Quark Hadron Duality Violation

an attempt to exorcise an old demon of the Heavy Quark Expansion

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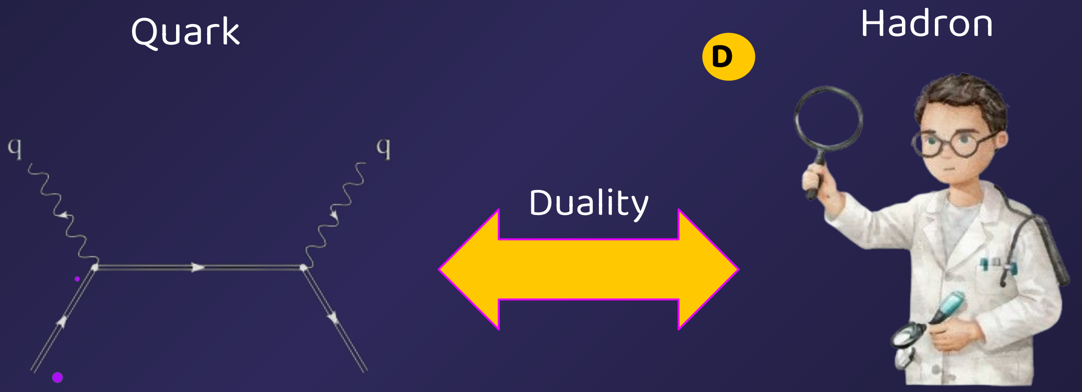
arXiv:2407.01473

Co-authors: T. Mannel, I. S. Milutin and K. K. Vos

Introduction

- Quark Hadron Duality (QHD) allows translation
  of predictions at the quark level to observables at the hadron level



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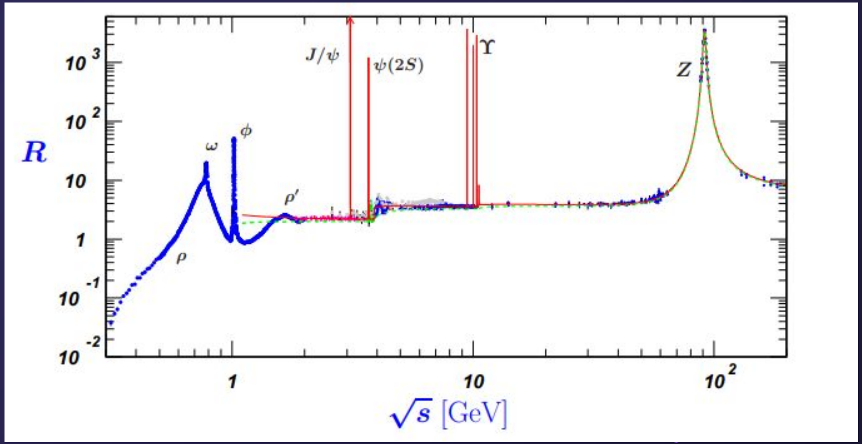
$$R = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)} = 3\sum_q e_q^2$$



Ratio of cross section e+ e- as a function of centre of mass energy,

M. Tanabashi et al. (PDG), 2019

Motivation

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- In the quest of sub-percent $|V_{cb}|$ precision QHD Violation (QHDV) might become the limiting factor

- Develop a model of QHDV in the context of the Heavy Quark Expansion (HQE)

- Apply our model to observables of Semi-leptonic inclusive B decays $B \to X_c \ell \overline{\nu}$

Modelling Duality Violation

- Asymptotic behaviour of the OPE expansion in $\frac{\Lambda_{QCD}}{Q}$ resulting in a non-converging series (like the perturbative case)

Chibisov et al. 1996, hep-ph/9605465

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- For example due to Instanton contributions,
  with ω inverse instanton size

$$\Pi(\tilde{Q}^2) \sim \exp\left(-\frac{1}{\omega}\sqrt{\tilde{Q}^2}\right)$$

$$\tilde{Q}^2 = -\tilde{q}^2$$

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- Cannot be expanded in $\frac{1}{\tilde{Q}^2}$ but are suppressed

- Would induce factorial growth of HQE coefficients

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Modelling Duality Violation

- Function with factorial growth
  Only converges if a_{2n} suppresses the factorial

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- Inverse Borel to re-obtain the function

$$F(\lambda) = \int_0^{\infty} \mathrm{d}M \, e^{-M} B[F](\lambda M)$$

Modelling Duality Violation

- Asymptotic function!

$$\tilde{B}[F](M) = \sum_{n=0}^{\infty} M^{2n} = \frac{1}{1-M^2} = \frac{1}{1+M}\frac{1}{1-M}$$

$$a_{2n} = 1$$

- One has to deal with the poles

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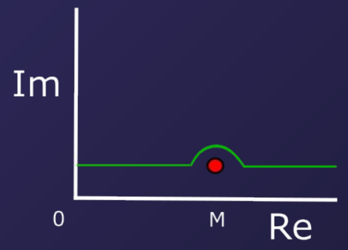

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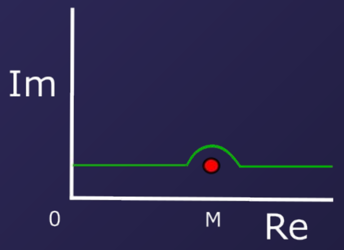

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- We identify this ambiguity with QHDV

$$\frac{1}{1-M+i\epsilon} - \frac{1}{1-M-i\epsilon} = 2i\pi\delta(1-M)$$

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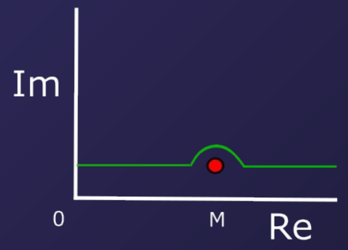

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- Why does this identification make sense?

Illustrative example

- Example fourier transform

$$f(Q) = \int_0^\infty \frac{1}{x^2 + r^2} e^{iQx} dx$$

- Singularity

$$x = \pm ir$$

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Illustrative example

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- Singularity $$x = \pm ir$$
- Exact solution $$f(Q) = \frac{\pi}{2r} e^{-Qr} + \frac{1}{2r}\left[e^{-Qr}\overline{\mathrm{Ei}}(Qr) - e^{Q\rho}\,\mathrm{Ei}(-Qr)\right]$$

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- Naive expansion is missing the exponential term

$$f(Q) = \frac{i}{r} \sum_{k=0}^{\infty} \frac{(2k)!}{(Qr)^{2k+1}}$$

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- Inserting into the inverse Borel gives us the missing exponential

$$\Delta_{\mathrm{DV}}F(\lambda)=2i\pi\int\limits_0^\infty \mathrm{d}M\, e^{-M}\frac{1}{1+M\lambda}\delta(1-\lambda M)=\frac{i\pi}{\lambda}\exp\left(-\frac{1}{\lambda}\right)$$

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- Now let's turn to the use case of the HQE...

Setting up the HQE

$$Q = m_b\, v - q$$
$$v = p/M_B$$

- Differential rate from leptonic tensor and hadronic correlation function via optical theorem

$$d\Gamma \propto L^{\mu\nu} \mathrm{Im}[T_{\mu\nu}(vQ, Q^2)]$$

$$T_{\mu\nu}(vQ, Q^2) = \int \mathrm{d}^4 x\, e^{-iQ\cdot x} \langle B(p)|T\{\bar{b}_v(x)\Gamma_\mu c(x)\, \bar{c}(0)\overline{\Gamma}_\nu b_v(0)\}|B(p)\rangle$$

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- Decompose in 5 scalar functions

$$T_{\mu\nu}(vQ, Q^2) = T_1\left(g_{\mu\nu} + \frac{Q_\mu v_\nu + Q_\nu v_\mu - i\epsilon_{\mu\nu\alpha\beta}Q^\alpha v^\beta}{vQ}\right) - T_2 g_{\mu\nu} + T_3 v_\mu v_\nu + T_4\frac{(Q_\mu v_\nu + Q_\nu v_\mu)}{vQ} + T_5\frac{Q_\mu Q_\nu}{(vQ)^2}$$

$$T_i \equiv T_i(vQ, Q^2)$$

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- Obtained by taking the forward matrix element

$$T_{\mu\nu}(Q) = \sum_{k=0}^{\infty} \left(\frac{1}{Q^2}\right)^{k+1} \langle B(v)|\bar{b}_v\Gamma_\mu\, \not{Q}[-(i\not{D})\,\not{Q}]^k\overline{\Gamma}_\nu b_v(0)|B(v)\rangle$$

Setting up the HQE

- Scalar Hadronic Structure functions

$$T_i(t, Q^2) = \frac{1}{\Lambda_{HQE}} \sum_{l=0}^{\infty} \left(\frac{\Lambda_{HQE}^2}{Q^2} \right)^{l+1} P_l^{(i)}(t)$$

$$P_l^{(i)}(t) = \sum_{n=0}^{l+1} t^{l+1-n} a_n^{(i,n+l)}$$

$$t \equiv \frac{vQ}{\Lambda_{HQE}}$$

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- $a_n^{(i,n+l)}$ can in principle be calculated from HQE parameters

Duality Violation model

- Factor out the expected factorial growth

$$P_l^{(i)}(t) = (2l)! \sum_{n=0}^{l+1} t^{l+1-n} b_n^{(i,n+l)}$$

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- If purely factorial

$$b_0^{(i,0+l)} \sim b_1^{(i,1+l)} \sim b_n^{(i,n+l)}$$

$$b_n^{(i,n+l)} = a_n^{(i,n+l)}/(2l)!$$

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- For the model we assume $b_n^{(i,n+l)} \sim 1$ except coefficients missing from the HQE i.e.

$$b_n^{(i,n+l)} = a_n^{(i,n+l)}/(2l)!$$

Duality Violation model

- Model ansatz polynomials based on the HQE parameters*

$$P_l^{(1,4)}(t) = (2l)! \sum_{m=1}^{l+1} t^m = (2l)! \frac{t - t + 2}{1 - t}$$

* Similar models for P 2,3 and 5

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$$P_l^{(1,4)}(t) = (2l)! \sum_{m=1}^{l+1} t^m = (2l)! \frac{t - t + 2}{1 - t}$$

- Model scalar hadronic structure functions

$$T_{1,4}(t, \lambda^2) \quad = \frac{1}{\Lambda_{HQE}} \frac{t\lambda^2}{1-t} \left(F_1(\lambda) - tF_2(\lambda)\right)$$

$$\lambda \equiv \frac{\Lambda_{HQE}}{\sqrt{Q^2}}$$

$$F_1(\lambda) \quad = \sum_{l=0}^{\infty} (2l)!(\lambda^2)^l \qquad\qquad F_2(\lambda) \quad = \sum_{l=0}^{\infty} (2l)!(t\lambda^2)^l$$

* Similar models for P 2,3 and 5

Duality Violation model

- Use optical theorem to obtain DV contribution to hadronic tensor

$$\hat{\Delta}_{\mathrm{DV}} W_{1,4}(vQ, Q^2) = -\frac{1}{\pi} \hat{\Delta}_{\mathrm{DV}} \mathrm{Im}\left[T_{1,4}(vQ, Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE} - vQ} \quad \frac{vQ}{\sqrt{Q^2}} \left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}} \right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}} \sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}} \sqrt{\frac{Q^2}{vQ}} \right) \right)$$

Duality Violation model

- OPE + DV model $$W_i \to W_i^{(\mathrm{OPE})} + 0.25\, \mathcal{C}_{\mathrm{DV}} \hat{\Delta}_{\mathrm{DV}} W_i(s, \hat{q}^2, \Lambda_{HQE})$$

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- Default scale choice $$\Lambda_{HQE} = 0.5\,\mathrm{GeV}$$
- Normalised so that partonic and DV contributions are equal for C_{DV}=1 (= 100% Duality Violation)

$$\frac{\Gamma}{\Gamma_0} = 0.657 + 0.657\,\mathcal{C}_{\mathrm{DV}} - 0.025|_{m_b^2} - 0.026|_{m_b^3} + 0.0003|_{m_b^4} + 0.007|_{m_b^5}$$

LLSA: arXiv:1407.4384

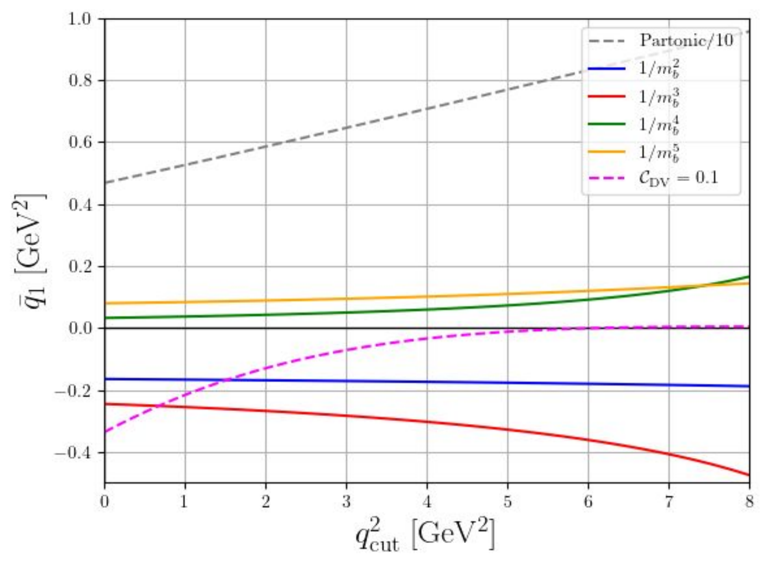
$m_c = 1.092\,\mathrm{GeV} \quad m_b = 4.573\,\mathrm{GeV}$

q^2 moments

$$Q_n(q^2_{\text{cut}}) \equiv \frac{1}{\Gamma_0} \int_{q^2_{\text{cut}}} \mathrm{d}q^2 \, (q^2)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}$$

$$\bar{q}_n \equiv \langle (q^2)^n \rangle_{q^2 \geq q^2_{\text{cut}}} \equiv \frac{Q_n(q^2_{\text{cut}})}{Q_0(q^2_{\text{cut}})}$$

- Non centralised q^2 moments
- DV most pronounced at low cut
- DV cut dependance differs slightly from power corrections
- Higher moments show a similar picture (see backup slides)



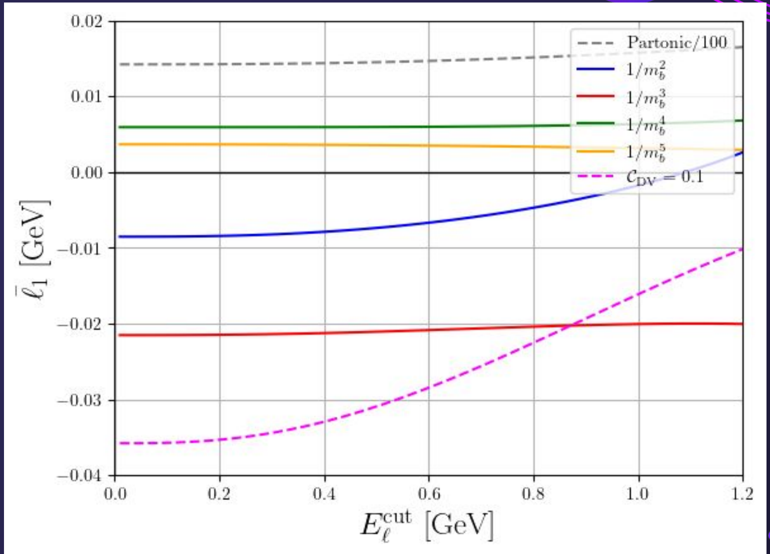
cut q^2 moments using LLSA values with DV contribution for Λ_{HQE} =0.5 GeV and C_{DV} = 0.1

Lepton energy moments

$$L_n(E_\ell^{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{E_\ell^{\rm cut}} {\rm d}E_\ell \, E_\ell^n \frac{{\rm d}\Gamma}{{\rm d}E_\ell}$$

$$\bar{\ell}_n \equiv \langle E_\ell^n \rangle_{E_\ell \geq E_\ell^{\rm cut}} \equiv \frac{L_n(E_\ell^{\rm cut})}{L_0(E_\ell^{\rm cut})}$$

- Non centralised moments
- DV most pronounced at low cut
- DV cut dependance does not differ significantly from power corrections
- Higher moments show a similar picture (see backup slides)
- DV may be difficult to disentangle from power corrections



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DV sensitive observables

- q^2 moment decomposition

$$\bar{q}_i = C_i^{(0)} + \frac{\mu_G^2}{m_b^2} C_i^{(2)} + \frac{\tilde{\rho}_D^3}{m_b^3} C_i^{(3)} + R_i$$

$$R_i = R_{DV} + \sum_{n=0}^{\infty} R_{m_b^{4+n}}$$

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- Construct observables depending only on R_i by cancelling lower order contributions

$$R_i = R_{DV} + \sum_{n=0}^{\infty} R_{m_b^{4+n}}$$

$$O_{\mathrm{DV}}^{(3)} = \xi_1 \frac{\bar{q}_1}{m_b^2} + \xi_2 \frac{\bar{q}_2}{m_b^4} + \xi_3 \frac{\bar{q}_3}{m_b^6} + \xi_4 \frac{\bar{q}_4}{m_b^8}$$

$$\xi_{(2..4)}(q_{cut}^2, \xi_1)$$

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$$\xi_{(2..4)}(q^2_{cut}, \xi_1)$$

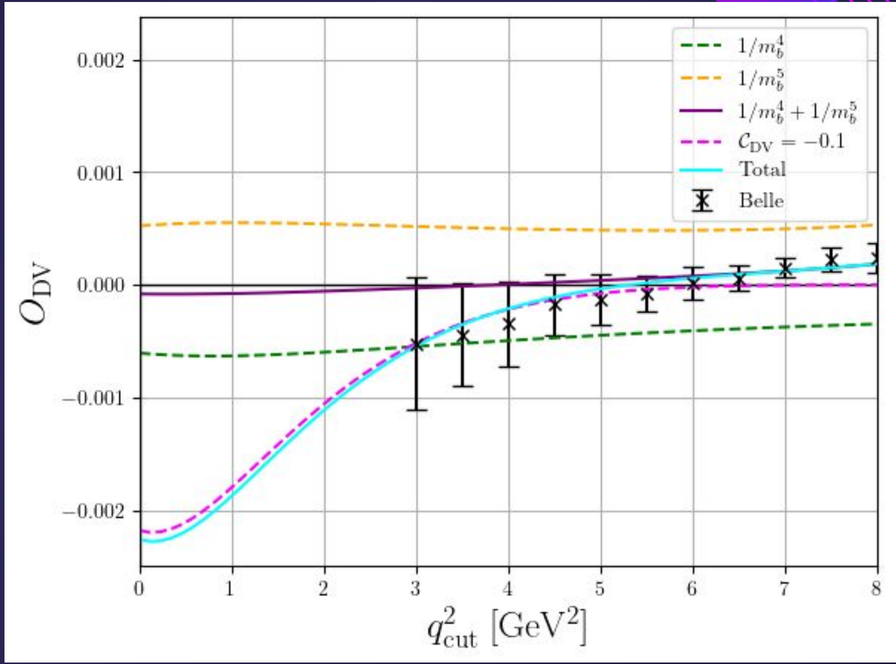
- No contribution from lower orders HQE

$$O_{\mathrm{DV}}^{(k)} \sim \Lambda_{HQE}^{k+1} / m_n^{k+1}$$

Data used from Belle collaboration, 2021
arxiv 2109.01685

QHDV from Belle data

- q^2 moment data from Belle electron channel (2021)



O_{DV} (q^2_{cut}) obtained from Belle data compared to theory and model predictions

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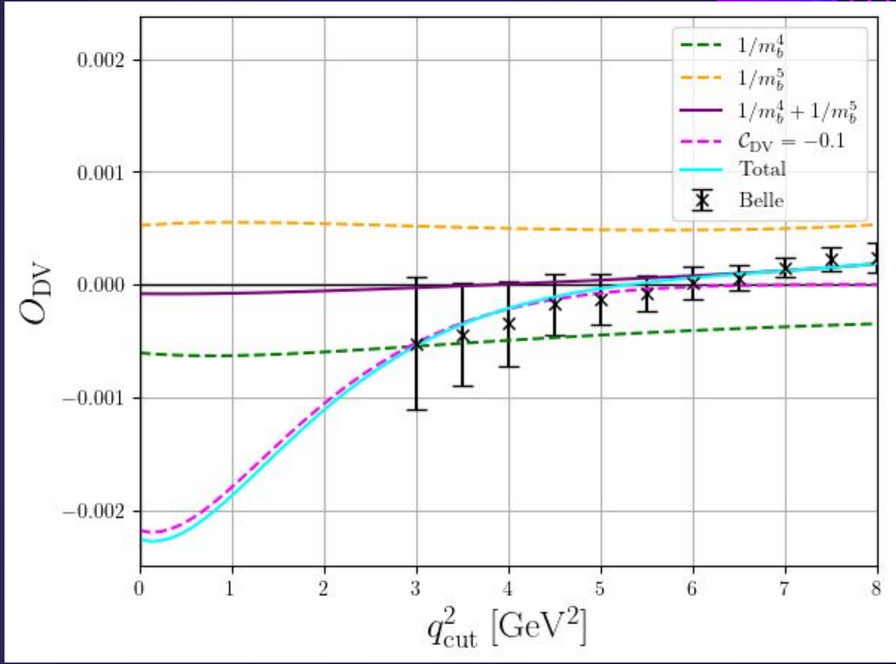
QHDV from Belle data

- q^2 moment data from Belle electron channel (2021)

- Comparison with theory (LLSA)

$$O^{(3)}_{\rm DV} = (5.182\, \mathcal{C}_{\rm DV} - 0.546|_{m_b^4} + 0.519|_{m_b^5}) \times 10^{-3}$$

$$(q^2_{\rm cut} = 3.0\ {\rm GeV}^2)$$



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QHDV from Belle data

- Determine C_{DV} from data using LLSA values

$$\mathcal{C}_{\rm DV} = -0.10 \pm 0.11 \qquad (q^2_{\rm cut} = 3.0~{\rm GeV}^2)$$
$$\mathcal{C}_{\rm DV} = -0.16 \pm 0.17 \qquad (q^2_{\rm cut} = 4.0~{\rm GeV}^2)$$
$$\mathcal{C}_{\rm DV} = -0.30 \pm 0.30 \qquad (q^2_{\rm cut} = 5.0~{\rm GeV}^2)$$

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$$\mathcal{C}_{\mathrm{DV}} = -0.30 \pm 0.30 \qquad (q^2_{\mathrm{cut}} = 5.0\ \mathrm{GeV}^2)$$

- Strongest constraint at low cuts
- Results consistent with $C_{DV} = 0$
- Combining different cuts and E_ℓ moments could further constrain

Conclusion and outlook

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- New DV sensitive observable build from kinetic moments can help constraint DV
  - Procedure could also constraint higher order

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    - Procedure could also constraint higher order
- Time for a full fit? .....

References presentation

- B. Chibisov, R. D. Dikeman, M. A. Shifman and N. Uraltsev, *Operator product expansion, heavy quarks, QCD duality and its violations*, Int. J. Mod. Phys. A 12 (1997) 2075–2133, [hep-ph/9605465]

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Illustrative example

- Expand at $x^2 = 0$ to form a kind of "OPE"

$$f(Q) = \int_0^\infty \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{r^{2k+2}} e^{iQx} dx$$

- Clearly missing the exponential term

$$f(Q) = \frac{i}{r} \sum_{k=0}^\infty \frac{(2k)!}{(Qr)^{2k+1}}$$

- Symmetric combination captures the uncertainty of the expansion coming from the singularity

$$\frac{f(Q) + f(-Q)}{2} = \frac{1}{2} \int_{-\infty}^\infty dx \frac{1}{x^2 + \rho^2} \mathrm{e}^{iQx}$$

$$= \frac{\pi}{2} \frac{\mathrm{e}^{-Q\rho}}{\rho}$$

- We found the lost exponential!

Chibisov et al. 1996
hep-ph/9605465

Duality Violation model

- Anzats model polynomials based on the HQE parameters

$$p_l^{(1,4)}(t) = \sum_{m=1}^{l+1} t^m = \frac{t - t^{l+2}}{1-t}$$

- Identifying the ambiguity through Borel transform

- Use optical theorem to obtain DV contribution to hadronic tensor

$$\hat{\Delta}_{\rm DV} W_{1,4}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{1,4}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE} - vQ} \quad \frac{vQ}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

Duality Violation model

- Choose polynomials based on calculating the parameters
up to l=5 and $1/m_b^5$

$$p_l^{(1,4)}(t) = \sum_{m=1}^{l+1} t^m = \frac{t-t^{l+2}}{1-t}$$

$$p_l^{(2,3)}(t) = \sum_{m=0}^{l} t^m = \frac{1-t^{l+1}}{1-t}$$

$$p_0^{(5)}(t) = 0 \qquad p_{l\geq 1}^{(5)}(t) = \sum_{m=2}^{l+1} t^m$$

$$\Rightarrow p_{l\geq 0}^{(5)}(t) = \frac{t^2-t^{l+2}}{1-t}$$

Duality Violation model

$$\hat{\Delta}_{\rm DV} W_{1,4}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{1,4}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ} \quad \frac{vQ}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{2,3}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{2,3}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ} \quad \frac{\Lambda_{HQE}}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right) - \sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{5}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{5}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ} \quad \frac{(vQ)^2}{\Lambda_{HQE}\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right) - \sqrt{\frac{\Lambda_{HQE}}{vQ}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

Differential rate

$$\frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{q}^2\mathrm{d}s\mathrm{d}y} = 48 m_b \Gamma_0 \left[\frac{2ys - y^2 - 2\hat{q}^2 + y\hat{q}^2}{1-s} W_1 + \hat{q}^2 W_2 + \frac{1}{2}\left(2ys - y^2 - \hat{q}^2\right) W_3 \right.$$

$$\left. + \frac{2ys - y^2 - \hat{q}^2}{1-s} W_4 + \frac{2ys - y^2 - \hat{q}^2}{2(1-s)^2} W_5 \right] \theta\left(\hat{q}^2\right) \theta\left(2ys - y^2 - \hat{q}^2\right)$$

$$\Gamma_0 = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} \qquad \hat{q}^2 = \frac{q^2}{m_b^2} \qquad s = \frac{v \cdot q}{m_b} \qquad y = \frac{2E_\ell}{m_b}$$

Our DV model

- OPE + DV model

$$W_i \to W_i^{(\mathrm{OPE})} + N\hat{\Delta}_{\mathrm{DV}} W_i(s, \hat{q}^2, \Lambda_{HQE})$$

- Scale of HQE chosen by taking the average of the HQE parameters

$$\Lambda_{HQE} = 0.5\,\mathrm{GeV}$$

- Normalised so that partonic and DV contributions are equal for C_{DV}=1, depends on scale

$$N = \frac{\Gamma_{\mathrm{P}}}{\Gamma_{\mathrm{DV}}} \mathcal{C}_{\mathrm{DV}} = 0.25\, C_{\mathrm{DV}}$$

- Breakdown of the normalised rate (using LSSA values)

$$\frac{\Gamma}{\Gamma_0} = 0.657 + 0.657\,\mathcal{C}_{\mathrm{DV}} - 0.025|_{m_b^2} - 0.026|_{m_b^3} + 0.0003|_{m_b^4} + 0.007|_{m_b^5}$$

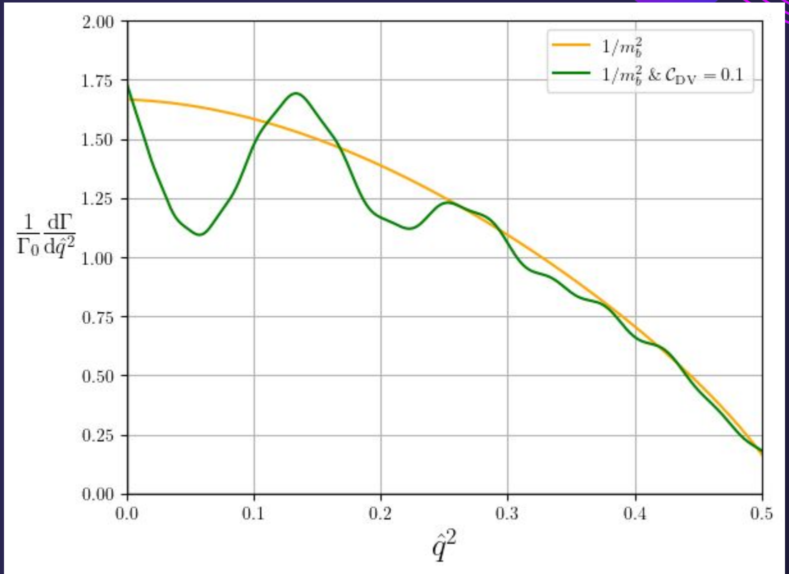
$$m_c = 1.092\,\mathrm{GeV} \quad m_b = 4.573\,\mathrm{GeV}$$

Instanton-like contribution

- Comparison with instanton terms motivates to keep the scale as a free fit parameter

- Choosing a small scale produces the expected 'wiggle' around the OPE

- For larger scale the period increases beyond the q^2 interval



Differential spectrum up to $1/m_b^2$ with DV for Λ DV = 10^{-4} GeV using N = 0.2508 C_{DV}

Kinematic moments

- q^2 moments

$$Q_n(q^2_{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{q^2_{\rm cut}} {\rm d}q^2\,(q^2)^n\,\frac{{\rm d}\Gamma}{{\rm d}q^2}$$

- Lepton Energy moments

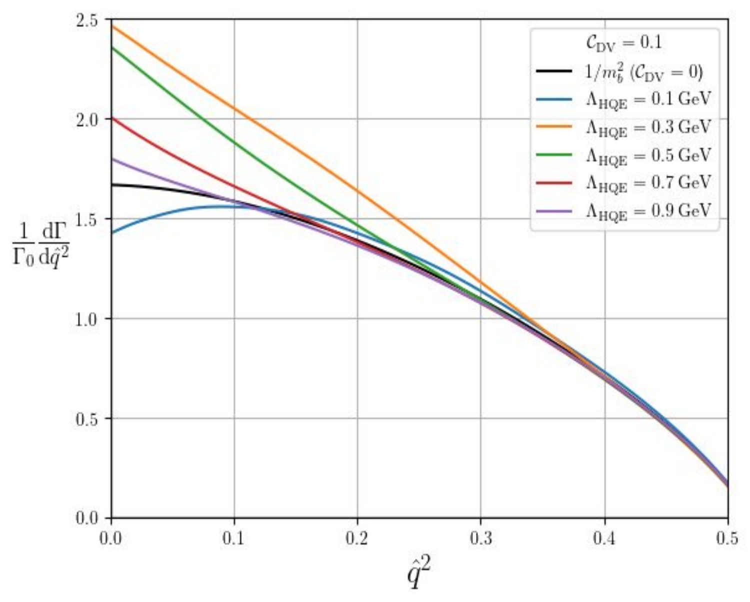
$$L_n(E_\ell^{\rm cut}) \equiv \frac{1}{\Gamma_0}\int_{E_\ell^{\rm cut}} {\rm d}E_\ell\,E_\ell^n\,\frac{{\rm d}\Gamma}{{\rm d}E_\ell}$$

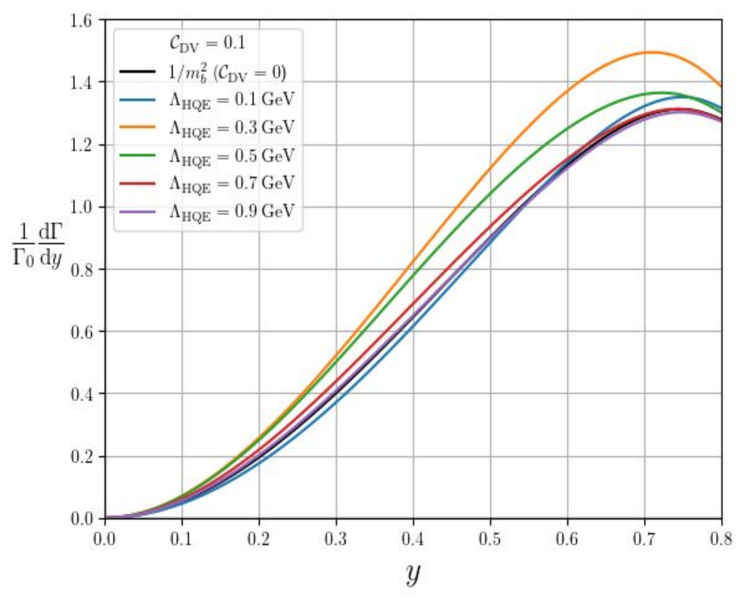
- Normalised and re-expanded in 1/mb and C_{DV} neglecting C_{DV}/mb terms

$$\bar{q}_n \equiv \langle (q^2)^n \rangle_{q^2 \geq q^2_{\rm cut}} \equiv \frac{Q_n(q^2_{\rm cut})}{Q_0(q^2_{\rm cut})}$$

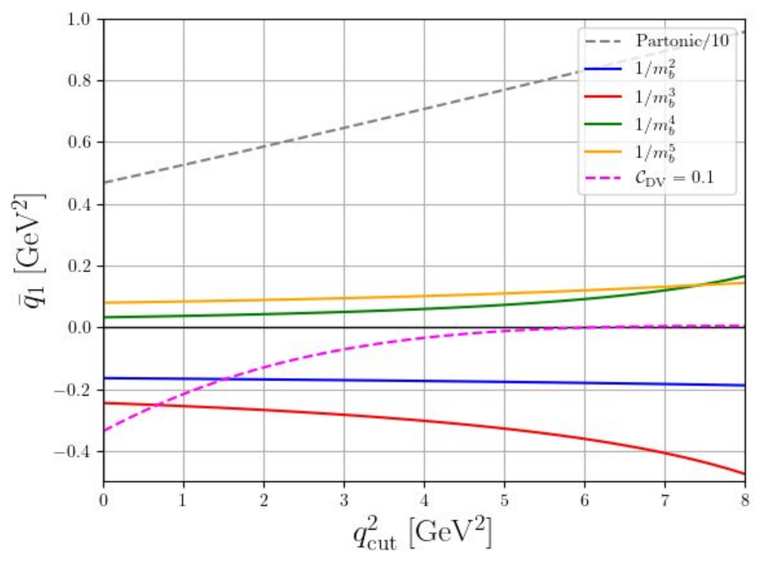
$$\bar{\ell}_n \equiv \langle E_\ell^n \rangle_{E_\ell \geq E_\ell^{\rm cut}} \equiv \frac{L_n(E_\ell^{\rm cut})}{L_0(E_\ell^{\rm cut})}$$

Effect of the scale parameter

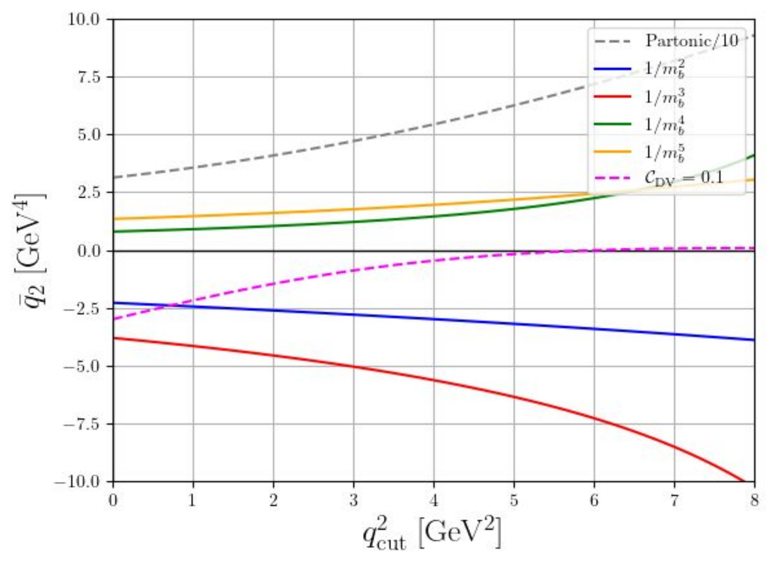




q^2 moments

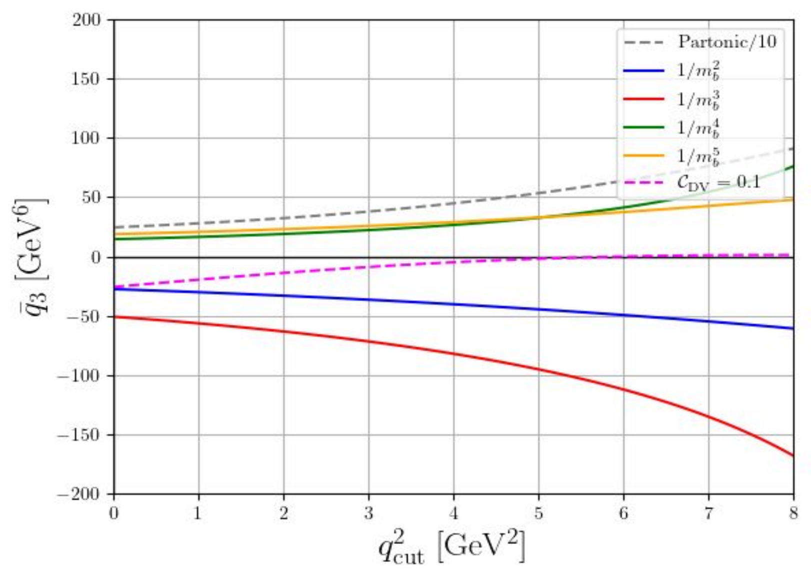


Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\mathrm{DV}} = 0.1$
$\bar{q}_1$ [GeV^2]
q^2_{cut} [GeV^2]

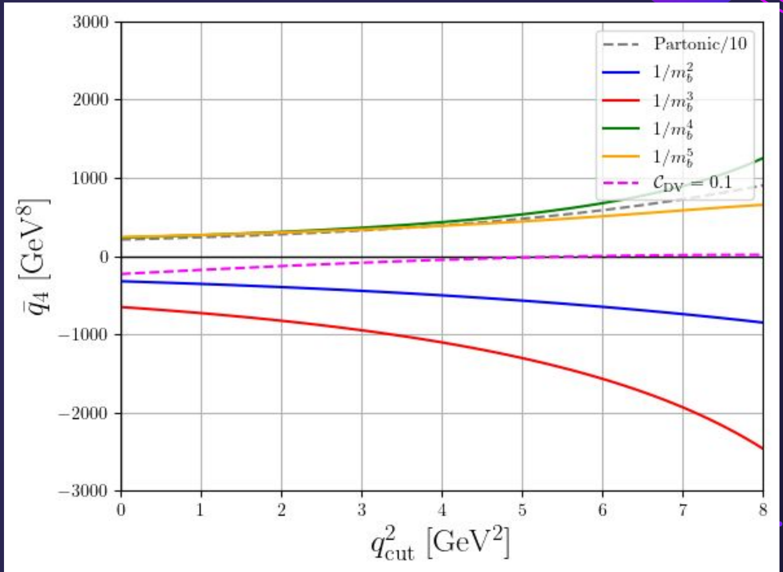


Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{\mathrm{DV}} = 0.1$
$\bar{q}_2$ [GeV^4]
q^2_{cut} [GeV^2]

q^2 moments

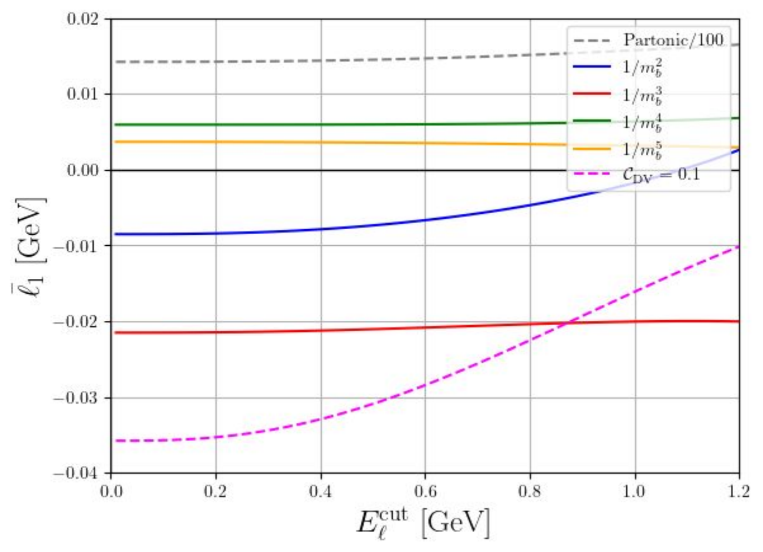


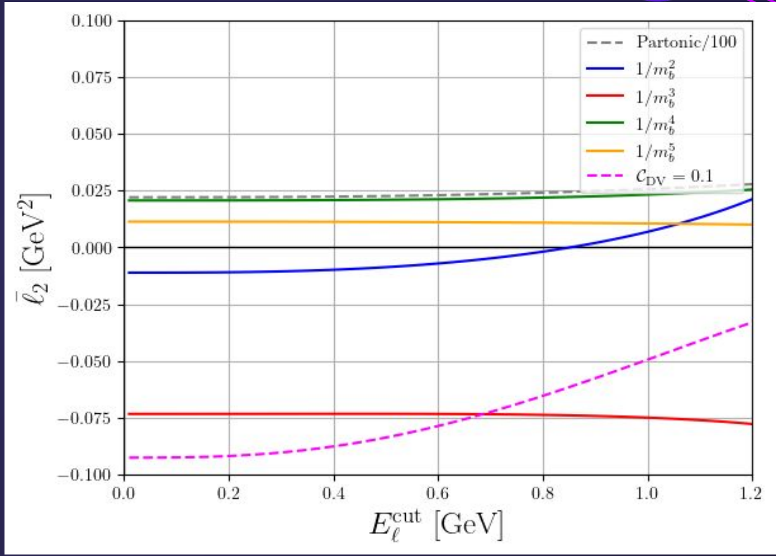
Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{DV} = 0.1$
$\bar{q}_3$ [GeV^6]
q^2_{cut} [GeV^2]



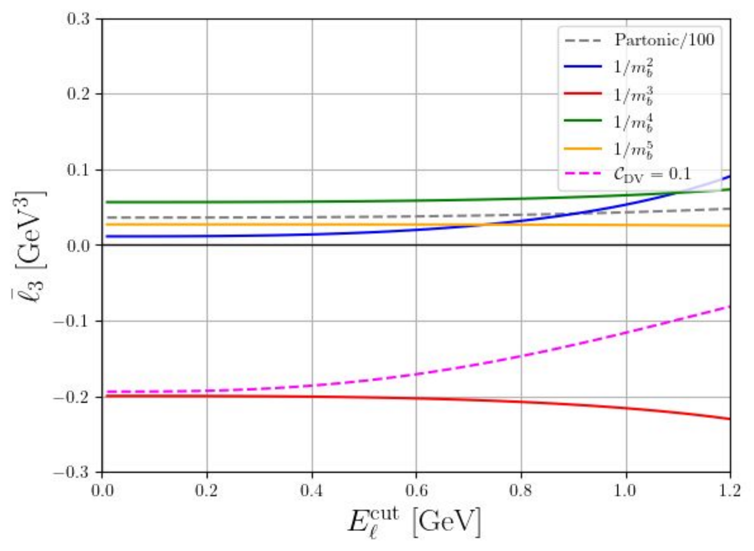
Partonic/10
$1/m_b^2$
$1/m_b^3$
$1/m_b^4$
$1/m_b^5$
$\mathcal{C}_{DV} = 0.1$
$\bar{q}_4$ [GeV^8]
q^2_{cut} [GeV^2]

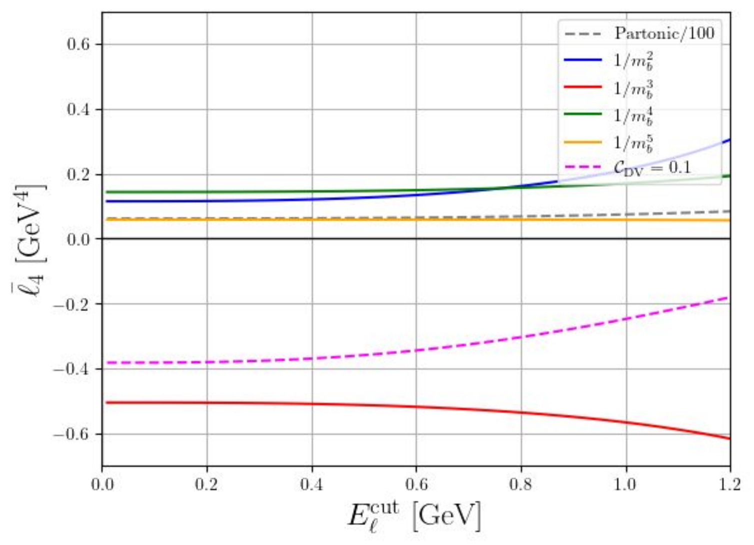
Lepton energy moments





Lepton energy moments





T_i				
$l=0$	$b_0^{(i,0)}$	$b_1^{i,1}$	-	-
$l=1$	$b_0^{(i,1)}$	$b_1^{i,2}$	$b_2^{i,3}$	-
$l=2$	$b_0^{(i,2)}$	$b_1^{i,3}$	$b_2^{i,4}$	$b_3^{i,5}$
$l=3$	$b_0^{(i,3)}$	$b_1^{i,4}$	$b_2^{i,5}$	$\mathcal{O}(1/m_b^6)$
$l=4$	$b_0^{(i,4)}$	$b_1^{i,5}$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$
$l=5$	$b_0^{(i,5)}$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$	$\mathcal{O}(1/m_b^6)$

T_1				
$l=0$	-0.5	0	-	-
$l=1$	0.032	-0.265	0	-
$l=2$	-0.052	0.050	0.002	0
$l=3$	-0.003	0.001	-0.0005	$\mathcal{O}$
$l=4$	-0.0002	0.0004	$\mathcal{O}$	$\mathcal{O}$
$l=5$	-0.000007	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_2				
$l = 0$	0	0.032	-	-
$l = 1$	0	-0.310	0.570	-
$l = 2$	0	-0.043	0.049	0.031
$l = 3$	0	-0.005	0.017	$\mathcal{O}$
$l = 4$	0	-0.0003	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_3				
$l = 0$	0	0.064	-	-
$l = 1$	0	-0.620	1.119	-
$l = 2$	0	-0.086	0.154	0.015
$l = 3$	0	-0.010	0.036	$\mathcal{O}$
$l = 4$	0	-0.0006	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_4				
$l = 0$	1	0	-	-
$l = 1$	-0.064	0.317	0	-
$l = 2$	0.103	-0.136	-0.004	0
$l = 3$	0.006	-0.007	0.001	$\mathcal{O}$
$l = 4$	0.0003	-0.001	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0.00001	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

T_5				
$l = 0$	0	0	-	-
$l = 1$	0.026	0	0	-
$l = 2$	0.003	0.035	0	0
$l = 3$	0.0003	0.001	0.001	$\mathcal{O}$
$l = 4$	0.00002	0.0002	$\mathcal{O}$	$\mathcal{O}$
$l = 5$	0	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

Heavy Quark Expansion

- Redefinition heavy quark field

$$b(x) = exp(-im_b v \cdot x) b_v(x)$$

- Operator Product Expansion of the Charm Propagator with m_c=0

$$\frac{1}{\gamma^\mu Q_\mu + i\gamma^\mu D_\mu} = \sum_{k=0}^{\infty} \left(\frac{1}{Q^2}\right)^{k+1} \gamma^\mu Q_\mu \left[-(i\gamma^\mu D_\mu)\gamma^\mu Q_\mu\right]^k$$

The model in HQE

$$T_{\mu\nu}(Q) = \int \mathrm{d}^4x\, e^{-iQ\cdot x} \langle B(p)|T\{\bar{b}_v(x)\Gamma_\mu c(x)\, \bar{c}(0)\overline{\Gamma}_\nu b_v(0)\}|B(p)\rangle$$

$$\frac{1}{\not{Q} + i\not{D}} = \sum_{k=0}^{\infty} \left(\frac{1}{Q^2}\right)^{k+1} \not{Q}[-(i\not{D})\not{Q}]^k$$

$$T_{\mu\nu}(Q) = \sum_{k=0}^{\infty} \left(\frac{1}{Q^2}\right)^{k+1} \langle B(v)|\bar{b}_v\Gamma_\mu \not{Q}[-(i\not{D})\not{Q}]^k\overline{\Gamma}_\nu b_v(0)|B(v)\rangle$$

$$\langle B(v)|\bar{b}_v \Gamma \not{Q} \overline{\Gamma} b_v|B(v)\rangle = a_0^{(i,0)}(vQ)$$

$$\langle B(v)|\bar{b}_v(-1)\Gamma \not{Q}(i \not{D}) \not{Q} \overline{\Gamma} b_v|B(v)\rangle = \Lambda_{HQE}\left(a_0^{(i,1)}(vQ)^2 + a_1^{(i,1)}Q^2\right)$$

$$\langle B(v)|\bar{b}_v \Gamma \not{Q}(i \not{D}) \not{Q}(i \not{D}) \not{Q} \overline{\Gamma} b_v|B(v)\rangle = \Lambda_{HQE}^2\left(a_0^{(i,2)}(vQ)^3 + a_1^{(i,2)}(vQ)Q^2\right)$$

$$T_{1,4}(t,\lambda^2) = \frac{1}{\Lambda_{HQE}}\frac{t\lambda^2}{1-t}\left(F_1(\lambda) - tF_2(\lambda)\right),$$

$$T_{2,3}(t,\lambda^2) = \frac{1}{\Lambda_{HQE}}\frac{\lambda^2}{1-t}\left(F_1(\lambda) - tF_2(\lambda)\right),$$

$$T_5(t,\lambda^2) = \frac{1}{\Lambda_{HQE}}\frac{t^2\lambda^2}{1-t}\left(F_1(\lambda) - F_2(\lambda)\right),$$

$$F_1(\lambda) = \sum_{l=0}^{\infty}(2l)!(\lambda^2)^l$$

$$F_2(\lambda) = \sum_{l=0}^{\infty}(2l)!(t\lambda^2)^l$$

Setting up the HQE

- Scalar Hadronic Structure functions

$$T_i(t, r^2) = \frac{1}{\Lambda_{HQE}} \sum_{l=0}^{\infty} \left(\frac{1}{r^2}\right)^{l+1} P_l^{(i)}(t)$$

$$r^2 \equiv \frac{Q^2}{\Lambda_{HQE}^2}$$

$$P_l^{(i)}(t) = \sum_{n=0}^{l+1} t^{l+1-n} a_n^{(i,n+l)}$$

$$t \equiv \frac{vQ}{\Lambda_{HQE}}$$

- $a_n^{(i,n+l)}$ can in principle be calculated from HQE parameters

Model expressions

$$\hat{\Delta}_{\rm DV} W_{1,4}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{1,4}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{vQ}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{2,3}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{2,3}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{\Lambda_{HQE}}{\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{vQ}{\Lambda_{HQE}}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$\hat{\Delta}_{\rm DV} W_{5}(vQ,Q^2) = -\frac{1}{\pi}\hat{\Delta}_{\rm DV}{\rm Im}\left[T_{5}(vQ,Q^2)\right] =$$

$$\frac{1}{\Lambda_{HQE}-vQ}\quad\frac{(vQ)^2}{\Lambda_{HQE}\sqrt{Q^2}}\left(\sin\left(\frac{\sqrt{Q^2}}{\Lambda_{HQE}}\right)-\sqrt{\frac{\Lambda_{HQE}}{vQ}}\sin\left(\frac{1}{\sqrt{\Lambda_{HQE}}}\sqrt{\frac{Q^2}{vQ}}\right)\right)$$

$$O_{\rm DV}^{(3)} = (5.182\,\mathcal{C}_{\rm DV} - 0.546|_{m_b^4} + 0.519|_{m_b^5}) \times 10^{-3} \qquad (q^2_{\rm cut} = 3.0\ {\rm GeV}^2)$$

$$O_{\rm DV}^{(3)} = (2.166\,\mathcal{C}_{\rm DV} - 0.494|_{m_b^4} + 0.499|_{m_b^5}) \times 10^{-3} \qquad (q^2_{\rm cut} = 4.0\ {\rm GeV}^2)$$

$$O_{\rm DV}^{(3)} = (0.751\,\mathcal{C}_{\rm DV} - 0.447|_{m_b^4} + 0.487|_{m_b^5}) \times 10^{-3} \qquad (q^2_{\rm cut} = 5.0\ {\rm GeV}^2)$$

Input values

Input values

m_b^{kin}	4.573 GeV	[20]
$\overline{m}_c(2\ \text{GeV})$	1.092 GeV	[20]
m_B	5.279 GeV	[30]
$\epsilon_{1/2}$	0.390 GeV	[23]
$\epsilon_{3/2}$	0.476 GeV	[23]
$(\mu_\pi^2)^\perp$	0.477 GeV^2	[20]
$(\mu_G^2)^\perp$	0.306 GeV^2	[20]

LSSA

LLSA approximation Historical basis

$(\rho_D^3)^\perp$	0.232 GeV^3
$(\rho_{LS}^3)^\perp$	-0.161 GeV^3
m_1	0.126 GeV^4
m_2	-0.112 GeV^4
m_3	-0.062 GeV^4
m_4	0.397 GeV^4
m_5	0.081 GeV^4
m_6	0.062 GeV^4
m_7	-0.039 GeV^4
m_8	-1.17 GeV^4
m_9	-0.393 GeV^4

LLSA approximation Historical basis

r_1	0.049 GeV^5
r_2	-0.106 GeV^5
r_3	-0.027 GeV^5
r_4	-0.043 GeV^5
r_5	0.00 GeV^5
r_6	0.00 GeV^5
r_7	0.00 GeV^5
r_8	-0.039 GeV^5
r_9	0.074 GeV^5
r_{10}	0.068 GeV^5
r_{11}	0.0059 GeV^5
r_{12}	0.010 GeV^5
r_{13}	-0.055 GeV^5
r_{14}	0.039 GeV^5
r_{15}	0.00 GeV^5
r_{16}	0.00 GeV^5
r_{17}	0.00 GeV^5
r_{18}	0.00 GeV^5

LLSA approximation RPI-basis

μ_π^2	0.477 GeV^2
μ_G^2	0.290 GeV^2
$\tilde{\rho}_D^3$	0.205 GeV^3
$\tilde{r}_E^4$	0.098 GeV^4
r_G^4	0.16 GeV^4
$\tilde{s}_E^4$	-0.074 GeV^4
s_B^4	-0.14 GeV^4
s_{qB}^4	-1.00 GeV^4
X_1^5	0.049 GeV^5
X_2^5	0.00 GeV^5
X_3^5	0.094 GeV^5
X_4^5	-0.41 GeV^5
X_5^5	-0.039 GeV^5
X_6^5	0.00 GeV^5
X_7^5	0.091 GeV^5
X_8^5	-0.0030 GeV^5
X_9^5	0.27 GeV^5
X_{10}^5	0.025 GeV^5