Universität Zürich[UZH]

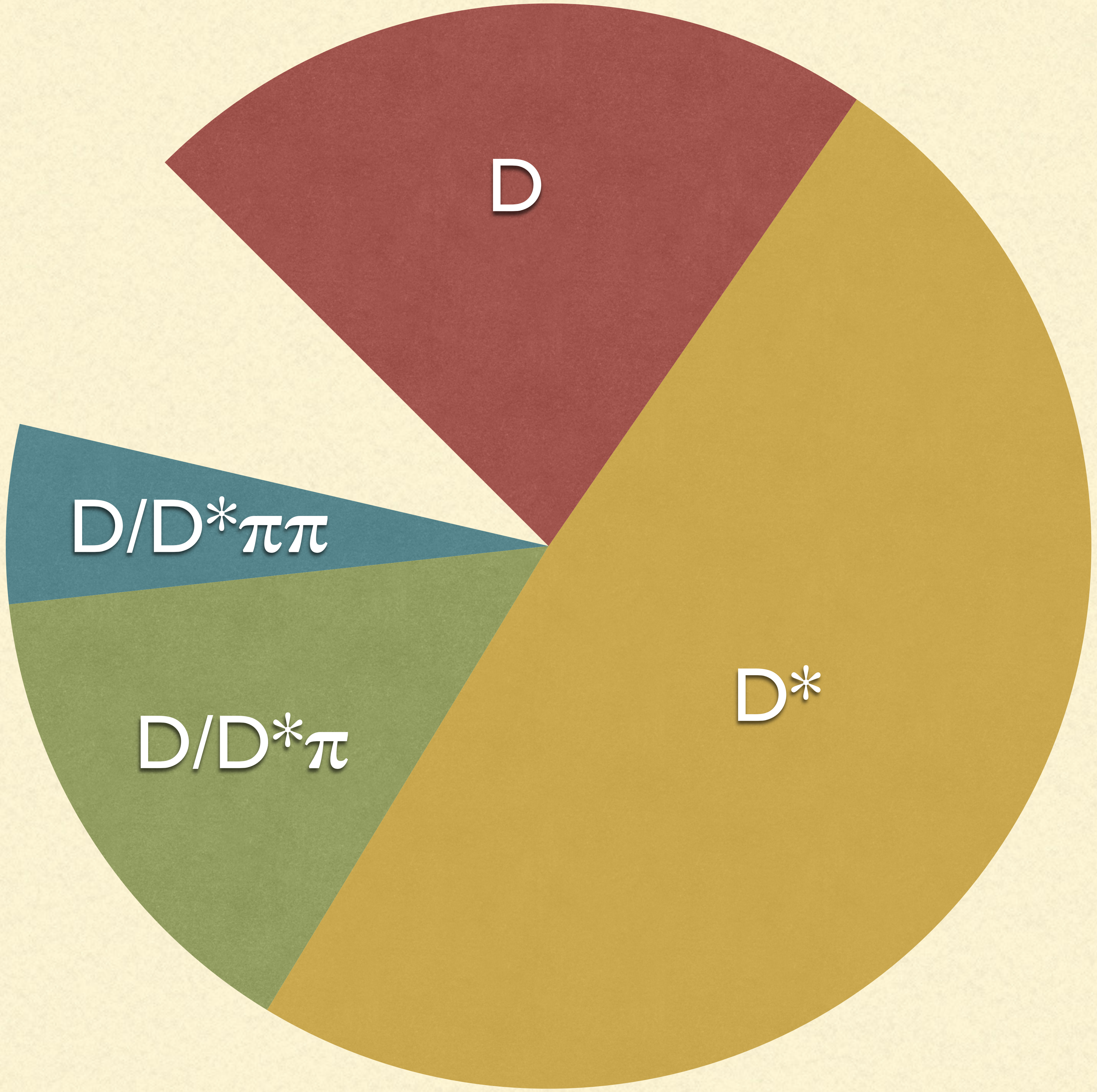
A model-independent parameterization of semileptonic B decays with two final-state hadrons

Based on 2311.00864, in collaboration with Erik Gustafson, Ruth Van de Water, Raynette van Tonder & Mike Wagman
WIP, in collaboration with Bastian Kubis, Ruth Van de Water & Raynette van Tonder

Outline

- Why do we care about more than one hadron?
- Theoretical fundamentals
- A new form factor parameterization
- Outlook

Why care about more hadrons?



- Semileptonic decays comprise more than 10% of all B-meson decays
- Ideal laboratory to determine $|V_{cb}|$ with multiple complementary approaches
- Allows for precise tests of lepton flavour universality
- Important background for $B \to X_u \ell \nu$ decays and other rare processes, such as $B \to K\nu\bar{\nu}$
- More than a quarter contain more than one final-state hadron (see Raynette's talk for issues regarding D^* and $D\pi$ pheno in general)

Why care about more hadrons?

Internal fit uncertainties	$\sigma_{\mathcal{R}(D^*)}(\times 10^{-2})$	$\sigma_{\mathcal{R}(D^0)}(\times 10^{-2})$
Statistical uncertainty	1.8	6.0
Simulated sample size	1.5	4.5
$B \to D^{(*)}DX$ template shape	0.8	3.2
$\overline{B} \to D^{(*)}\ell^-\overline{\nu}_\ell$ form-factors	0.7	2.1
$\overline{B} \to D^{**}\mu^-\overline{\nu}_\mu$ form-factors	0.8	1.2
$\mathcal{B}(\overline{B} \to D^* D_s^-(\to \tau^-\overline{\nu}_\tau)X)$	0.3	1.2
MisID template	0.1	0.8
$\mathcal{B}(\overline{B} \to D^{**}\tau^-\overline{\nu}_\tau)$	0.5	0.5
Combinatorial	< 0.1	0.1
Resolution	< 0.1	0.1
Additional model uncertainty	$\sigma_{\mathcal{R}(D^*)}(\times 10^{-2})$	$\sigma_{\mathcal{R}(D^0)}(\times 10^{-2})$
$B \to D^{(*)}DX$ model uncertainty	0.6	0.7
$\overline{B}_s^0 \to D_s^{**}\mu^-\overline{\nu}_\mu$ model uncertainty	0.6	2.4
Baryonic backgrounds	0.7	1.2
Coulomb correction to $\mathcal{R}(D^{*+})/\mathcal{R}(D^{*0})$	0.2	0.3
Data/simulation corrections	0.4	0.8
MisID template unfolding	0.7	1.2
Normalization uncertainties	$\sigma_{\mathcal{R}(D^*)}(\times 10^{-2})$	$\sigma_{\mathcal{R}(D^0)}(\times 10^{-2})$
Data/simulation corrections	$0.4 \times \mathcal{R}(D^*)$	$0.6 \times \mathcal{R}(D^0)$
$\tau^- \to \mu^-\nu\overline{\nu}$ branching fraction	$0.2 \times \mathcal{R}(D^*)$	$0.2 \times \mathcal{R}(D^0)$
Total systematic uncertainty	2.4	6.6
Total uncertainty	3.0	8.9

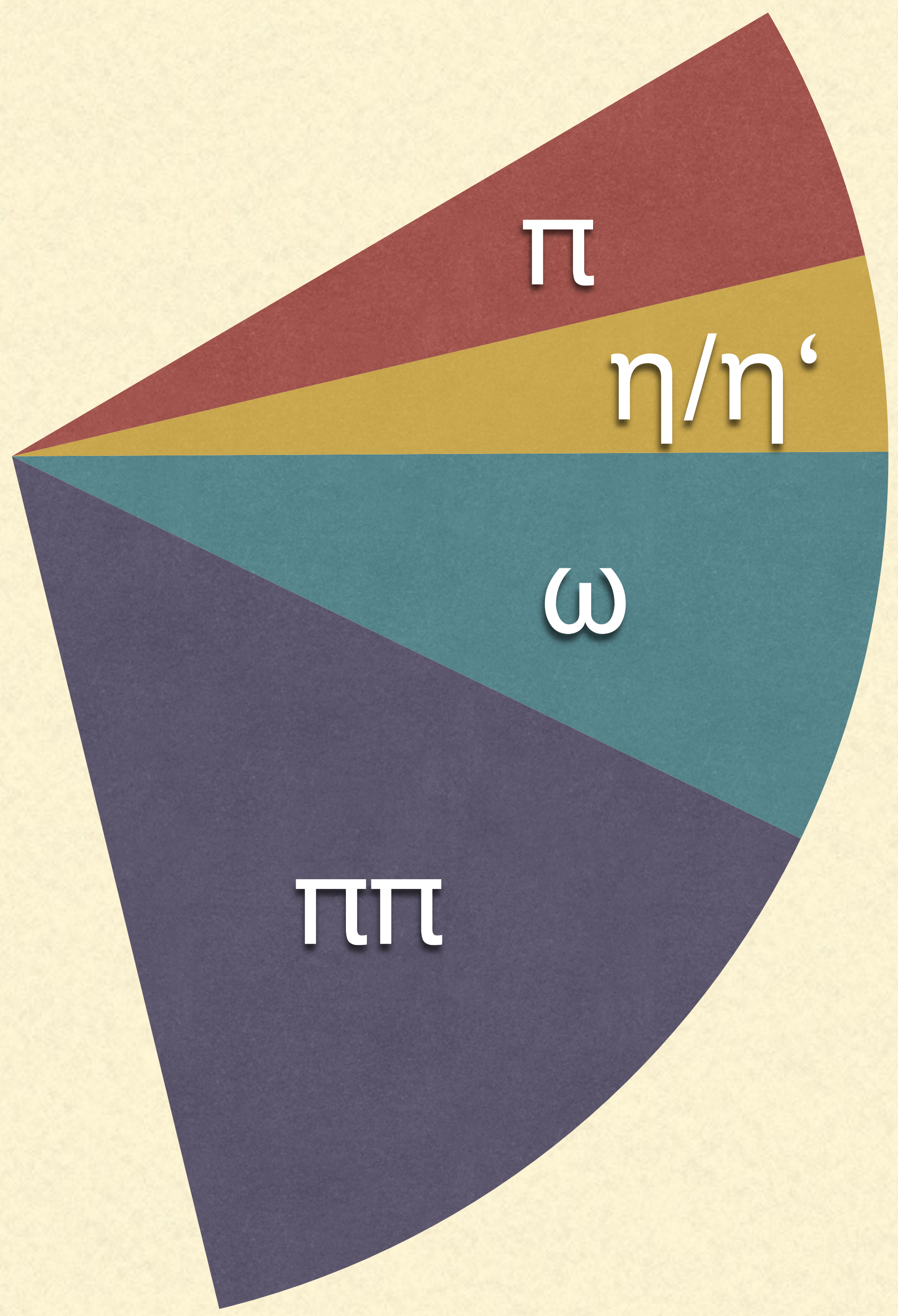
LHCb Collaboration, PRL 131 111802 (2023)

- Relevant for $R(D^{(*)})$ & $R(X)$ measurements

Source	Uncertainty [%]		
	e	μ	ℓ
Experimental sample size	8.8	12.0	7.1
Simulation sample size	6.7	10.6	5.7
Tracking efficiency	2.9	3.3	3.0
Lepton identification	2.8	5.2	2.4
$X_c\ell\nu$ M_X shape	7.3	6.8	7.1
Background (p_ℓ, M_X) shape	5.8	11.5	5.7
$X\ell\nu$ branching fractions	7.0	10.0	7.7
$X\tau\nu$ branching fractions	1.0	1.0	1.0
$X_c\tau(\ell)\nu$ form factors	7.4	8.9	7.8
Total	18.1	25.6	17.3

Belle II Collaboration, PRL 132 211804 (2024)

Why care about more hadrons?



- Situation worse for $b \to u\ell\nu$ transitions
- The exclusive modes we know only make up a third of all semileptonic decays
- Separating $B \to \rho\ell\nu$ from other $B \to \pi\pi\ell\nu$ decays highly model dependent
- Lattice QCD results for $B \to \pi\pi\ell\nu$ on the horizon (see L. Leskovec et al. 2403.19543)

Why care about more hadrons?

Source	$q1$	$q2$	$q3$	$q4$	$q5$	$q6$	$q7$	$q8$	$q9$	$q10$
					$B^+ \to \rho^0 \ell^+ \nu_\ell$					
Detector effects	2.8	2.0	1.6	1.1	1.7	1.9	2.4	1.4	1.4	1.6
Beam energy	2.1	1.9	1.9	1.5	1.3	1.1	1.0	0.9	0.8	0.5
Simulated sample size	14.1	7.8	7.4	6.3	6.3	5.2	6.4	5.6	6.2	7.3
BDT efficiency	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6
Physics constraints	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8
Signal model	0.7	0.2	0.2	0.2	0.3	0.4	0.5	0.3	1.8	2.4
ρ lineshape	1.7	1.6	2.0	1.0	1.9	1.8	1.4	0.9	1.6	1.7
Nonresonant $B \to \pi\pi\ell\nu_\ell$	5.6	6.3	6.7	8.6	9.3	10.7	10.1	7.0	7.8	11.8
DFN parameters	3.6	5.5	4.1	3.5	1.1	1.2	2.7	1.7	1.9	2.3
$B \to X_u\ell\nu_\ell$ model	1.7	3.0	3.8	5.0	5.8	6.1	6.3	1.9	7.2	12.4
$B \to X_c\ell\nu_\ell$ model	1.8	1.9	1.7	1.1	1.4	1.7	0.9	0.9	1.9	2.6
Continuum	31.5	24.3	17.0	19.6	13.2	14.8	16.0	16.6	15.2	18.7
Total systematic	35.6	27.5	21.0	23.5	18.8	20.5	21.6	19.4	20.2	27.0
Statistical	30.0	17.5	20.8	14.4	12.4	13.6	14.1	10.4	12.2	11.8
Total	46.6	32.6	29.6	27.6	22.6	24.6	25.8	22.0	23.6	29.5

Belle II collaboration, 2407.17403

- Continuum & Simulated sample size can be improved in the future (see Jochen's talk?)
- Modelling itself does not simply get better
- Lineshapes generally process-dependent
- What on earth is nonresonant $B \to \pi\pi\ell\nu$?

What do we need going forward?

- Model-independent parameterizations like BGL and its modifications such as BCL have played a crucial role in the past three decades
- Surprisingly simple form (although issues with truncation, see Talks by Florian, Stefan, Andreas)
- Allow to connect theoretical & experimental information from different kinematical regions
- General, but still allow to impose symmetry constraints (e.g. HQET, see Nico's Talk)
- In use beyond semileptonic B-decays: Pion VFF, Lepton-Nucleon scattering, …

→ We want a model-independent parameterization for two-hadron final states that has the same strengths as the BGL expansion

Theoretical fundamentals: Unitarity bounds

$$\Pi^{L/T}_{(J)}(q) \equiv i \int d^4x \, e^{iq\cdot x} \langle 0 | \, J^{L/T}(x) \, J^{L/T}(0) \, | 0 \rangle$$

$$\chi^L_{(J)}(Q^2) \equiv \frac{\partial \Pi^L_{(J)}}{\partial q^2}\bigg|_{q^2=Q^2} = \frac{1}{\pi}\int_0^\infty \mathrm{d}q^2 \frac{\mathrm{Im}\Pi^L_{(J)}(q^2)}{(q^2-Q^2)^2}$$

$$\chi^T_{(J)}(Q^2) \equiv \frac{1}{2}\frac{\partial^2 \Pi^T_{(J)}}{\partial (q^2)^2}\bigg|_{q^2=Q^2} = \frac{1}{\pi}\int_0^\infty \mathrm{d}q^2 \frac{\mathrm{Im}\Pi^T_{(J)}(q^2)}{(q^2-Q^2)^3}$$

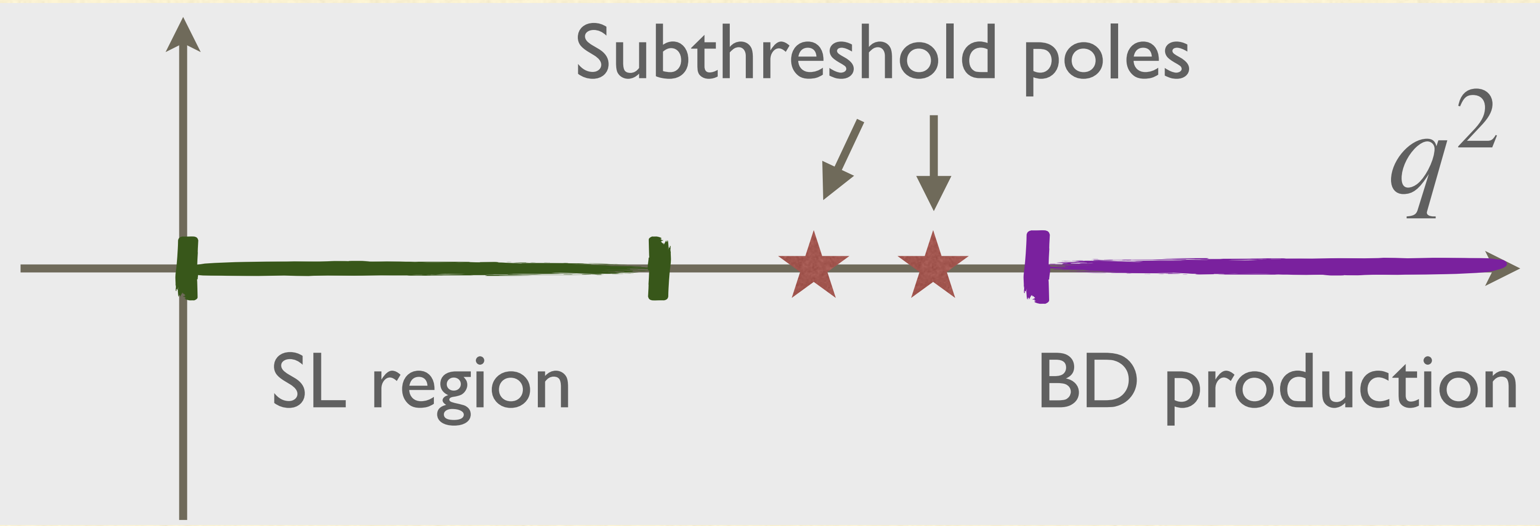
$$\mathrm{Im}\Pi^{T/L}_{(J)} = \frac{1}{2}\sum_X \int \mathrm{dPS}\, P^{\mu\nu}_{T/L} \left\langle 0 \left| J_\mu \right| X \right\rangle \left\langle X \left| J_\nu \right| 0 \right\rangle \delta^{(4)}(q-p_X)$$

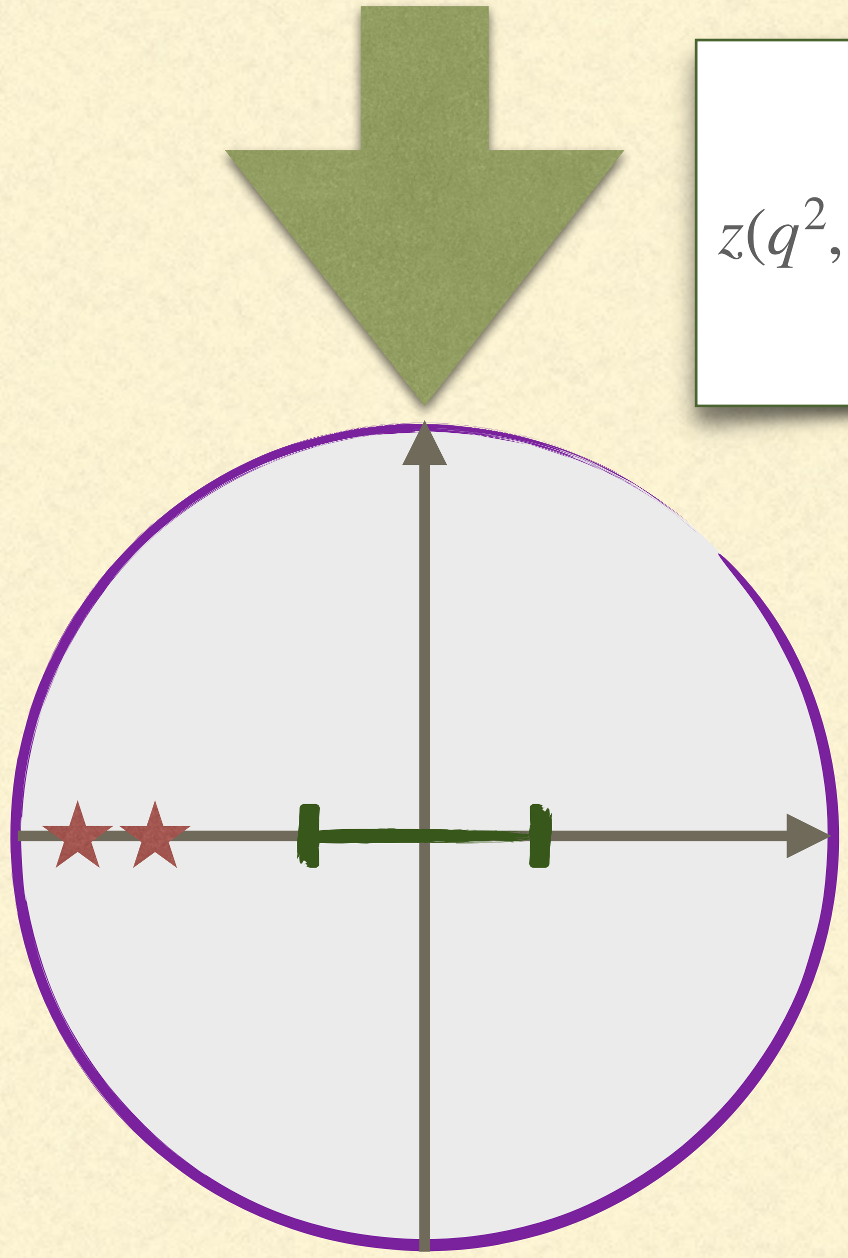
$$\mathrm{Im}\Pi^T_{(V)}|_{BD} = K(q^2)\,|f_+(q^2)|^2$$

- Starting point: once and twice subtracted dispersion relations [Boyd, Grinstein, Lebed; Caprini; …]
- Susceptibilities perturbatively computable for large space-like Q^2 or at $Q^2 = 0$ if heavy quarks involved; also on the Lattice! (Martinelli, Simula, Vittorio; Harrison)
- Optical theorem allows to write the imaginary part as sum over all possible final states
- Neglecting a final state leads to an inequality

Ingredient 1: Unitarity bounds

Theoretical fundamentals: Unitarity bounds





$$z(q^2, q_0^2) = \frac{\sqrt{q_+^2 - q^2} - \sqrt{q_+^2 - q_0^2}}{\sqrt{q_+^2 - q^2} + \sqrt{q_+^2 - q_0^2}}$$

$$1 \geq \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z} \left| B(z)\Phi(z)f(z) \right|^2$$

$$f(z) = \frac{1}{\Phi(z)B(z)} \sum_{i=0}^{\infty} a_i z^i \qquad 1 \geq \sum_{i=0}^{\infty} |a_i|^2$$

- Mapping q^2 to the dimensionless variable z transforms integration region to unit circle
- In this form it is evident that our FFs live in the Hardy space H^2
- Insert Blaschke products to get rid of subthreshold poles and zeroes in kinematic factors
- Series expansion (or orthogonal polynomials)
- Semileptonic region: $|z| < 1$

Ingredient 2: Convergent expansion

Theoretical Fundamentals: $2 \to 2$ scattering

$$\left\langle p_3 p_4; b \,\middle|\, \mathcal{S} - 1 \,\middle|\, p_1 p_2; a \right\rangle = i(2\pi)^4 \delta^{(4)}\left(\sum p_i\right) \mathcal{M}_{ba}(\{p_i\})$$

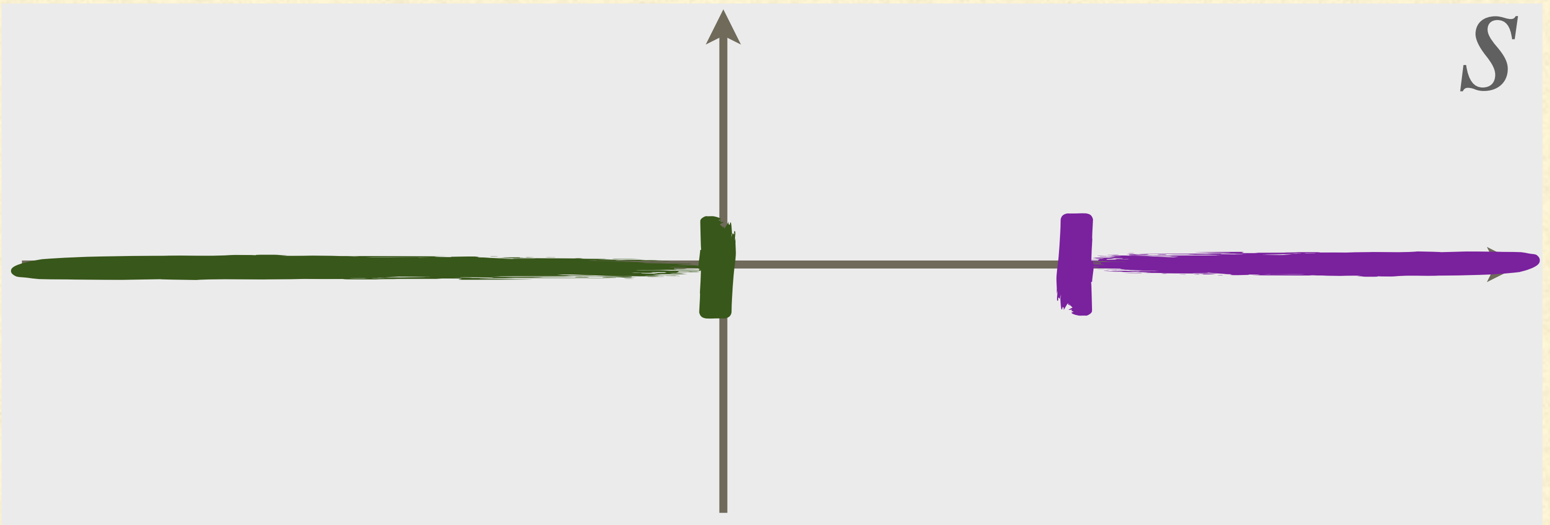
$$\mathcal{M}_{ab} - \mathcal{M}^*_{ba} = i(2\pi)^4 \sum_c \int \mathrm{d}\Phi_c \mathcal{M}_{ca} \mathcal{M}^*_{cb}$$

$$\mathcal{A}_a - \mathcal{A}^*_a = i(2\pi)^4 \sum_c \int \mathrm{d}\Phi_c \mathcal{M}^*_{ca} \mathcal{A}_c$$

- Simplest scattering process with nontrivial kinematic dependence
- Described by unitary operator $\mathcal{S}$
- Scattering amplitude $\mathcal{M}$ depends on 2 independent Mandelstam variables
- $\mathcal{M}$ real below lowest threshold, imaginary part constrained by Unitarity above
- Two-particle production amplitude $\mathcal{A}$ shares phase with $\mathcal{M}$, e.g. pion production in lepton collisions

Partial-wave expansion for dummies

$$\mathscr{M}_{ba}(s,t) = \sum_l P_l(\cos\theta)\sqrt{\rho_b}^{-1} f_{ba}(s)\sqrt{\rho_a}^{-1}$$

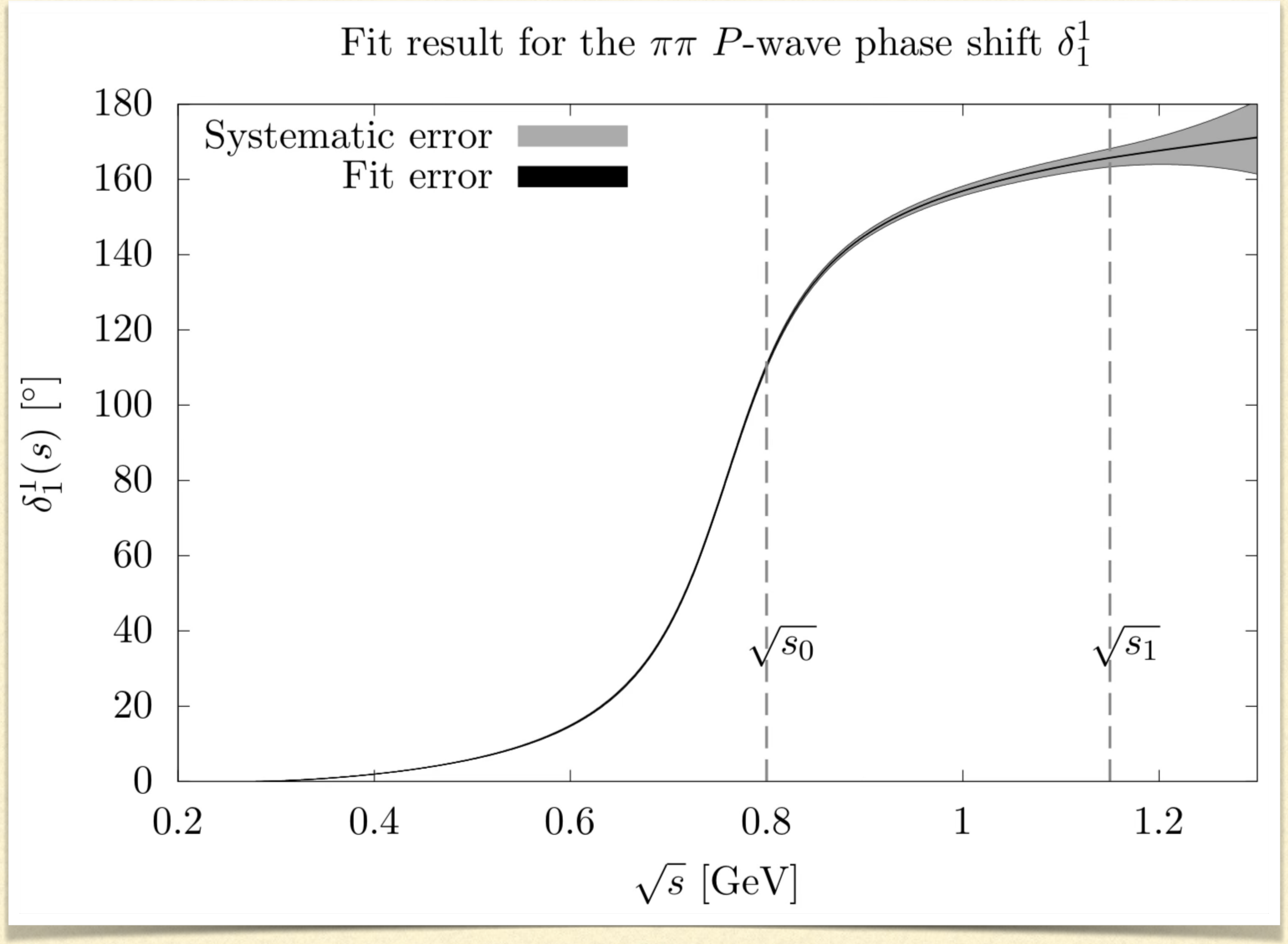


$$f^l_{aa}(s) = \frac{\eta_l(s)e^{2i\delta_l(s)} - 1}{2i}$$

- Resonances have well-defined spin, their poles only occur in a specific partial wave of $\mathscr{M}$
- Partial-wave expansion conveniently separates different resonances, e.g. in pion scattering: $\rho, f_0(500), f_0(980), f_2(1270)$
- Partial-wave expanded amplitudes have left-handed branch cuts which are remnants of branch cuts in other Mandelstam variables
- Diagonal elements can be expressed through scattering phase δ_l and inelasticity η_l

Ingredient 3: Left-hand cuts

Partial-wave expansion for dummies



- Resonances have well-defined spin, their poles only occur in a specific partial wave of $\mathscr{M}$
- Partial-wave expansion conveniently separates different resonances, e.g. in pion scattering: $\rho, f_0(500), f_0(980), f_2(1270)$
- Partial-wave expanded amplitudes have left-handed branch cuts which are remnants of branch cuts in other Mandelstam variables
- Diagonal elements can be expressed through scattering phase δ_l and inelasticity η_l

Phase-shifts are everything

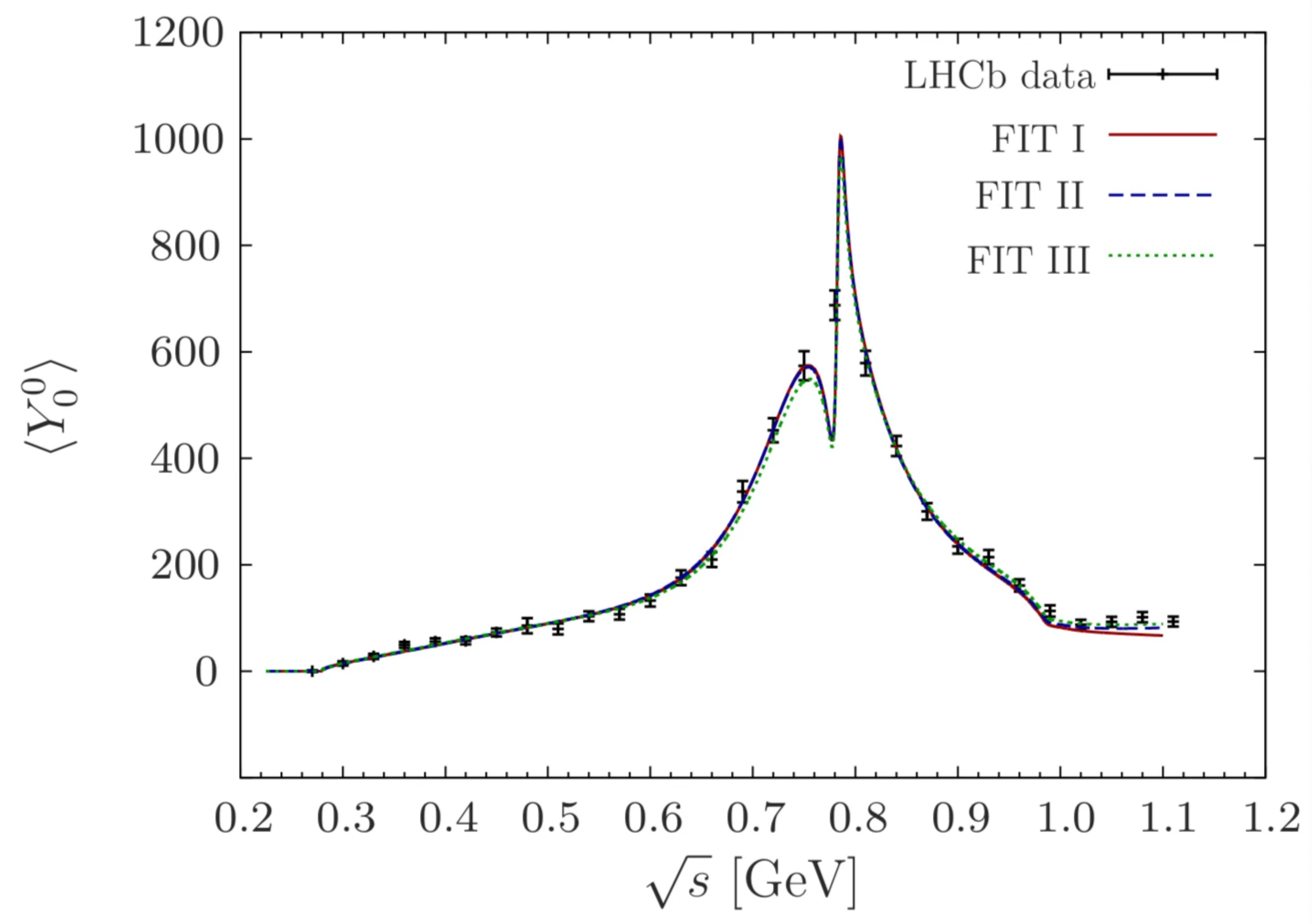
$$\ln \Omega_l(s) = \frac{s}{\pi} \int ds' \frac{\delta_l(s')}{s'(s'-s)}$$

$$F(s) = \Omega(s) \sum a_i z(s, s_{in})^i$$

$$\mathrm{Im}\Omega(s+i\epsilon) = \frac{1}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} \frac{T^*(s')\Sigma(s')\Omega(s')}{s'-s-i\epsilon} ds'$$

- Below the first inelastic threshold, the elastic scattering phase is universal
- Omnès function is a model-independent way to transport this information
- Common treatment of lineshapes in $e^+e^- \to \pi^+\pi^-, \tau \to \pi^-\pi^0\nu_\tau, K \to \pi\pi\ell\nu$, $B_{(s)} \to J/\Psi\pi^+\pi^-, \ldots$
- Works best for light mesons, $\pi\pi$, $K\pi$, but also S-wave $D\pi$
- Extensions beyond first inelastic threshold clear

Phase-shifts are everything



Ingredient 4: Omnès functions for lineshapes

- Below the first inelastic threshold, the elastic scattering phase is universal
- Omnès function is a model-independent way to transport this information
- Common treatment of lineshapes in $e^+e^- \to \pi^+\pi^-, \tau \to \pi^-\pi^0\nu_\tau, K \to \pi\pi\ell\nu$, $B_{(s)} \to J/\Psi\pi^+\pi^-, \dots$
- Works best for light mesons, $\pi\pi$, $K\pi$, but also S-wave $D\pi$
- Extensions beyond first inelastic threshold clear

Theoretical fundamentals: Three-body decays

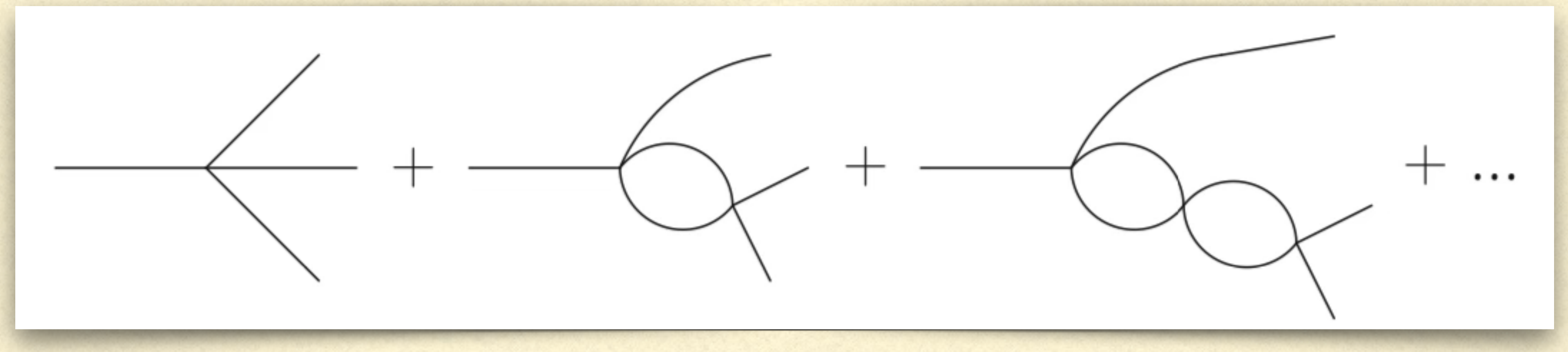
$$\mathrm{Im}\Pi^{T/L}_{(J)} = \frac{1}{2}\sum_X \int \mathrm{dPS}\, P^{\mu\nu}_{T/L} \left\langle 0 \left| J_\mu \right| X \right\rangle \left\langle X \left| J_\nu \right| 0 \right\rangle \delta^{(4)}(q - p_X)$$

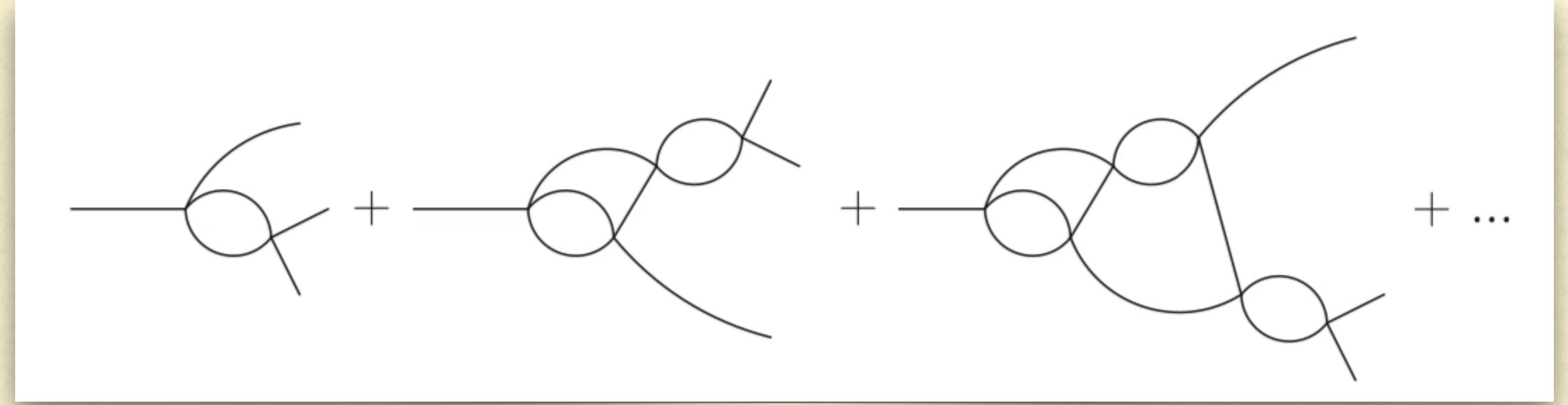
$$\mathcal{F}(s,t,u) = \sum_{x\in\{s,t,u\}} \sum_l F^{(x)}_l(x) P_l(\cos\theta_x)$$

$$F^{(l)}_{(s)}(s) = \Omega^{(l)}_{(s)}(s)\left(Q^{(l)}_{(s)}(s) + \frac{s^n}{\pi}\int \frac{\mathrm{d}x}{x^n} \frac{\sin\delta^{(l)}_{(s)}(s)\hat{F}^{(l)}_{(s)}(x)}{|\Omega^{(l)}_{(s)}(x)|(x-s)}\right)$$

- Amplitudes relevant for Unitarity bounds are $1 \to n$ amplitudes of particle with mass q^2
- Khuri-Treiman formalism already has 2 of our ingredients built in (PR 119 1115-1121 (1960))
- Write decay amplitude as sum of 3 partial-wave expanded amplitudes
- Fixed s, t & u dispersion-relations lead to coupled system of integral equations
- The two other channels enter via hat functions

Theoretical fundamentals: Three-body decays





Taken from: EPJC 83 (2023) 6, 510

$$F^{(l)}_{(s)}(s) = \Omega^{(l)}_{(s)}(s)\left(Q^{(l)}_{(s)}(s) + \frac{s^n}{\pi}\int \frac{\mathrm{d}x}{x^n}\frac{\sin\delta^{(l)}_{(s)}(s)\hat{F}^{(l)}_{(s)}(x)}{|\Omega^{(l)}_{(s)}(x)|(x-s)}\right)$$

- Amplitudes relevant for Unitarity bounds are $1 \to n$ amplitudes of particle with mass q^2
- Khuri-Treiman formalism already has 2 of our ingredients built in (PR 119 1115-1121 (1960))
- Write decay amplitude as sum of 3 partial-wave expanded amplitudes
- Fixed s, t & u dispersion-relations lead to coupled system of integral equations
- The two other channels enter via hat functions (here we could use $B^* \to D^{(*)}$ FFs)

A new parameterization

$$\mathrm{Im}\Pi^{T/L}_{(J)} = \frac{1}{2}\sum_X \int \mathrm{dPS}\, P^{\mu\nu}_{T/L} \left\langle 0 \left| J_\mu \right| X \right\rangle \left\langle X \left| J_\nu \right| 0 \right\rangle \delta^{(4)}(q - p_X)$$

$$\mathcal{F}(s,t,u) = \sum_{x\in\{s,t,u\}} \sum_l F^{(l)}_{(x)}(x,q^2) P_l(\cos\theta_x)$$

$$F^{(l)}_{(s)}(s,q^2) = \Omega^{(l)}_{(s)}(s)\left(f^{(l)}_{(s)}(s,q^2) + \frac{s^n}{\pi}\int \frac{\mathrm{d}x}{x^n} \frac{\sin\delta^{(l)}_{(s)}(s)\hat{F}^{(l)}_{(s)}(x,q^2)}{|\Omega^{(s)}_l(x)|(x-s)} \right)$$

- Amplitudes implicitly depend on mass
- s-dependence not polynomial above inelastic thresholds
- Find unitarity bound and parameterization for $f(s,q^2)$
- The hat functions now depend on $B^* \to \pi/D^{(*)}$ FFs

A new parameterization

$$\mathrm{Im}\Pi(q^2)\Big|_{M_1M_2M_3} = \sum_x \int_{x_+}^{(\sqrt{q^2}-m_y)^2} \mathrm{d}x \sum_l \frac{K_l(q^2,x)}{2l+1} |F^{(l)}_{(x)}(x,q^2)|^2$$

$$\chi \geq \frac{1}{\pi}\int_0^\infty \mathrm{d}q^2 \int_{s_+}^{s_-(q^2)} \mathrm{d}s \frac{K(s,q^2)}{q^{2n}} |\Omega(s) f(s,q^2)|^2$$

$$\chi \geq \frac{1}{\pi}\int_{s_+}^\infty \mathrm{d}s \hat{K}(s) \int_{q_+^2(s)}^\infty \mathrm{d}q^2 \frac{\tilde{K}(s,q^2)}{q^{2n}} |f(s,q^2)|^2$$

- Unitarity bounds in general off-diagonal
- Off-diagonal terms small, ignore for derivation of parameterization
- Similar to KT treatment: ignore left-hand cuts and add them back later
- Crucial: change integration order!
- In NWA: $\hat{K}(s) \to \delta(s - M_R^2)$

A new parameterization

$$f(s, q^2) = \frac{1}{B(q^2)\phi(q^2; s)} \sum_i a_i(s) z^i(q^2, q_+^2(s))$$

$$\chi \geq \frac{1}{\pi} \sum_i \int_{s_+}^{\infty} \mathrm{d}s \hat{K}(s) \, | \, a_i(s) \, |^2$$

$$a_i(s) = \frac{1}{\tilde{B}(s)\tilde{\phi}(s)} \sum_j b_{ij} y^j$$

- q^2-integration as in standard BGL
- If $q_+^2(s_+)$ larger than lowest two-body threshold: $z^i \to p_i(z)$
- Now we can treat every a_i as an s-dependent FF
- Follow Caprini's treatment of pion VFF, (EPJ C 13 471-484 (2000))
- Alternative: BCL-like expansion

$$y = \frac{\sqrt{s_{in} - s} - \sqrt{s_{in}}}{\sqrt{s_{in} - s} + \sqrt{s_{in}}}$$

Putting it all together

- A model-independent parameterization of $1 \to 2$ decays is possible, building on 60+ years of dispersion theory
- Bound on $b^{(l)}_{ij,(x)}$ quadratic, but not diagonal
- In heavy-to-heavy decays the left-hand cuts are far from the semileptonic region, so we can ignore integrals over hat functions
- Simplified application to $B \to D\pi\ell\nu$ successful and $B \to \pi\pi\ell\nu$ including implementation in EOS underway (→ Raynette's Talk)
- Powerful framework for many future phenomenological applications

$$F^{(l)}_{(s)}(s,q^2) = \Omega^{(l)}_{(s)}(s)\left(\frac{1}{B_{(s)}(q^2)\tilde{B}^{(l)}_{(s)}(s)\phi^{(l)}_{(s)}(q^2)\tilde{\phi}^{(l)}_{(s)}(s)}\sum_{i,j} b^{(l)}_{ij,(s)} z^i_{(s)} y^j_{(s)} + \frac{s^n}{\pi}\int \frac{\mathrm{d}x}{x^n}\frac{\sin\delta^{(l)}_{(s)}(s)\hat{F}^{(l)}_{(s)}(x,q^2)}{|\Omega^{(s)}_l(x)|(x-s)}\right)$$

Putting it all together

- A model-independent parameterization of $1 \to 2$ decays is possible, building on 60+ years of dispersion theory
- Bound on $b^{(l)}_{ij,(x)}$ quadratic, but not diagonal
- In heavy-to-heavy decays the left-hand cuts are far from the semileptonic region, so we can ignore integrals over hat functions; for narrow enough resonances, we can approximate lineshapes
- Simplified application to $B \to D\pi\ell\nu$ successful and $B \to \pi\pi\ell\nu$ including implementation in EOS underway (→ Raynette's Talk)
- Powerful framework for many future phenomenological applications

$$F^{(l)}_{(s)}(s, q^2) = \frac{\Omega^{(l)}_{(s)}(s)}{B_{(s)}(q^2)\tilde{B}^{(l)}_{(s)}(s)\phi^{(l)}_{(s)}(q^2)\tilde{\phi}^{(l)}_{(s)}(s)} \sum_{i,j} b^{(l)}_{ij,(s)} z^i_{(s)} y^j_{(s)} \to \frac{\Omega^{(l)}_{(s)}(s)}{B_{(s)}(q^2)\phi^{(l)}_{(s)}(q^2)} \sum_i b^{(l)}_i z^i_{(s)}$$

Putting it all together

- A model-independent parameterization of $1 \to 2$ decays is possible, building on 60+ years of dispersion theory
- Bound on $b^{(l)}_{ij,(x)}$ quadratic, but not diagonal
- In heavy-to-heavy decays the left-hand cuts are far from the semileptonic region, so we can ignore integrals over hat functions; for narrow enough resonances, we can approximate lineshapes
- Simplified application to $B \to D\pi\ell\nu$ successful and $B \to \pi\pi\ell\nu$ including implementation in EOS underway ($\to$ Raynette's Talk)
- Powerful framework for many future phenomenological applications

$$F^{(l)}_{(s)}(s, q^2) = \frac{g^{(l)} F^{(l)}(s, r_{BW})}{(s - M^2_{R,l}) + i M_{R,l} \Gamma_R(s)} \frac{1}{B_{(s)}(q^2) \phi^{(l)}_{(s)}(q^2)} \sum_i b^{(l)}_{i,(s)} z^i_{(s)}$$