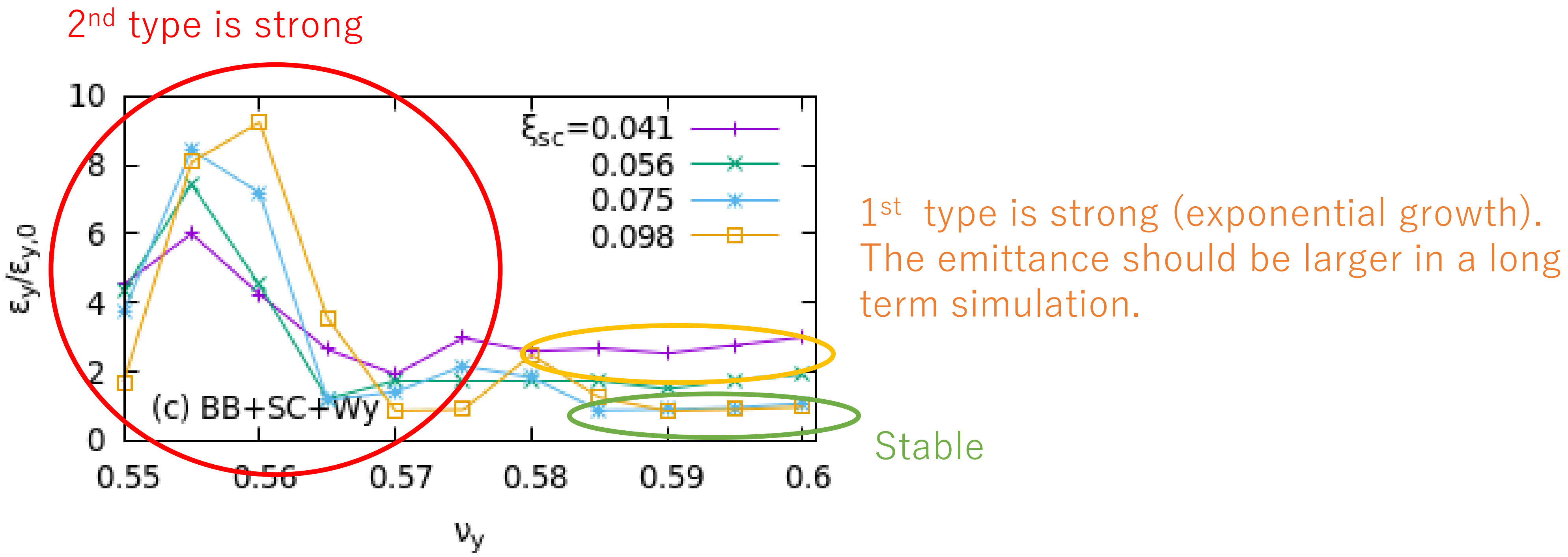
Simulation considering beam-beam, space charge and impedance

- Low space charge tune shift
  - Saw-tooth behavior at ν_y=0.550 and 0.555. 2nd type of instability is dominant.
  - Exponential growth is seen at $\nu_y \geq$0.560.
- High space charge tune shift
  - Saw-tooth behavior is seen wide range of tune.
  - 1st type of instability is cancelled by space charge.
  - 2nd type is enhanced by lack of Landau damping due to the tune shift (spread) cancellation.
  - Stable solution is found at high tune $\nu_y \geq$0.585.



Summary of simulation considering beam-beam, space charge and impedance

- Stable solution is found at high tune ν_y ≥0.585 for a high space charge tune shift.



More Realistic case: asymmetric collision

- E(e+) 4 GeV x E(e-) 7 GeV. $N_p=7\times10^{10}$, $N_e=4\times10^{10}$.
- β_x =0.08, β_y =0.001, ξ_{bb}~0.063 for both beams,
- $\beta_x=\beta_y$=12m, ξ_{sc} =0.0384, β_x =8, β_y=20m, ξ_{sc} =0.0583,
- No space charge for e- beam.

- Arc of LER is tracked by the same way as the mode analysis.
  - Divided in to n=8, $T = T_{bb}(UT_{sc}U)^n$

- Wake data "Integrated wake potentials in LER/HER" in https://kds.kek.jp/event/40318/

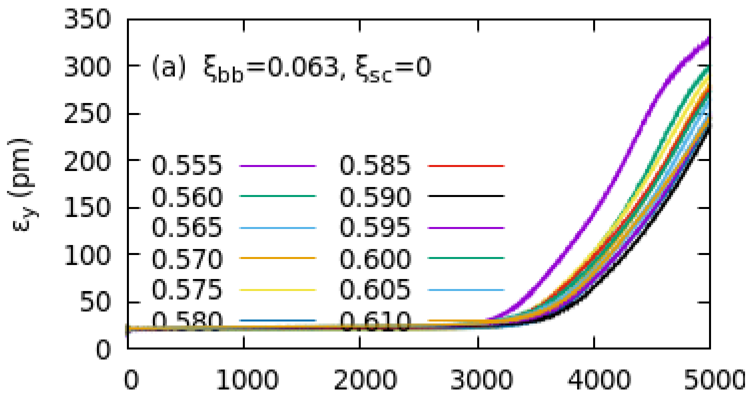
- sed 's/skbts_12/skbts_23/;s/43.555/43.610/' skbts.pl | perl -f

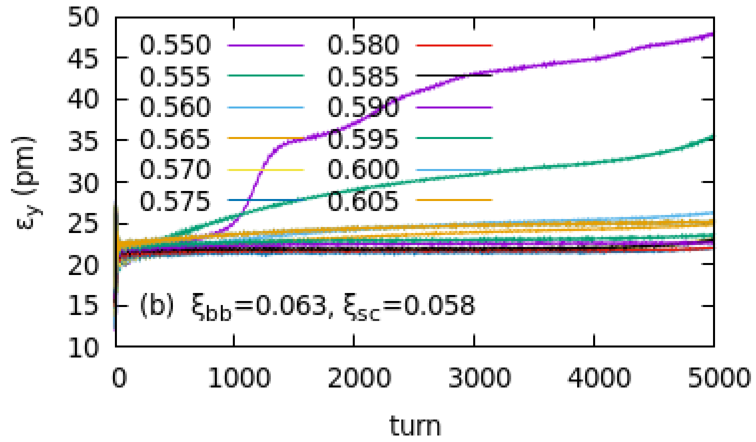
Simulation for an asymmetric collider
considering beam-beam, space charge and impedance

- ξ_{sc} =0.
  - Exponential growth is seen at all tune.

- ξ_{sc} =0.0583
  - Saw-tooth behavior is seen wide range of tune at ν_y=0.550 and 0.555. 2nd type of instability.
  - 1st type of instability is cancelled by space charge.
  - Stable at $\nu_y \geq$0.560.

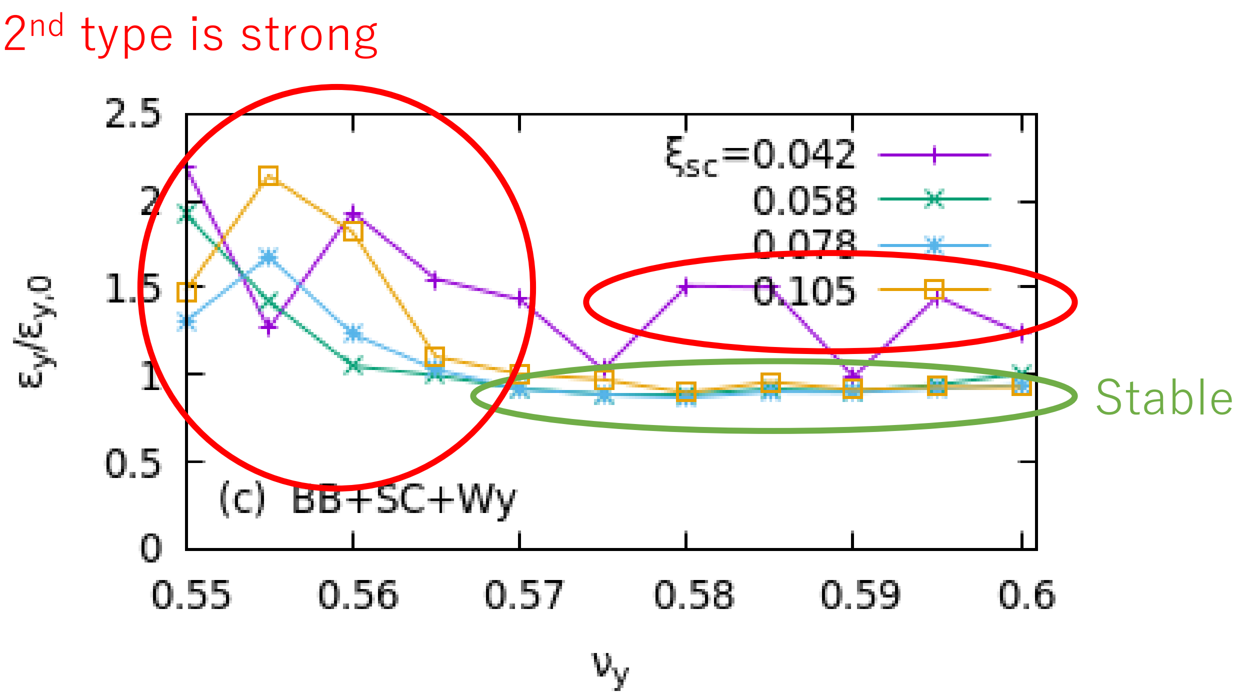
- Better situation than symmetric collision even space charge only in e+ beam





Summary of simulation for an asymmetric collider

- Stable area increases compare with the symmetric case.
- Coherence of the two beams



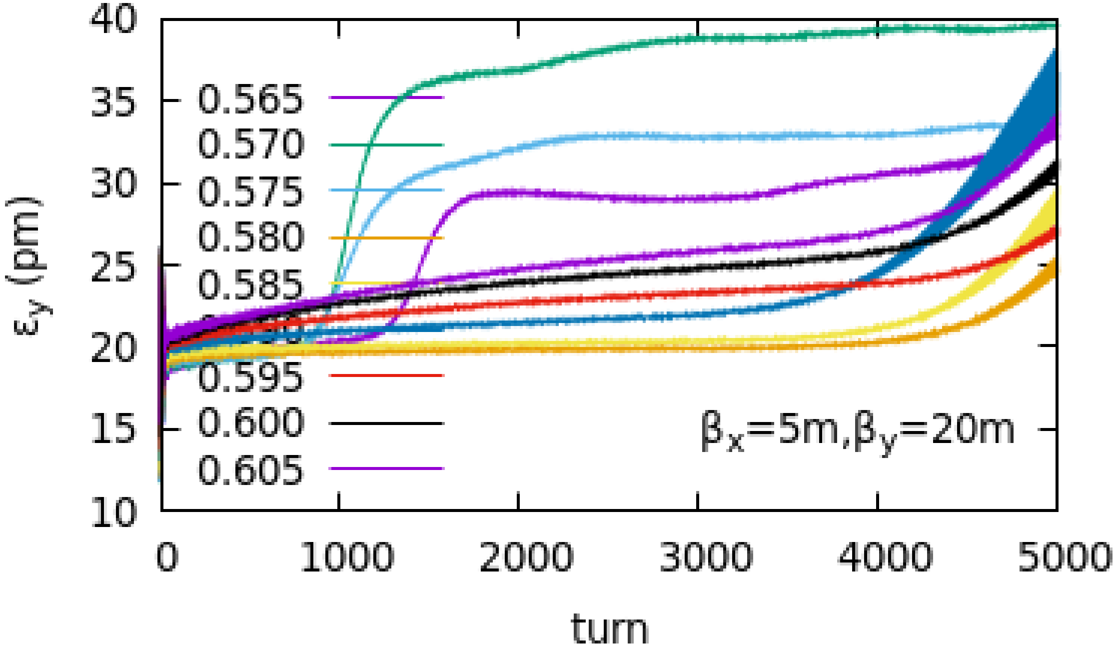
Summary

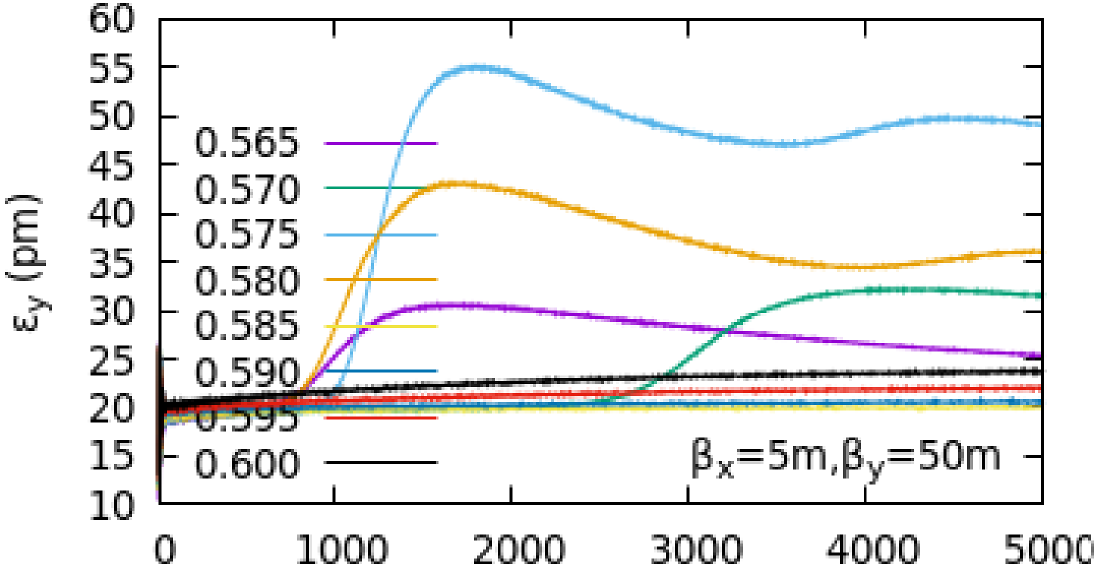
- Combined effects of beam-beam, space charge and impedance were studied.
- The cross wake for a large Piwinski angle collision is similar to the space charge wake.
- Type 1 TMCI: all wakes can be considered uniformly distributed in a ring.
  - Typical case: beam-beam modes 0 and -1 are coupled by the impedance. This instability is suppressed by the space charge force.
- Type 2 TMCI: localized beam-beam+uniform space charge+(impedance)
  - Typical case: mode couples to that wrapped at (half)-integer. High order mode head-tail. The space charge force can not compensate for this instability.
- Strong-strong simulation with beam-beam space charge and impedance
  - These two types of TMCI are reproduced.
  - Space charge compensation worsened the instability due to the decrease of tune spread.
  - Asymmetric collision showed better stability than symmetric collision. The coherence of the two beams may be broken.

Thank you for your attention

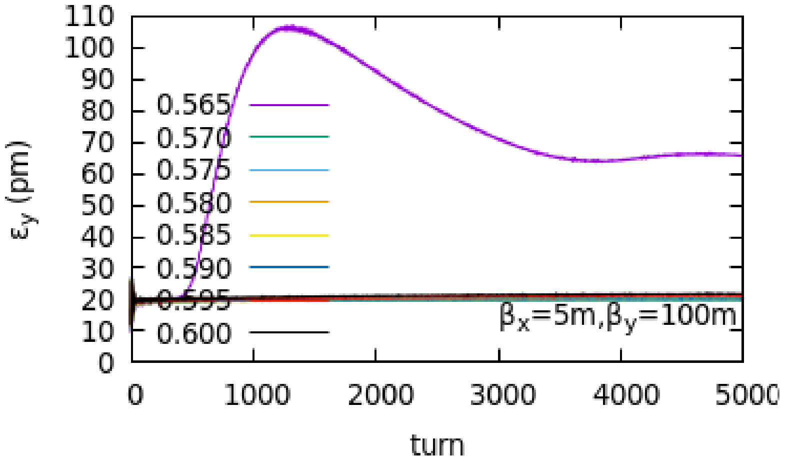
Another emittance growth induced by space charge

- Vertical beam-beam head-tail (Y-Z) instability with the same mechanism as X-Z.



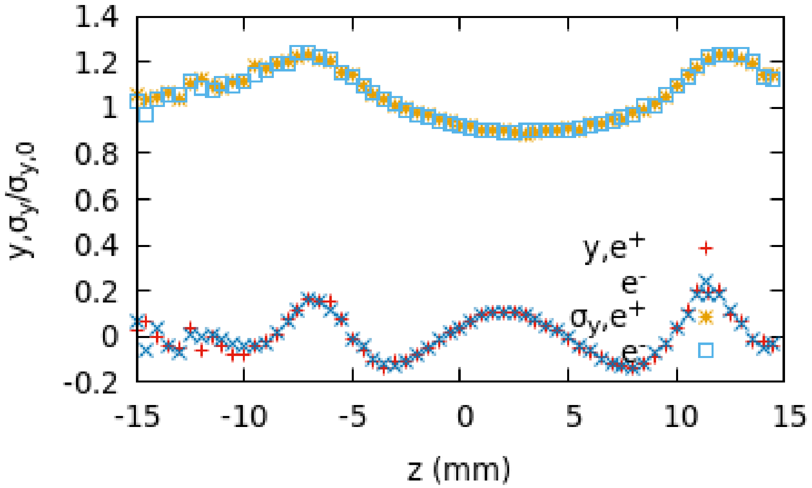


- Y-Z is enhanced by small tune spread of BB+SC



Bunch profile at Y-Z instability

- Y and sy as function of z at 1100-th turn
- Ny=0.575, bx=5m, by=20m

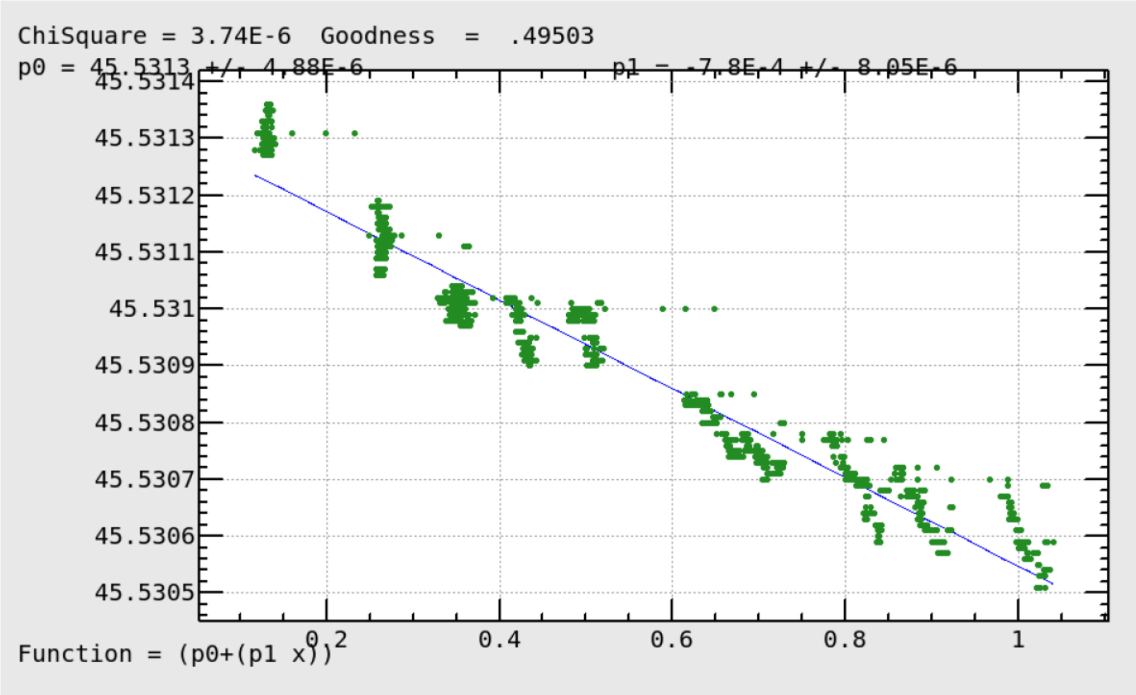


- High order mode instability, m~3.

Summary of simulations

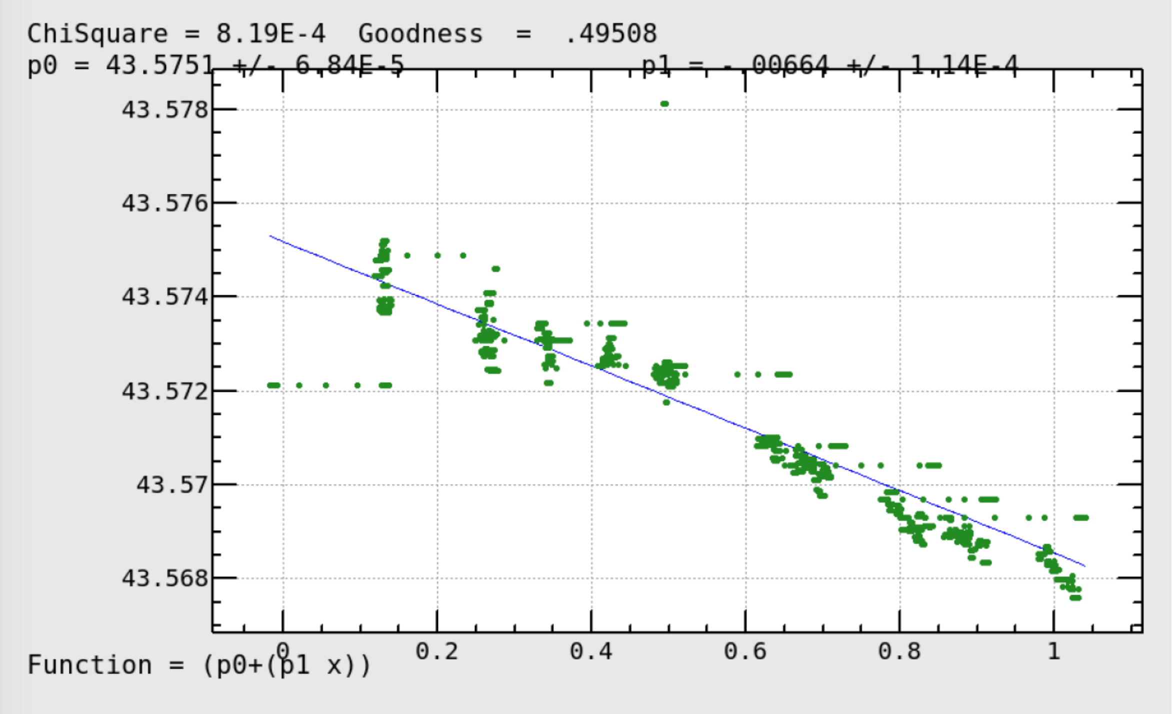
- Symmetric Beam-beam, SC, wake interactions.
- For wo space charge or wo wake, vertical beam-beam head-tail (Y-Z) instability is seen ny<0.560. The same mechanism as X-Z. BBYZ, which is caused by localized BB, is not weakened by uniform space charge.
- For wo space charge and w wake, vertical beam-beam mode coupling instability is seen independent of ny.
- Increasing space charge tune shift, BBMC weaken, but Y-Z is seen lower ny<=0.590. Lack of tune spread of BB+SC.

March 25, 2021



I_b (mA)

$\Delta\nu_x$ = -0.00078 (1/mA)



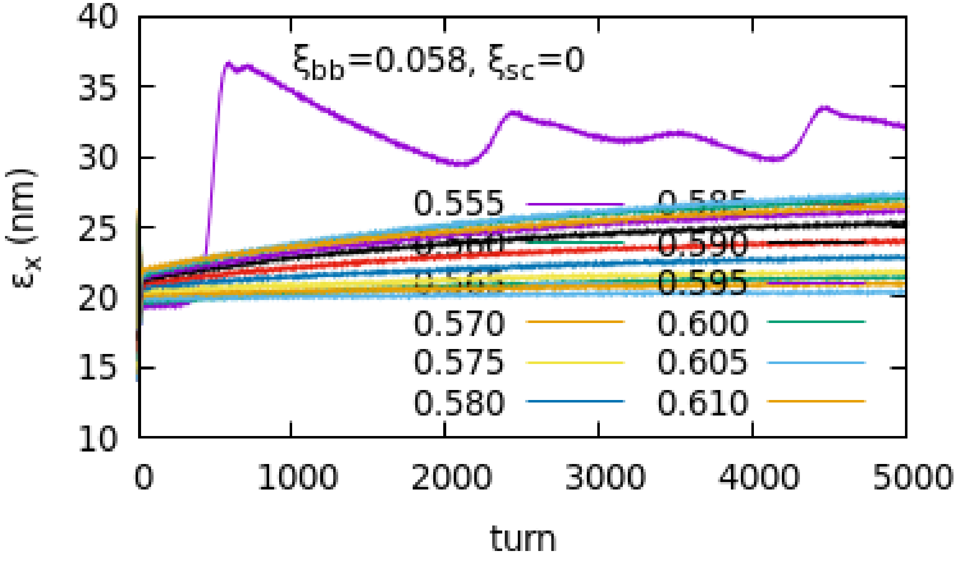
I_b (mA)

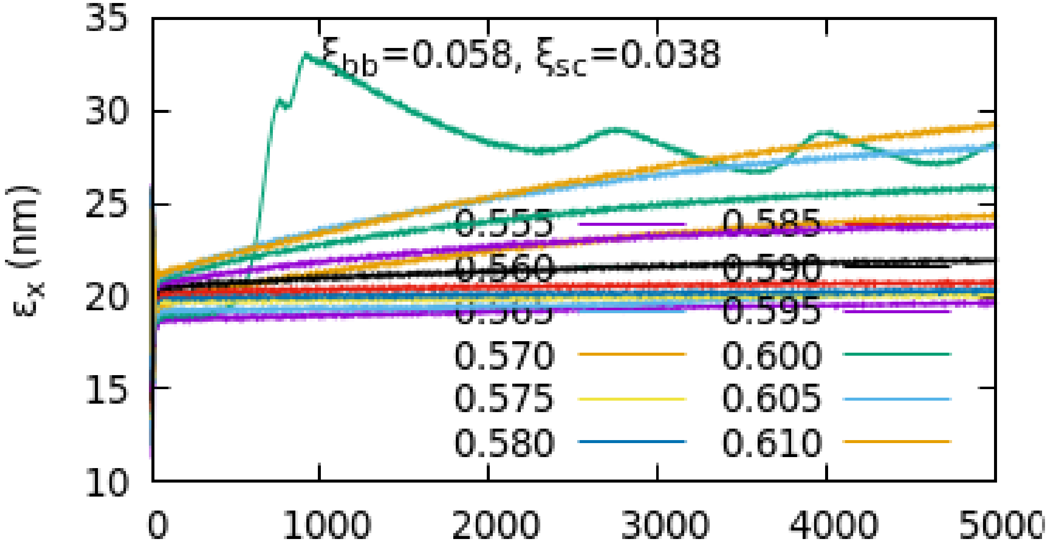
$\Delta\nu_y$ = -0.00664 (1/mA)

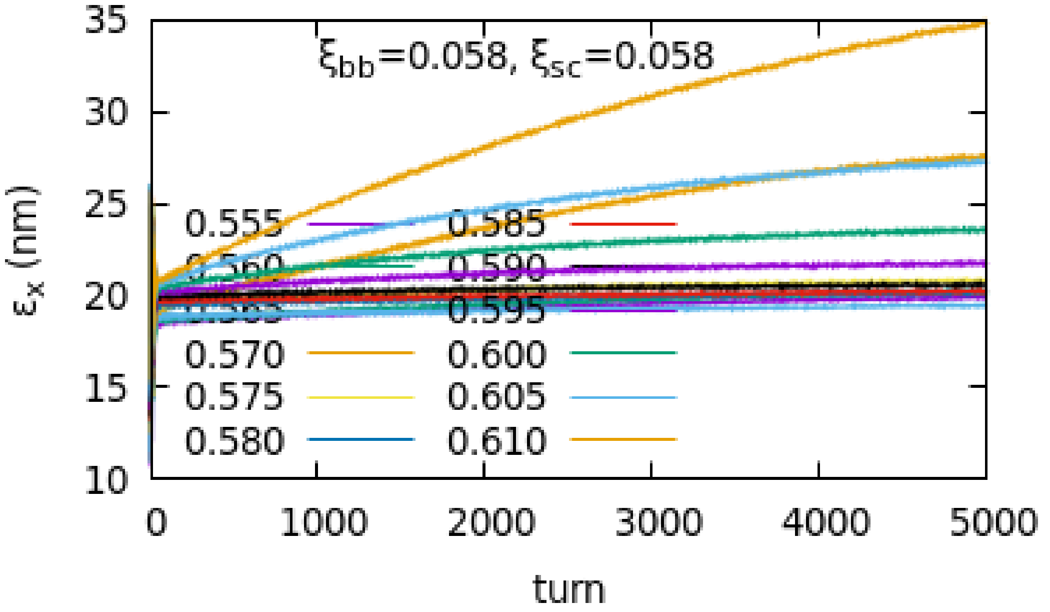
Wye=Wyp

Simulation result

- NoWk

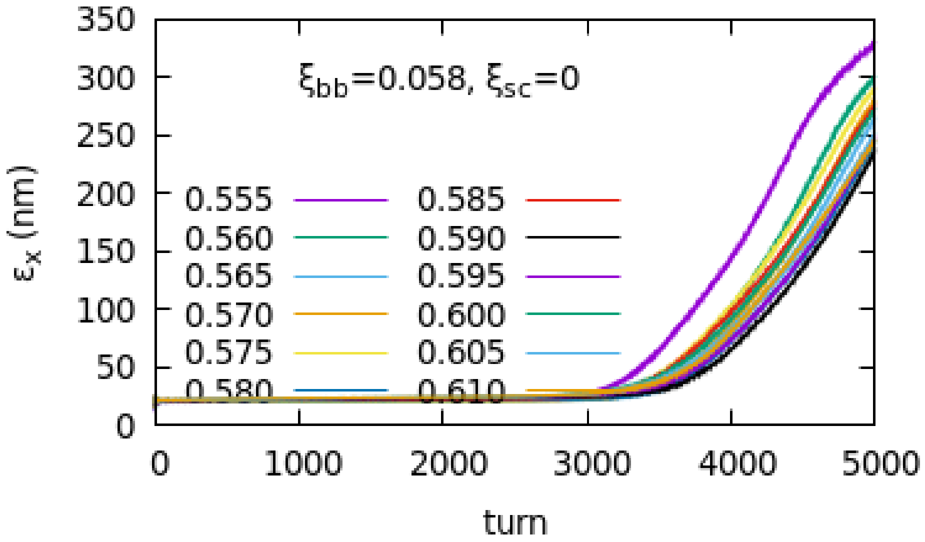


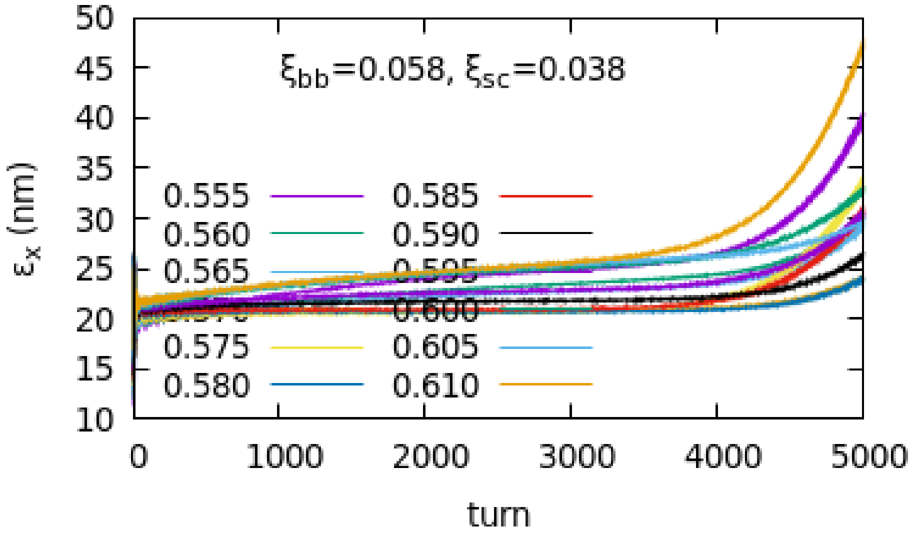


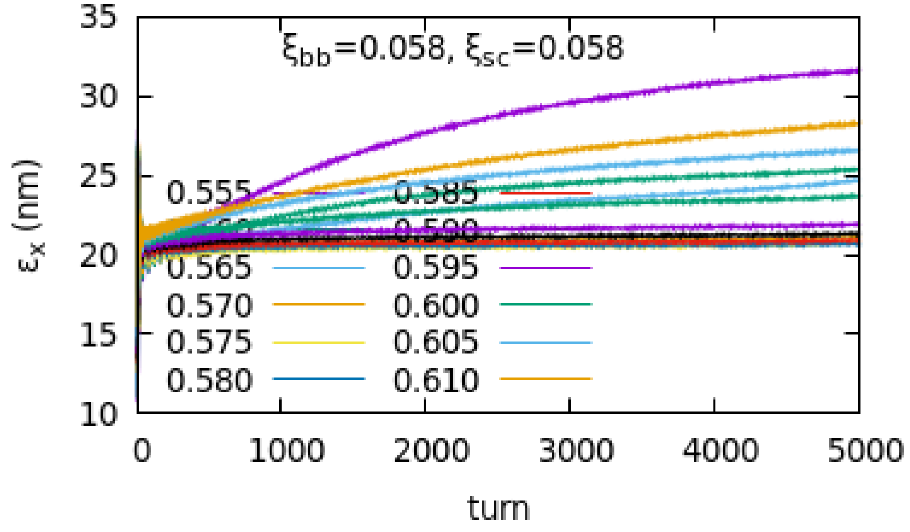


Simulation result

- Wake







Space charge force for dipole moment distribution in the longitudinal phase space

- Momentum change in the longitudinal phase space

$$\Delta p_y(z,\delta) = k_{sc}\rho(z)[y(z,\delta) - y(z)]\Delta s \qquad k_{sc}(s) = \frac{2Nr_e}{\beta^2\gamma^3}\frac{1}{\sigma_y(\sigma_x+\sigma_y)}$$

$$y(z) = \int y(z,\delta)\psi(z,\delta)d\delta/\rho(z) \qquad \rho(z) = \int \psi(z,\delta)d\delta$$

- Expression using Wake

$$W_{Q/D}(z-z') = \mp k_{sc}\delta(z-z') \qquad Z_{Q/D}(\omega) = \mp i\frac{k_{sc}}{c}$$

- One turn matrix $e^{-i\mu}$, square matrix with order

$$\bar{W}_{l,l'}(J) = -\frac{k_{sc}i^{l-l'}}{\sqrt{2\pi}\sigma_z}e^{r^2/4}I_{(l-l')/2}\left(\frac{r^2}{4}\right)$$

$$W_{l,l'}(J,J')\psi(J')\Delta J = \frac{k_{sc}i^{l-l'}}{2\pi\sigma_z}\int dk_\sigma J_l(k_\sigma r)J_{l'}(k_\sigma r')e^{-r'^2/2}r\Delta r$$

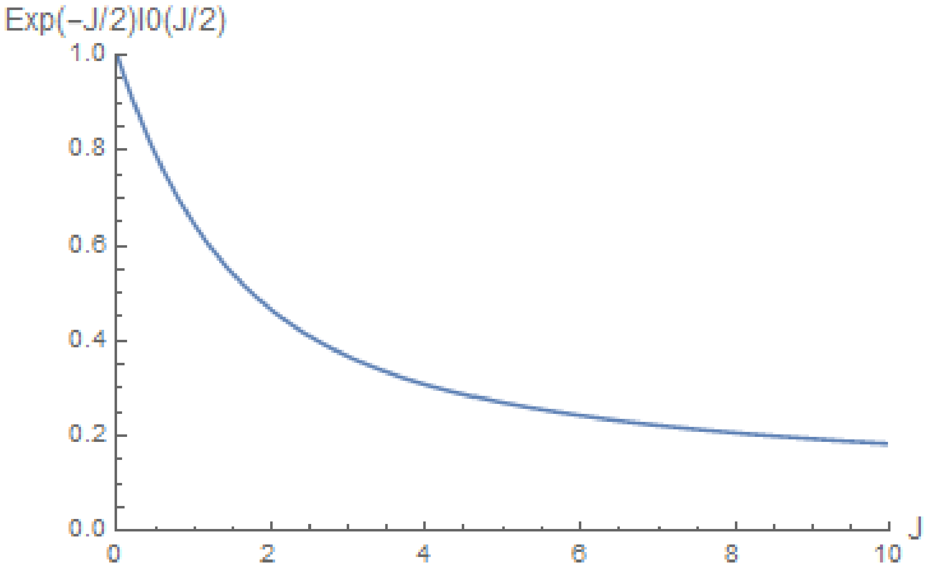
$$r = \sqrt{2J/\varepsilon_z} \qquad k_\sigma = \omega\sigma_z/c$$

Tune shift of space charge mode

- Tunes shift for J dependent modes

$$\nu_{y,l}(J) = \frac{\beta_y}{4\pi}\bar{W}_{ll}(J) = -\frac{\beta_y}{4\pi}\frac{k_{sc}}{\sqrt{2\pi}\sigma_z}e^{r^2/4}I_0\left(\frac{r^2}{4}\right)$$

$$\xi = \frac{\beta_y}{4\pi}\frac{k_{sc}}{\sqrt{2\pi}\sigma_z}$$



$J = J/\varepsilon_z$

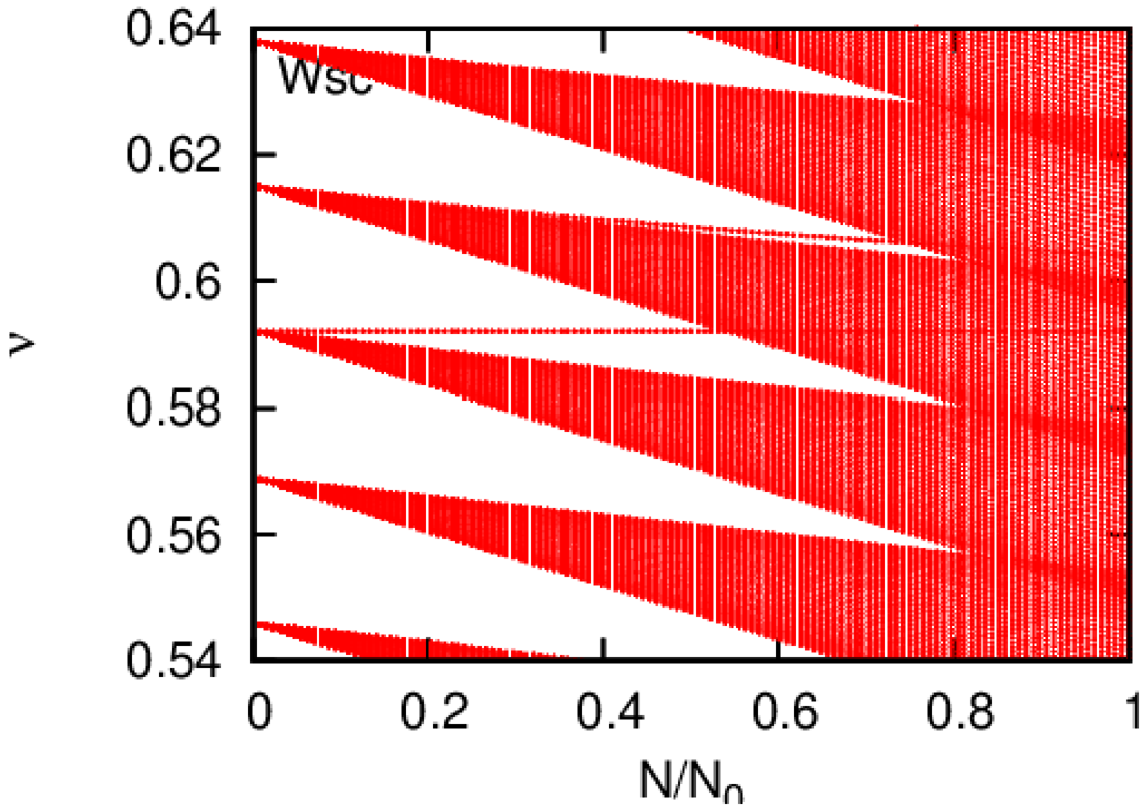
- Tune shift for simple radial mode

$$\int_{-\infty}^{\infty} dk\left[-n!L_n(k_\sigma^2/2) + (k_\sigma^2/2)^n\right]e^{-k_\sigma^2}$$

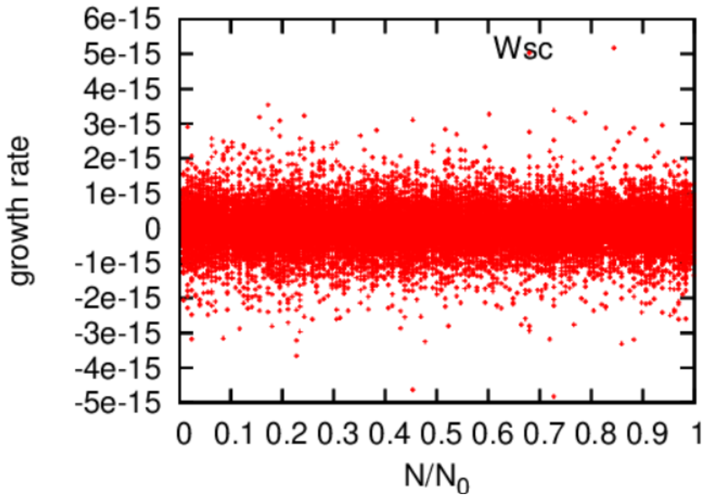
$$= \left(0, -\frac{\sqrt{\pi}}{2}, -\sqrt{\pi}, -\frac{87\sqrt{\pi}}{32}\right) \qquad \text{for } n = \pm l = (0,1,2,3)$$

Mode analysis solving eigenvalue problem

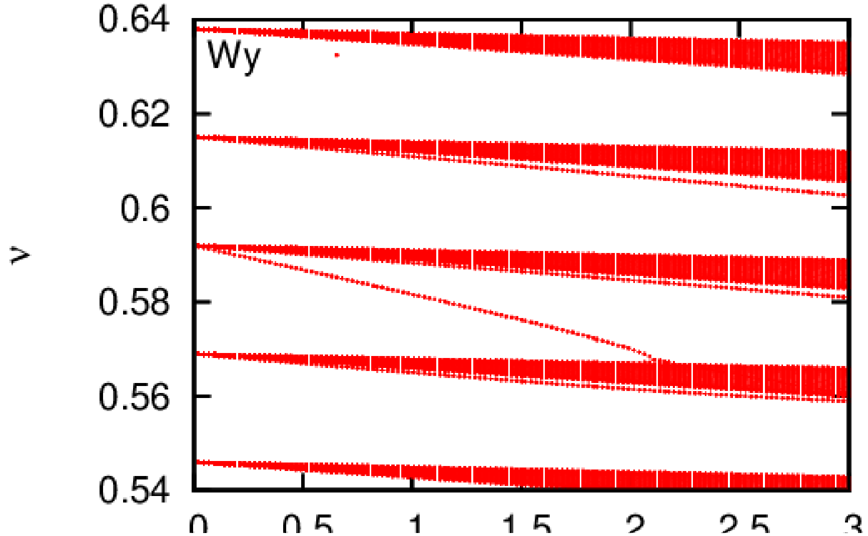
- Eigen modes in the presence of space charge force
- All modes are stable, because of pure imaginary impedance.

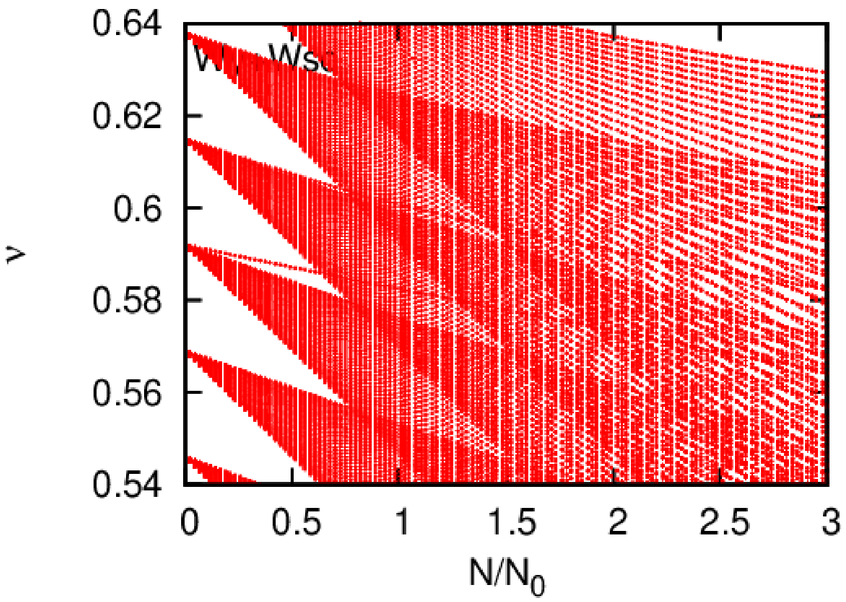


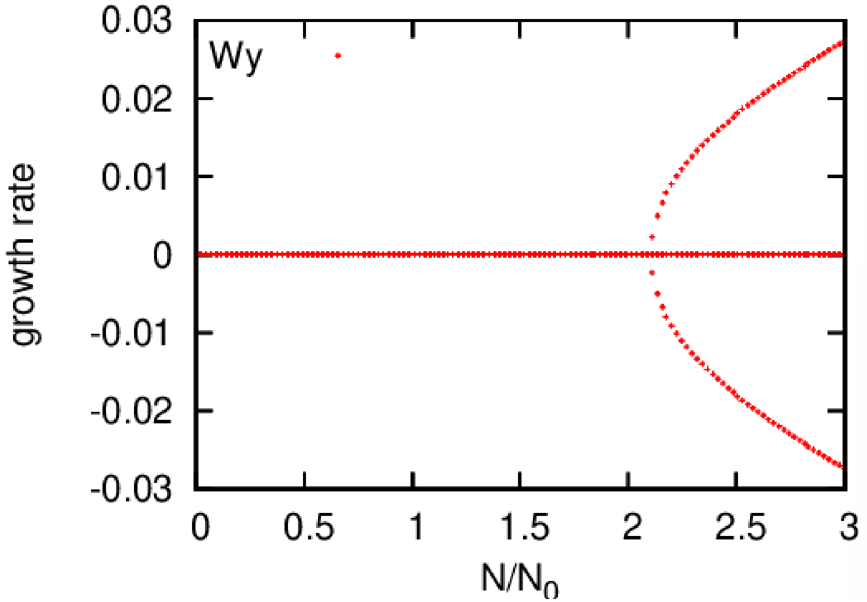
$$Z_{Q/D}(\omega) = \mp i \frac{k_{sc}}{c}$$

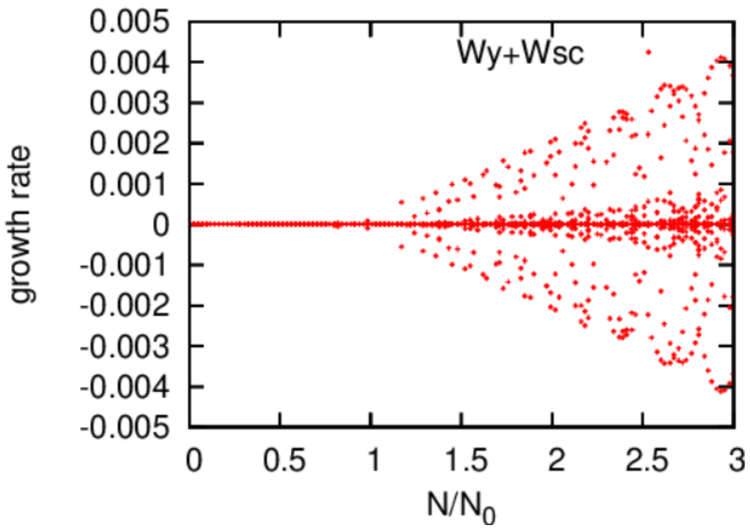


Transverse wake









Head-tail motion inner bunch

- Dipole and quadrupole wake force

$$\delta p_y(z,\delta) = -\int [W_Q(z-z')y(z,\delta)+W_D(z-z')y(z')]\rho(z')dz'$$

$$y(z) = \int y(z,\delta)\psi(z,\delta)d\delta/\rho(z) \qquad \rho(z) = \int \psi(z,\delta)d\delta$$

- Radial mode expansion

$$\frac{y(z,\delta)}{\sqrt{\beta_y}} = \sum_{l=-\infty}^{\infty} y_l(J)e^{il\phi} \qquad \sqrt{\beta_y}p_y(z,\delta) = \sum_l p_l(J)e^{il\phi}$$

- Dipole and quadrupole wake force

$$\delta p_l(J) = -\beta_y \left[\sum_{l'} \bar{W}_{ll'}(J) y_{l'}(J) + \int_0^\infty \sum_{l'} W_{ll'}(J, J') \psi(J') y_{l'}(J') dJ' \right]$$

$$\bar{W}_{ll'}(J) = \frac{1}{2\pi} \int \psi(J') dJ' d\phi d\phi' W_Q(z - z') e^{-i(l-l')\phi} = \frac{i^{l-l'-1}}{2\pi} \int d\omega Z_Q(\omega) J_{l-l'}(k_\sigma r) e^{-k_\sigma^2/2}$$

$$W_{ll'}(J, J') = \frac{1}{2\pi} \int d\phi d\phi' W_D(z - z') e^{-il\phi + il'\phi'} = i^{l-l'-1} \int d\omega Z_D(\omega) J_l(k_\sigma r) J_{l'}(k_\sigma r')$$

- Discretize for J

$$\delta p_l(J) = -2 \sum_{J'} \sum_{l'} M_{W,lJ,l'J'} y_{l'}(J')$$

$$M_{W,lJl'J'} = \frac{\beta_y}{2} \left[\bar{W}_{ll'}(J) \delta_{JJ'} + W_{ll'}(J, J') \psi(J') \Delta J \right]$$

Revolution matrix

- Arc

$$\begin{pmatrix} \bar{x}_l(J,\phi) \\ \bar{p}_l(J\phi) \end{pmatrix} = M_0 \begin{pmatrix} x_l(J,\phi) \\ p_l(J,\phi) \end{pmatrix} \qquad M_0 = e^{-2\pi i l \nu_s} \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix}$$

- Wake

$$M_W = \begin{pmatrix} 1 & 0 \\ -2M_{W,lJl'J'} & 1 \end{pmatrix}$$

- Revolution matrix, square matrix with order $2 \times (2l_{max}+1) \times n_J$

$$M_{rev} = M_0 M_W$$

Uniformly distributed wake source

- Base $(x_l(J) + ip_l(J))$
- Wake $\delta(x_l(J) + ip_l(J)) = -iM_{w,lJ,l'J'}(x_{l'}(J') + ip_{l'}(J'))$
- Synchro-betatron motion $\delta(x_l(J) + ip_l(J)) = -i(\phi_y + l\phi_s)(x_{l'}(J') + ip_{l'}(J'))$
- One turn matrix $e^{-i\mu}$, square matrix with order $(2l_{max+1} + 1) \times n_J$

$$(x_l(J) + ip_l(J))_{t+1} = e^{-i\mu}(x_l(J) + ip_l(J))_t$$

$$\mu = \mu_y + l\mu_s + M_W$$

Simple radial mode

- Flat distribution +azimuthal modulation

$$y_l(J) = y_{l0}\hat{J}^{|l|/2}/\sqrt{|l|!}$$

$$\hat{J} = J/\varepsilon_z$$

$$\delta p_{l0} = -2\sum M_{W,ll'}y_{l'0}$$

$$M_{W,ll'} = \frac{-i\beta_y^*}{4\pi\sqrt{|l|!|l'|!}}\int_{-\infty}^{\infty} d\omega$$

$(2l_{max+1}+1)$-th order square matrix

$$\left[i^{l-l'}Z_Q(\omega)\int_0^\infty dJ e^{-J}J^{(|l|+|l'|)/2}J_{l-l'}(k_\sigma r)e^{-k_\sigma^2/2}\right.$$

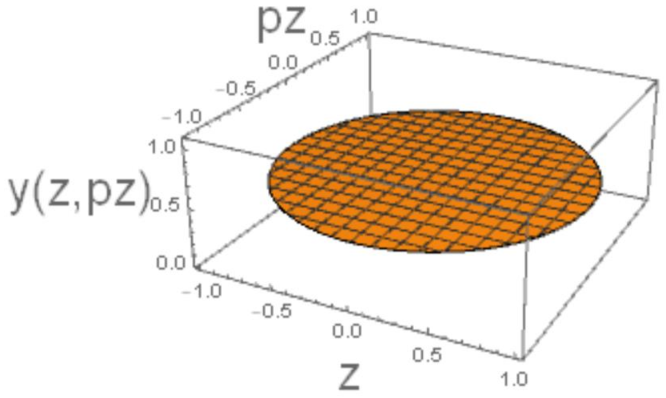
$$\left.+i^{|l|-|l'|}Z_D(\omega)(k_\sigma/\sqrt{2})^{|l|+|l'|}e^{-k_\sigma^2}\right],$$

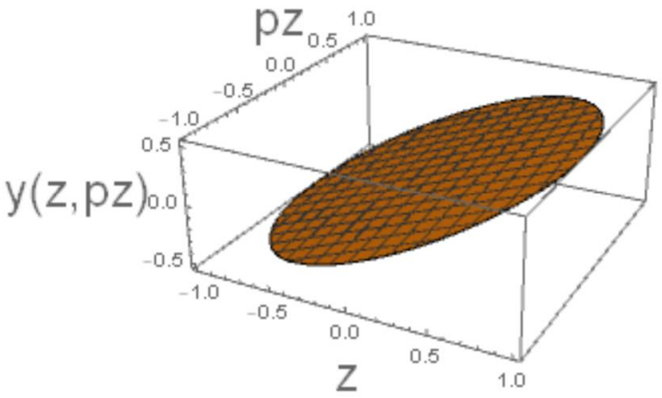
$$r = \sqrt{2J/\varepsilon_z} \qquad k_\sigma = \omega\sigma_z/c$$

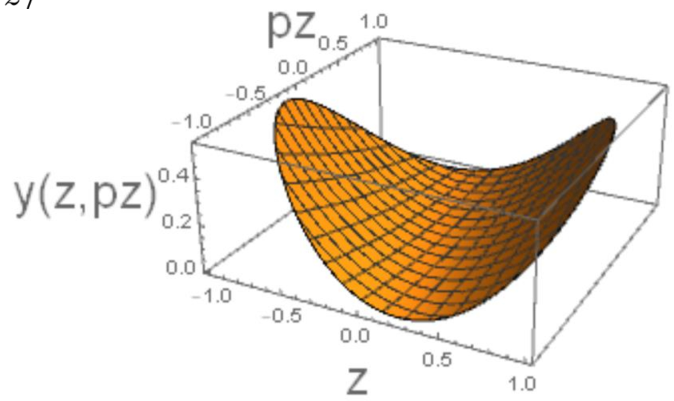
- Diagonal term, tune shift, M/(2π)

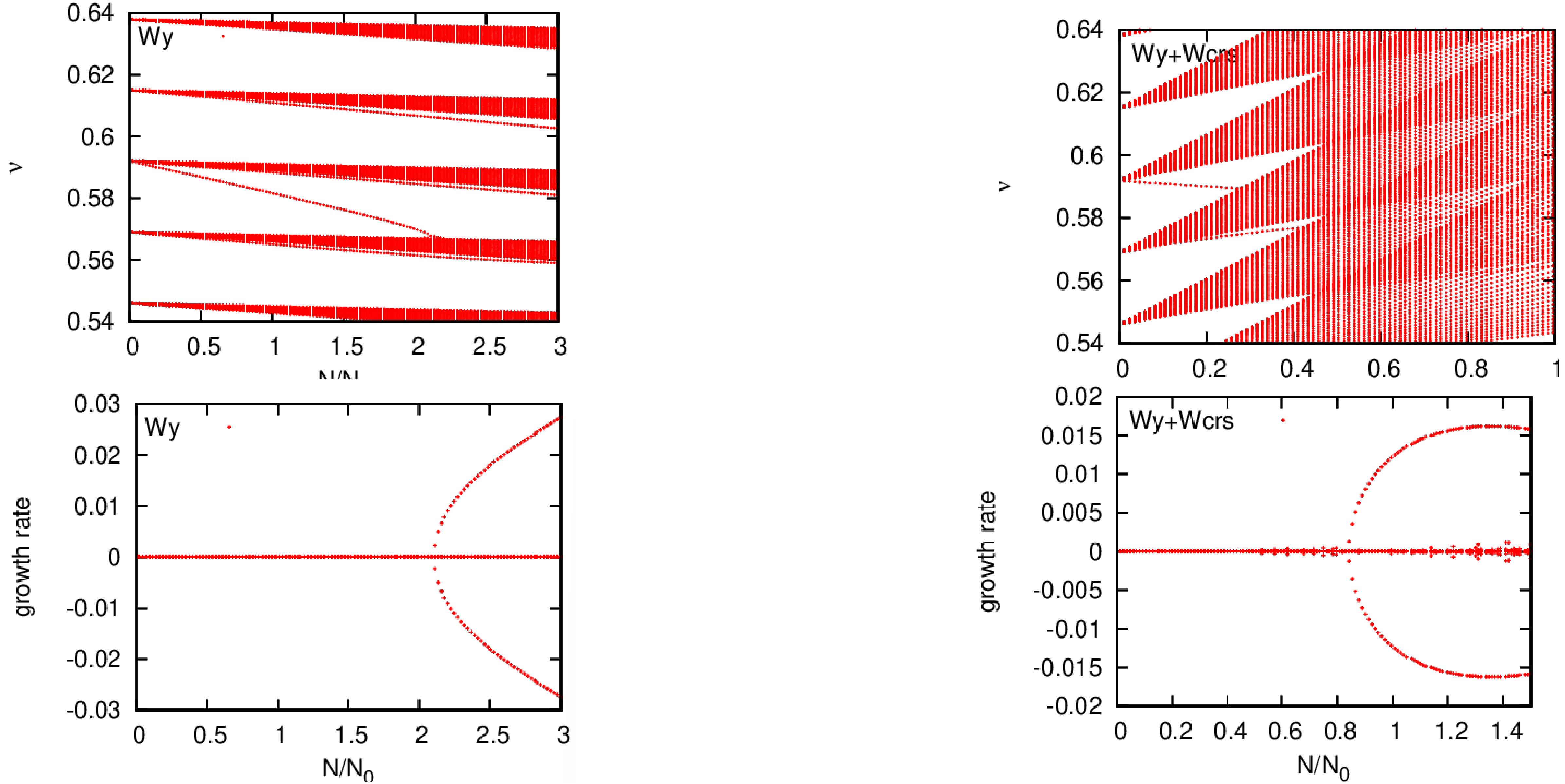
$$\Delta\nu_{y,\pm n} = -\frac{\beta_y^*}{4\pi}\frac{i}{2\pi n!}\int_{-\infty}^{\infty} d\omega$$

$$\left[Z_Q(\omega)n!L_n(k_\sigma^2/2) + Z_D(\omega)(k_\sigma^2/2)^n\right]e^{-k_\sigma^2}$$



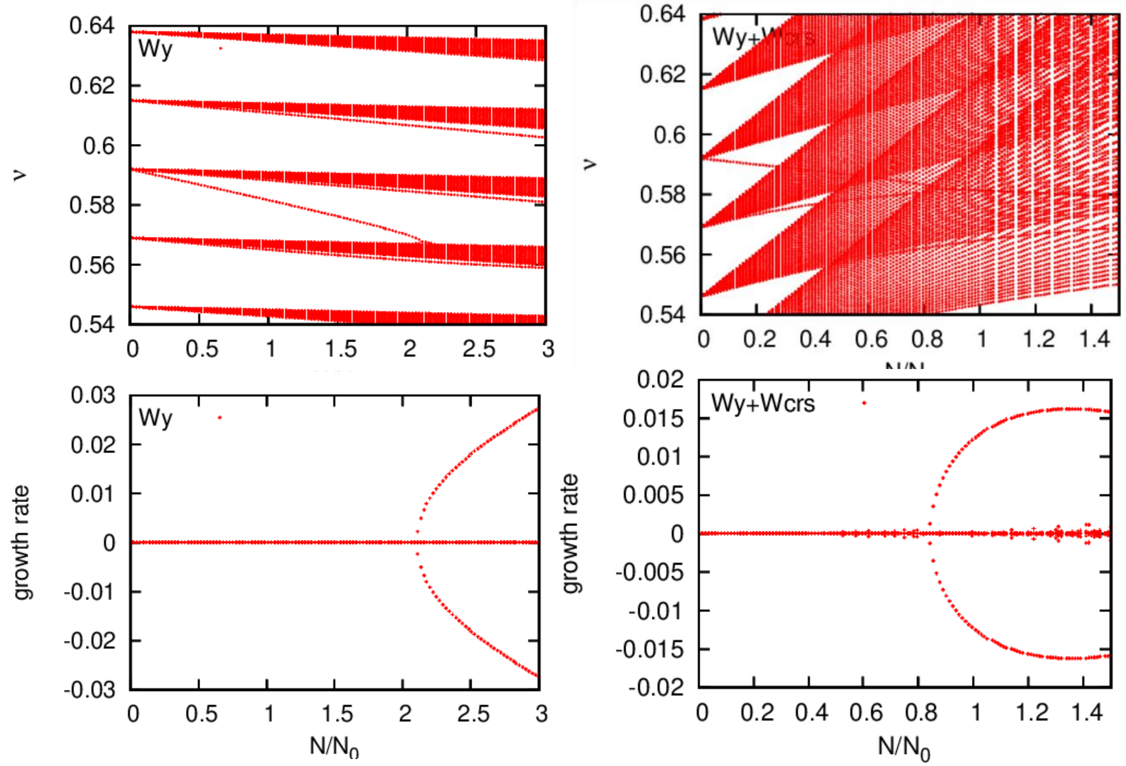






Wy
ν
0.64
0.62
0.6
0.58
0.56
0.54
0 0.5 1 1.5 2 2.5 3
Wy+Wcrs
ν
0.64
0.62
0.6
0.58
0.56
0.54
0 0.2 0.4 0.6 0.8 1
Wy
growth rate
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
0 0.5 1 1.5 2 2.5 3
N/N0
Wy+Wcrs
growth rate
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0 0.2 0.4 0.6 0.8 1 1.2 1.4
N/N0

Wy+WBB+Wsc


Wy
ν
0.64
0.62
0.6
0.58
0.56
0.54
0
0.5
1
1.5
2
2.5
3
Wy+Wcrs
Wy
growth rate
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
N/N0
Wy+Wcrs
growth rate
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
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1
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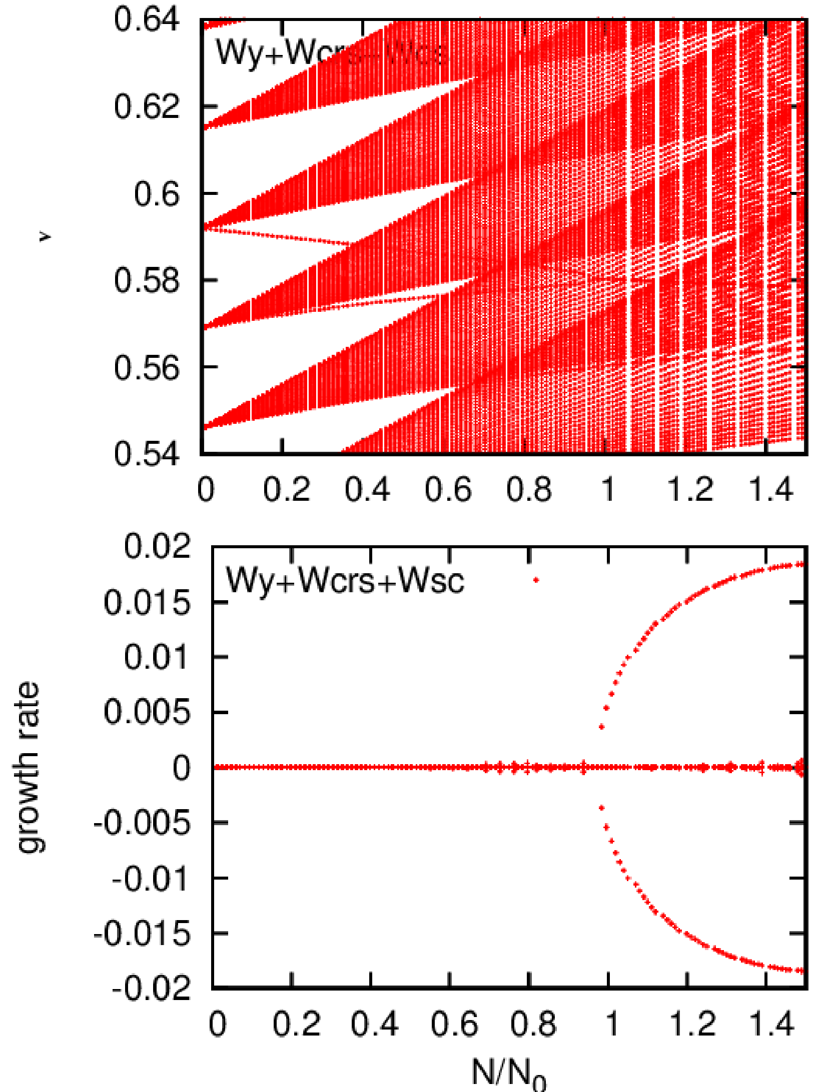
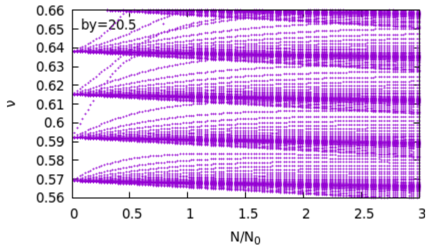
N/N0


Wy+Wcrs+Wcs
ν
0.64
0.62
0.6
0.58
0.56
0.54
0
0.2
0.4
0.6
0.8
1
1.2
1.4
Wy+Wcrs+Wsc
growth rate
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
N/N0

Pi mode

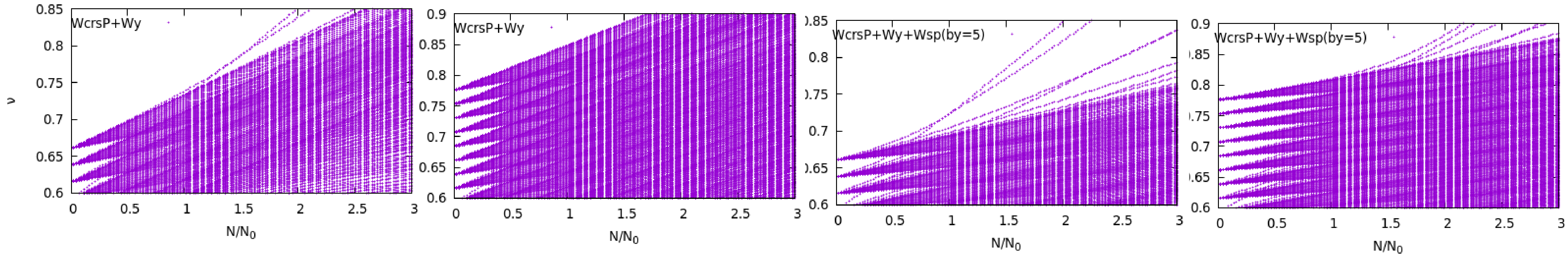


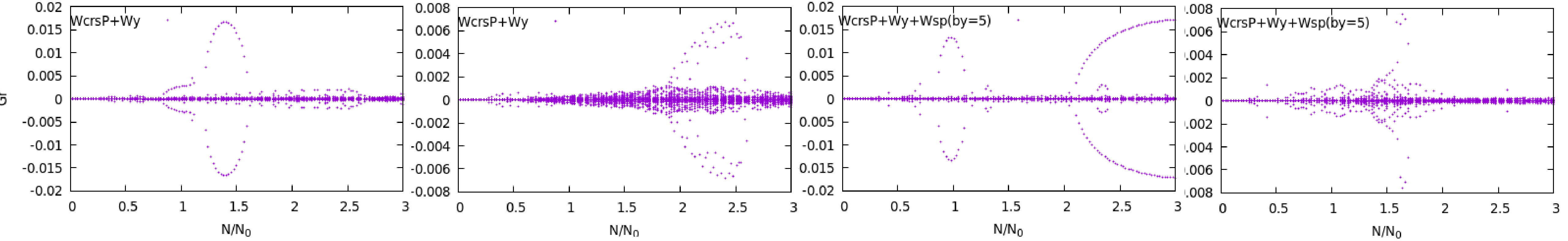
Wcrs+Wy

Lmax=3 Lmax=8

Wcrs＋Wsp＋Wy

Lmax=3 Lmax=8





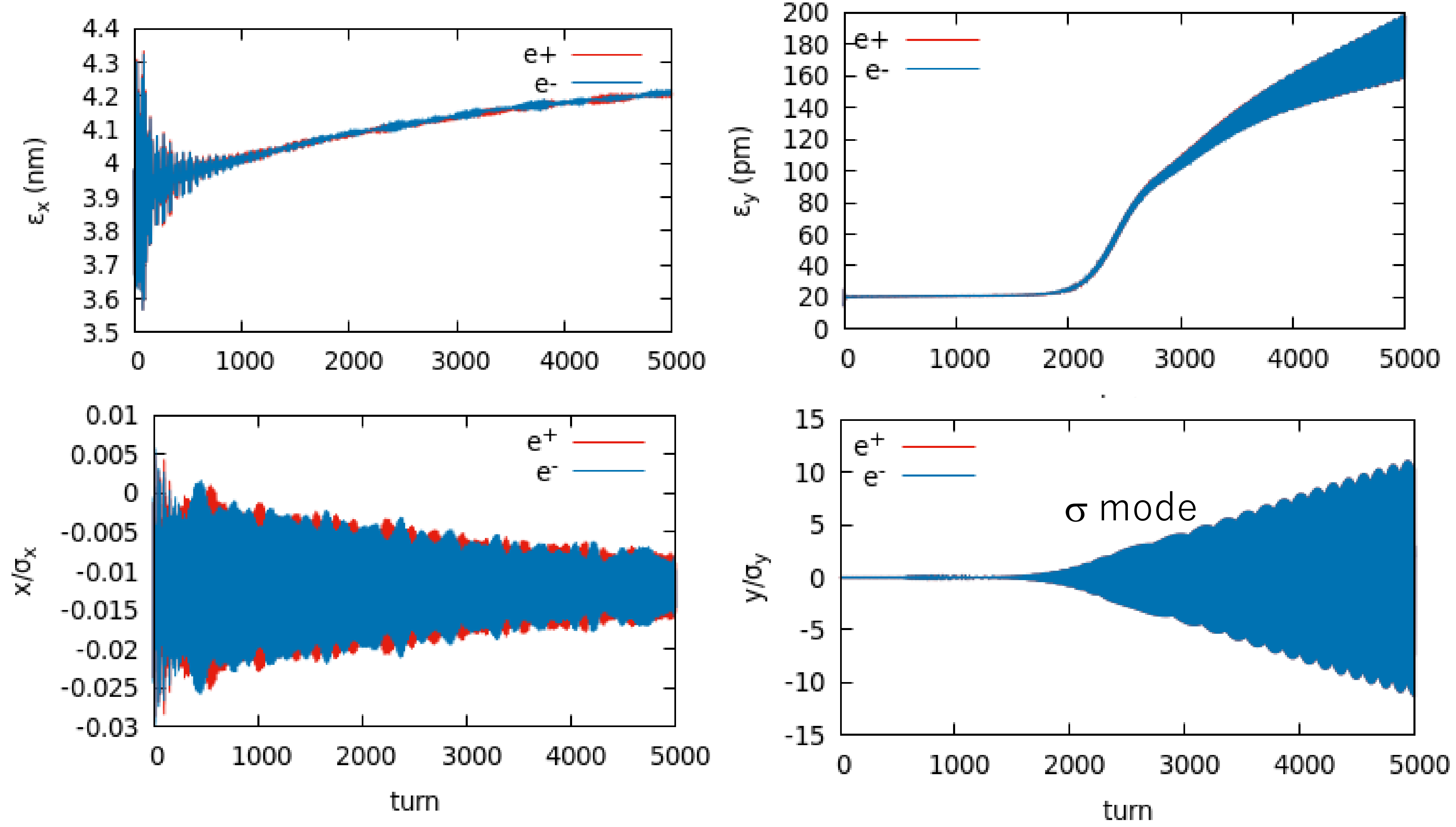
There are no clear growth in pi mode.

Strong-strong simulation (no space charge)

- Transparency model, E=4GeV, N=5x10^{10}.
- ε_x=4nm, ε_y=25pm, β_x=80mm, β_y=1mm, σ_z=6mm, θ_{hcrs}=41.5mrad
- $\Delta\nu_y$=0.0726

$$\Delta\nu_{yL} = \frac{N_e\ r_e}{2\pi\gamma_p}\ \frac{\beta_y}{\sigma_{xz}\ \sigma_y}$$

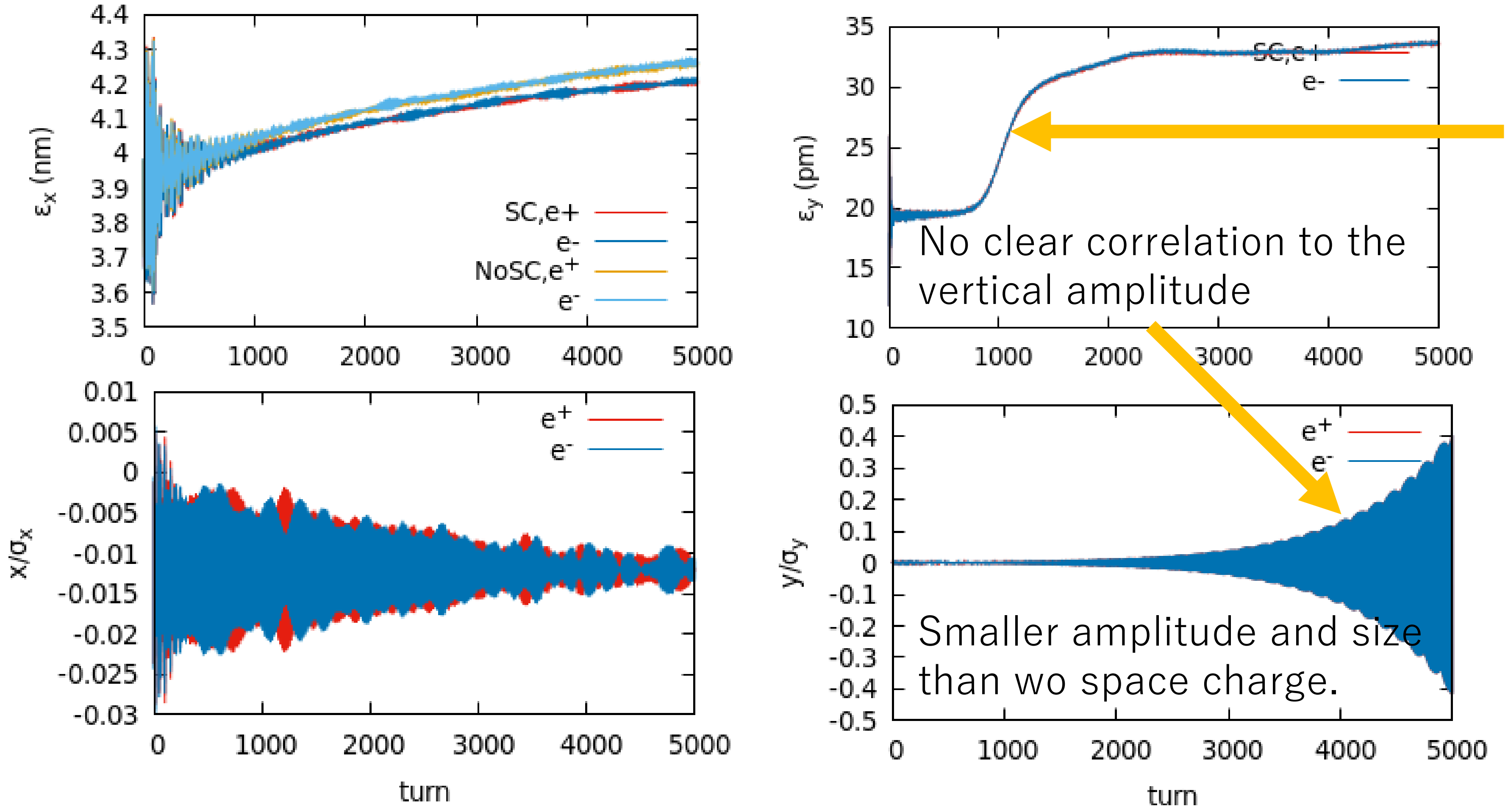
$$\sigma_{xz} = \sqrt{\epsilon_x \beta_x + (\theta_{crs}\ \sigma_z)^2}$$

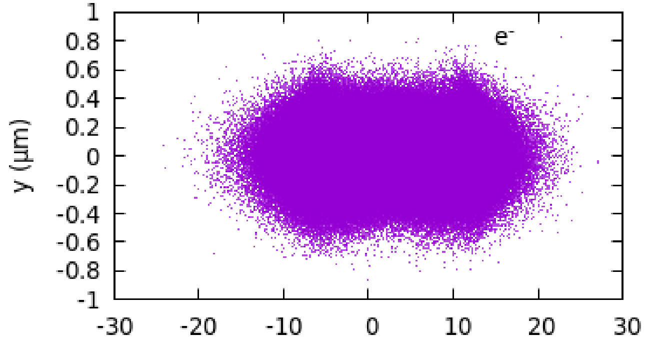


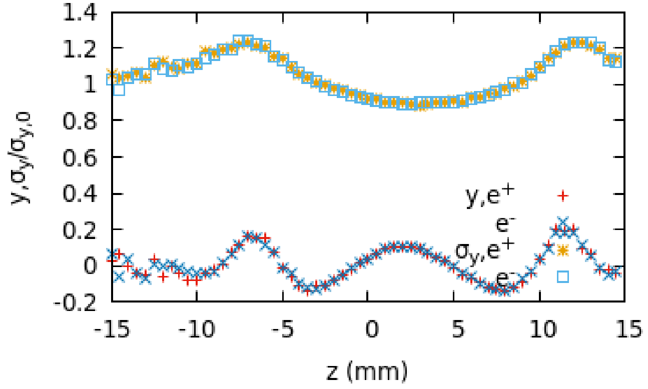
Strong-strong simulation (with space charge)

$$\Delta\nu_y = -\frac{\lambda_p r_e \beta_y L}{2\pi\sigma_y(\sigma_x+\sigma_y)\beta_p^2\gamma_p^3} \qquad \lambda_p = \frac{N_p}{\sqrt{2\pi}\,\sigma_z}$$

- $\Delta\nu_y$=0.0512 for $<\beta_x>$=5m, $<\beta_y>$=20m



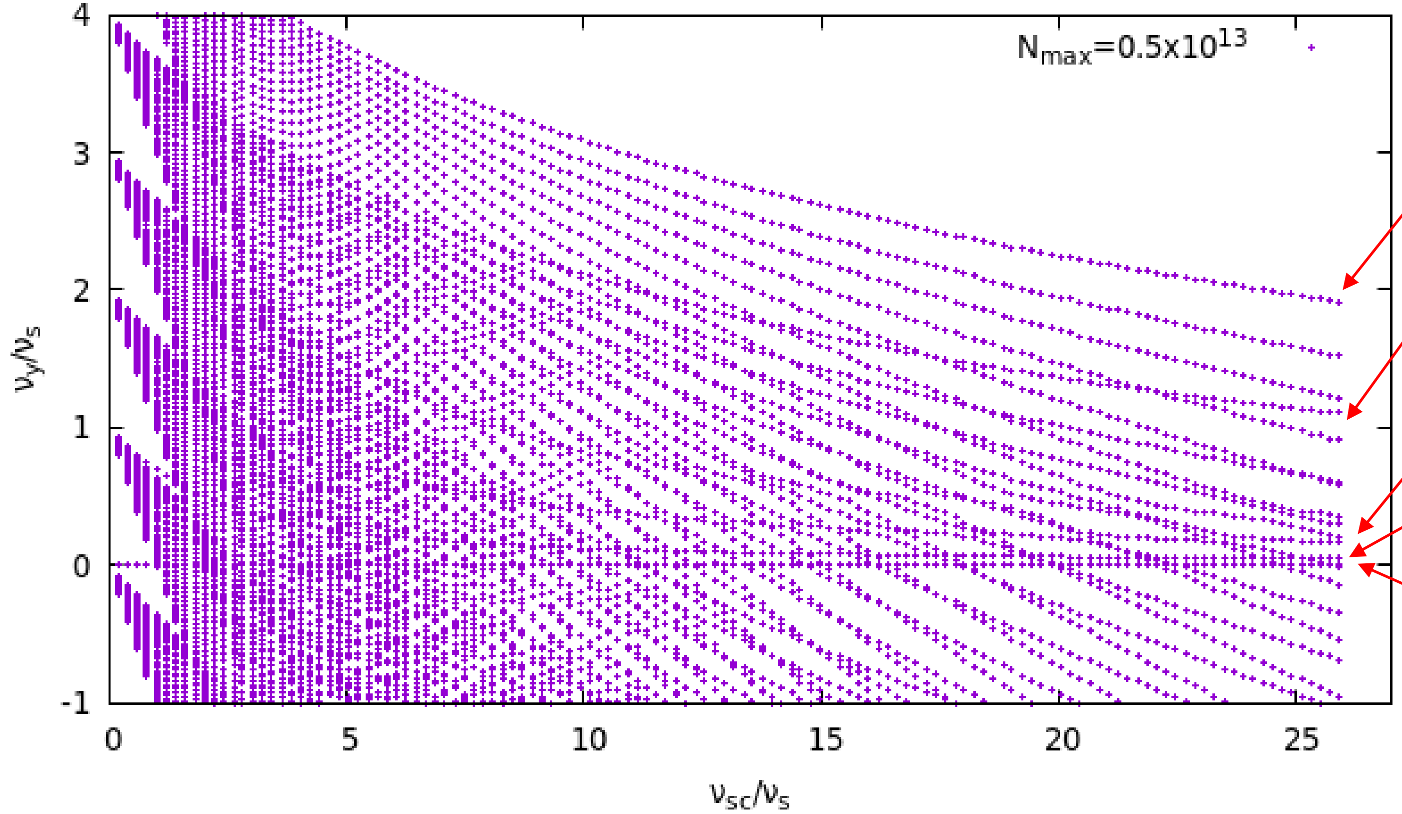


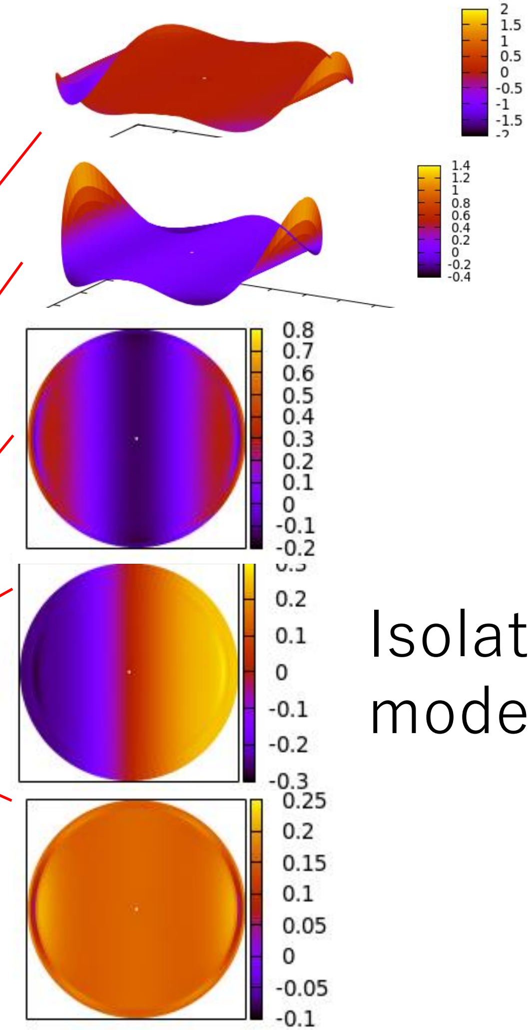


It seems a high order coherent mode

Space charge mode (J-PARC MR)

- E=3.938GeV, σ_x=10mm, $<\beta_x>$=11.8m, σ_z =10m, σ_δ =0.16%, ν_s=0.0024.
- $\Delta\nu_x = -\dfrac{\lambda_p\, r_p\, \beta_x\, L}{2\pi\, \sigma_x\, (\sigma_x + \sigma_y)\, \beta_p^2\, \gamma_p^3}$; $\Delta\nu_{sc}$=0.124 at N_p=10^{13}. $\lambda_p = \dfrac{N_p}{\sqrt{2\pi}\,\sigma_z}$;



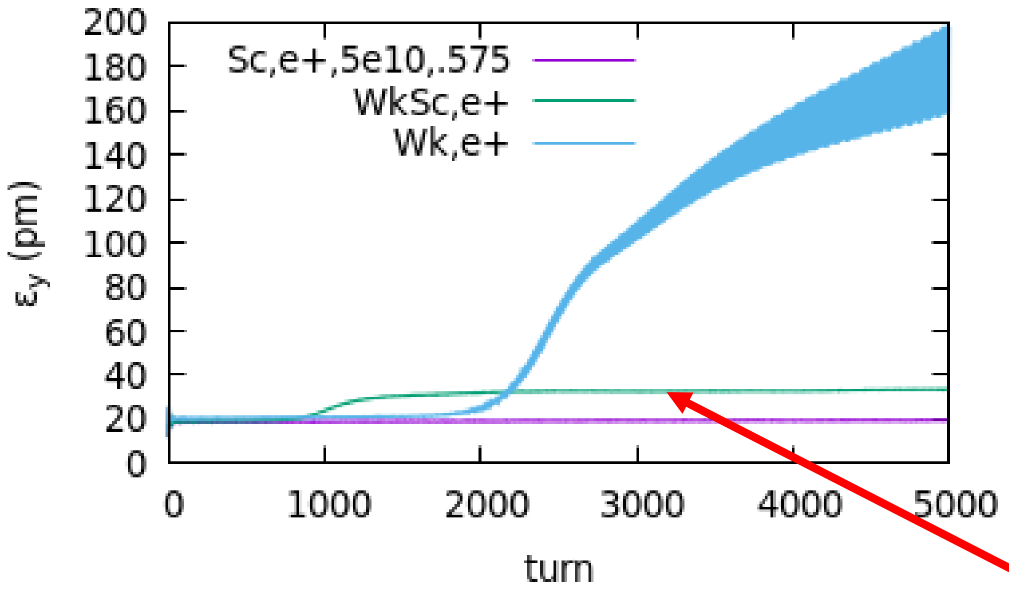


Isolated modes

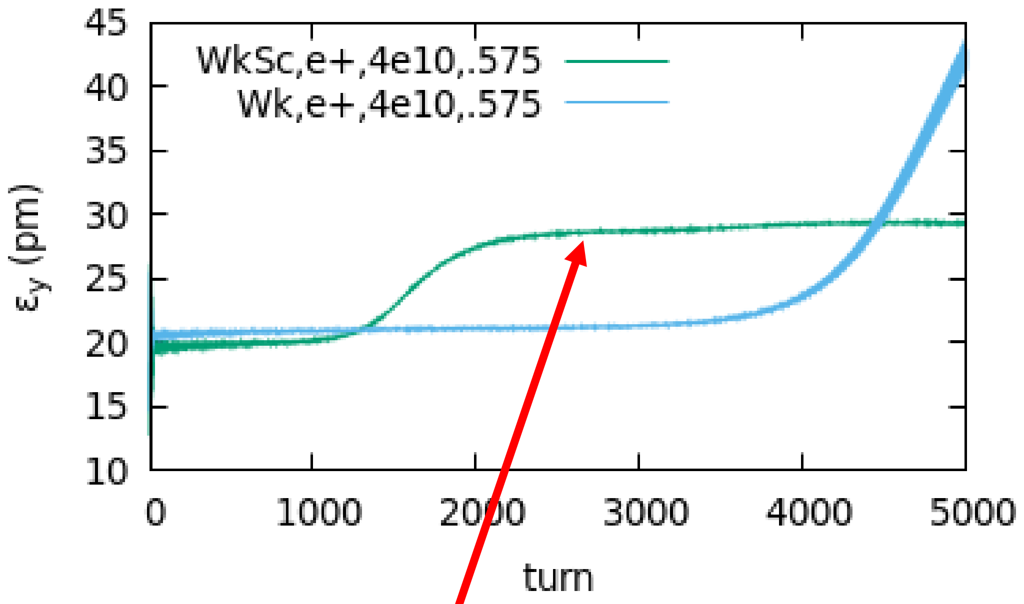
Strong-strong simulation with space charge

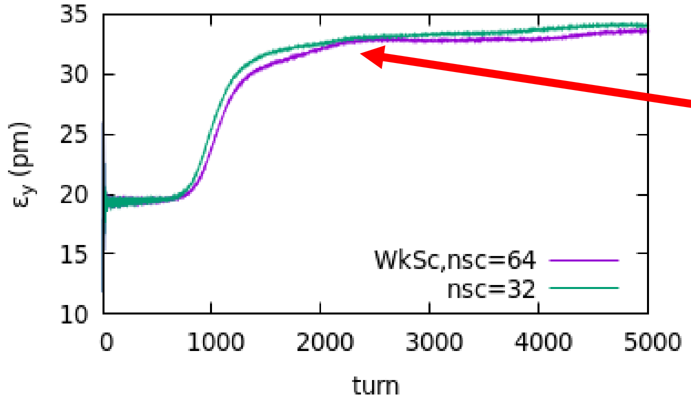
Space charge force is applied at $\beta_{x,sc}$=5m, $\beta_{y,sc}$=20m with Narc=8 points in arc.

Np=5e10



4e10





- Space charge force suppresses the beam-beam mode coupling instability.
- Another emittance growth appears due to space charge.
- Number of slice in space charge calculation

Weak space charge

- BBMC instability