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Inclusive $B \to X_u \ell \nu$: Towards NNLO Extractions of V_{ub}

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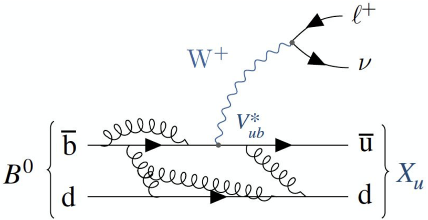
In collaboration with:
A. Broggio, P. Gambino and A. Ferroglia; arXiv:241x.xxxx

$B \to X_u \ell \nu$ Decay Distribution: Optical Theorem



Decay distribution $(B \to X_u \ell \nu)$

$$\frac{d^3\Gamma}{dq^2\, dE_\ell\, dE_\nu} \sim \sum_{X_u} \sum_{\text{pols.}} \frac{|\langle X_u \ell \nu | \mathcal{H}_{\text{eff}} | B \rangle|^2}{2 m_B} \delta^4(p_B - p_{X_u} - q)$$
$$= \frac{G_F^2\, |V_{ub}|^2}{8\pi^3} L_{\mu\nu} W^{\mu\nu}$$

Aneesh V. Manohar, Mark B. Wise; hep-ph/9308246

⇒ **Inclusive decays**: inclusive quantities do not depend on the hadronic final state

⇒ $L^{\mu\nu}$ **leptonic tensor** and $W^{\mu\nu}$ **hadronic tensor**

$$L^{\mu\nu} = 2\left(p_\ell^\mu p_{\nu_\ell}^\nu + p_\ell^\nu p_{\nu_\ell}^\mu - g^{\mu\nu} p_\ell p_{\nu_\ell} + i\epsilon^{\mu\nu\eta\rho} p_{\ell\eta} p_{\nu_\ell \rho}\right)$$

⇒ **Optical Theorem**: $d\Gamma \sim B$-meson forward scattering amplitude

$$W^{\mu\nu} \sim \text{Im} \int d^4x\, \mathrm{e}^{-iq\cdot x} \langle \bar{B} | T\left\{\bar{b}(x)\gamma_\mu(1-\gamma_5)u(x)\bar{u}\gamma^\nu(1-\gamma_5)b\right\} |\bar{B}\rangle$$

J. Chay, H. Georgi, A. Vainshtein; Phys. Lett. B247 (1992) 399
Aneesh V. Manohar, Mark B. Wise; hep-ph/9308246

$B \to X_u \ell \nu$ Decay Distribution: Form Factors

⇒ **Form factors** govern the structure of the decay distribution

$$m_b\, W^{\mu\nu} = -\, g^{\mu\nu}\, W_1 + v^\mu v^\nu\, W_2 + i\epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma\, W_3 + \hat{q}^\mu \hat{q}^\nu\, W_4 + (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu) W_5$$

J. Chay, H. Georgi, A. Vainshtein; Phys. Lett. B247 (1992) 399
Aneesh V. Manohar, Mark B. Wise; hep-ph/9308246
I. I. Bigi, N. G. Uraltsev; hep-ph/9310285

with $v^\mu = \dfrac{p^\mu}{m_b}$ and $q^\mu = p_\ell^\mu + p_{\nu_\ell}^\mu$ dilepton momentum

⇒ $B \to X_u \ell \nu$ decay distribution in terms of form factors and **massless leptons**

$$\frac{d^3\Gamma}{d\hat{E}_\ell\, d\hat{q}_0\, d\hat{q}^2} = \frac{G_F^2 m_b^5 |V_{ub}|^2}{16\pi^3} \theta(\hat{E}_\ell)\theta(\hat{q}^2)\theta\left(\hat{q}_0 - \hat{E}_\ell - \frac{\hat{q}^2}{4\hat{E}_\ell}\right) \times \left\{\hat{q}^2\, W_1 - \left[2\hat{E}_\ell^2 - 2\hat{E}_\ell \hat{q}_0 + \frac{\hat{q}^2}{2}\right] W_2 + \hat{q}^2 (2\hat{E}_\ell - \hat{q}_0)\, W_3\right\}$$

Aneesh V. Manohar, Mark B. Wise; hep-ph/9308246

where "hat quantities" are $\hat{x} = \dfrac{x}{m_b}$

$B \to X_u \ell \nu$ Decay Distribution: Heavy Quark Expansion

⇒ **Heavy Quark Expansion** (HQE): OPE in $1/m_b$ leading to the expression for W_i

$$W_i = W_i^{(0)} + W_i^{(\pi)} \frac{\mu_\pi^2}{m_b^2} + W_i^{(G)} \frac{\mu_G^2}{m_b^2} + W_i^{(D)} \frac{\rho_D^3}{m_b^3} + W_i^{(LS)} \frac{\rho_{LS}^3}{m_b^3} + \dots$$

$$W_i^{(j)} = \sum_n \left(\frac{\alpha_s}{\pi}\right)^n W_i^{(j,n)}$$

⇒ $W_i^{(j)}$ are **perturbatively calculable** coefficients

Theory status of the form factors in GGOU

P. Gambino, P. Giordano, G. Ossola, and N. Uraltsev; arXiv:0707.2493

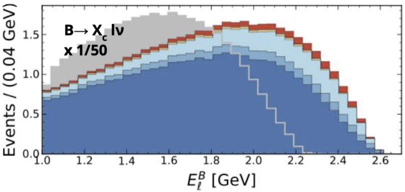
⇒ $W_i^{(0)}$: $b \to uW$ partonic decay up to NNLO BLM corrections $\sim O(\beta_0 \alpha_s^2)$

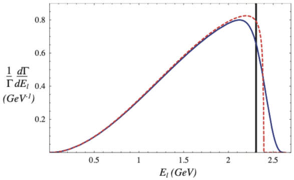
I. I. Y. Bigi, N. G. Uraltsev and A. I. Vainshtein; hep-ph/9207214
I. I. Y. Bigi, M. A. Shif- man, N. G. Uraltsev and A. I. Vainshtein; hep-ph/9304225
F. de Fazio, M. Neubert; hep-ph/9905351
V. Aquila, P. Gambino, G. Ridolfi, N. Uraltsev; hep-ph/0503083

⇒ $W_i^{(\pi,G,D,LS)}$: $(b\gamma^\mu P_L u)$ current QCD to HQET matching at LO

B. Blok, L. Koyrakh, M. A. Shifman and A. I. Vainshtein; hep-ph/9307247
Aneesh V. Manohar, Mark B. Wise; hep-ph/9308246

$b \to c$ Backgrounds: Phase Space Cuts





Belle; arXiv:2102.00020
M. Neubert; hep-ph/9311325

$B \to X_c \ell \nu$ very CKM favoured w.r.t. $B \to X_u \ell \nu$ ($|V_{cb}/V_{ub}| \sim 10$)

- $\Rightarrow$ Large charm backgrounds
- $\Rightarrow$ $B \to X_u \ell \nu$ signal difficult to measure
- $\Rightarrow$ Need to impose kinematic cuts to separate signal from background

$$\frac{m_b}{2} \sim E_\ell^{\max} \sim E_\ell > \frac{m_B^2 - m_D^2}{2m_B} \quad \text{and} \quad 0 \sim m_X^2 < m_D^2$$

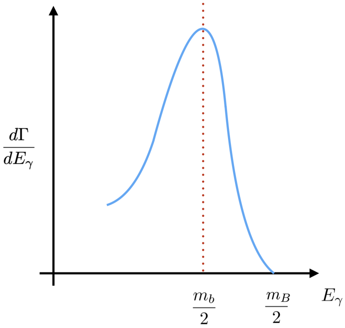
Convergence of the local OPE is destroyed within the region allowed by the kinematic cuts

- $\Rightarrow$ $(m_b v + k - q)^2 = (m_b v - q)^2 + O(m_b \Lambda_{\rm QCD}) + O(\Lambda_{\rm QCD}^2) \not\approx (m_b v - q)^2$ since $(m_b v - q)^2 \sim 0$
- $\Rightarrow$ Region very sensitive to non-perturbative effects of $O(k) \sim O(\Lambda_{\rm QCD})$

M. Neubert; hep-ph/9311325
M. Luke; hep-ph/0307378

Shape Function(s): $B \to X_s\gamma$

The residual $\sim \Lambda_{\rm QCD}$ momentum of the b-quark in the B-meson cannot be encoded into the non-perturbative matrix elements of the OPE. Needs to be resumed into a non-perturbative **Shape Function**



Partonic decay (tree level)

- ⇒ $b(p) \to s(p')\gamma(q)$ with $p = m_b v$
- ⇒ Infinitely narrow photon line at $E_\gamma^{(0)} = \frac{m_b}{2}$

Hadronic level

- ⇒ $B(p_B) \to X_s(p_{X_s})\gamma(q)$
- ⇒ Hadronic kinematic boundary at $E_\gamma^{\rm max} = \frac{m_B}{2}$
- ⇒ Partonic vs hadronic dynamics: $E_\gamma^{\rm max} - E_\gamma^{(0)} = \frac{m_B - m_b}{2} \sim \frac{\Lambda_{\rm QCD}}{2}$
- ⇒ Partonic dynamics: $b(p) \to s(p')\gamma(q)$ with $p = m_b v + k$ and $k \sim \Lambda_{\rm QCD}$

Decay distribution $d\Gamma/dE_\gamma$ is smeared due to purely non-perturbative effects

$$\frac{d\Gamma}{dE_\gamma} = \int dk_+\, F(k_+) \frac{d\Gamma^{\rm pert}}{dE_\gamma}\left(E_\gamma - \frac{k_+}{2}\right)$$

Bigi, Shifman, Uraltsev, Vainshtein; hep-ph/9312359

Shape Function(s) in the GGOU Framework

The **GGOU approach** models the **resummation** of **power corrections** as a convolution with non-perturbative **Shape Functions** (SFs)

$$W_i(q_0, q^2) = \int dk_+ F_i(k_+) W_i^{\mathrm{pert}} \left[q_0 - \frac{k_+}{2} \left(1 - \frac{q^2}{m_b M_B} \right), q^2 \right]$$

- ⇒ The SFs are the parton distribution functions for the b quark in the B meson
- ⇒ In the $m_b \to \infty$ limit, the SFs $F_i(k_+)$ reduce to a **single** and **universal** SF (for radiative and semileptonic decays)
- ⇒ At finite m_b **non-universal subleading** SFs emerge
- ⇒ **SFs modelling** becomes an **irreducible systematic** to $|V_{ub}|$ determinations

P. Gambino, P. Giordano, G. Ossola, and N. Uraltsev; arXiv:0707.2493

SFs in the GGOU Framework

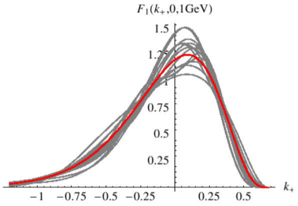
Subleading $O(1/m_b)$ corrections are absorbed into non-universal q^2-dependent SFs

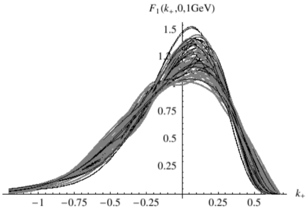
$$W_i(q_0, q^2) = \int dk_+ F_i(k_+, q^2) W_i^{\rm pert}\left[q_0 - \frac{k_+}{2}\left(1 - \frac{q^2}{m_b M_B}\right), q^2\right]$$

SFs can be constrained by matching with the q_0-moments of the OPE for the structure functions:

$$\int dk_+ k_+^n F_i(k_+, q^2) = \left(\frac{2}{\Delta}\right)^n \left[\delta_{n0} + \frac{J_i^{(n,0)}}{I_i^{(0,0)}}\right]$$

- ⇒ Matching consistency implies W_i up to $O(1/m_b^3)$ and $W_i^{\rm pert}$ at tree-level in the convolution formula
- ⇒ $I_i^{(n,0)}$ and $J_i^{(n,0)}$ the nth central q_0-moments of $W_i^{\rm tree}$ and $W_i^{\rm pow}$ (up to $O(1/m_b^3)$)
- ⇒ Different parametric families for $F_i(k_+, q^2)$ used to estimate the theoretical errors





P. Gambino, P. Giordano, G. Ossola, and N. Uraltsev: arXiv:0707.2493

Improvements on the GGOU Approach

Improving constraints and modelling of SFs

⇒ α_s/m_b^2 and α_s/m_b^3 corrections extend theoretical constraints on SF moments:

$$\int dk_+ k_+^n F_i(k_+, q^2) = \left(\frac{2}{\Delta}\right)^n \left[\delta_{n0} + \frac{J_i^{(n,0)}}{I_i^{(0,0)}} + O(\alpha_s)\right]$$

B. Capdevila, P. Gambino, S. Nandi; arXiv:2102.03343

⇒ Switch from parametric families to model-independent **Neural Networks** (NN) (**NN$\boldsymbol{V_{ub}}$ project**), akin to NNPDF

P. Gambino, K. Healey, C. Mondino; arXiv:1604.0759

⇒ Ongoing work: improve NNV_{ub} with new theoretical constraints and 1D experimental data for the NN trainings

Improving theory precision

⇒ From NNLO BLM to **full NNLO** to refine perturbative kernels used in observable calculations.

⇒ The improvements will allow for a more precise extraction of V_{ub} by refining the theoretical framework

A. Broggio, B. Capdevila, A. Ferroglia, P. Gambino,; arXiv:241x.xxxx

Open Challenges in Analytic NNLO Calculations

$$W_i^{(2)}(\hat{q}_0, \hat{q}^2) = w_i^{(2,\delta)}(\hat{q}^2)\,\delta(1+\hat{q}^2-2\hat{q}_0) + \sum_{m=0}^{3} w_i^{(2,+)}(\hat{q}_0,\hat{q}^2)\left[\frac{\ln^m(1+\hat{q}^2-2\hat{q}_0)}{1+\hat{q}^2-2\hat{q}_0}\right]_+ + \mathcal{R}_i^{(2)}(\hat{q}_0,\hat{q}^2)$$

⇒ $w_i^{(2,\delta)}$ **virtual corrections** known NNLO results

R. Bonciani, A. Ferroglia; arXiv:0809.4687

But full analytic results for the complete calculation still require additional work

⇒ **Missing topologies in double real radiation corrections**

- ⇒ Master Integrals (MIs) arising from these topologies include square root terms in the denominator
- ⇒ Non trivial to factorise into generalised harmonic polylogarithms

R. Bonciani, A. Broggio, L. Cieri, A. Ferroglia; arXiv:1807.01681

⇒ **One real, one virtual corrections** contributions at NNLO remain uncalculated

⇒ **Potential semi-analytic approach:** Missing MIs might be calculated using methods like the “expand and match” approach based on `AMFlow`

M. Fael, F. Lange, K. Schönwald and M. Steinhauser; arXiv:2106.05296
X. Liua, Yan-Qing Ma; arXiv:2201.11669

Numerical Approach to $b \to uW^*$ Calculations

⇒ **Numerical approach adopted due to analytic challenges**

Repurposed a code for $t \to bW^*$ observables at NNLO and adapted it for $b \to uW^*$

J. Gao, C. S. Li, H. X. Zhu; arXiv:1210.2808

⇒ Promote m_W from an on-shell value to $\hat{q}^2$

⇒ **Numerical calculation** for $b \to uW + \text{two}$ jets at LO and $b \to uW + \text{one}$ jet at NLO

⇒ The **Catani-Seymour method** employed to **cancel infrared divergences** in phase space (PS) integrals

S. Catani, M. H. Seymour; hep-ph/9605323

⇒ **Slicing method** to regulate the **unresolved collinear divergences** in the integrals using SCET inputs. Implemented **analytic** NNLO structures for the $b \to uW^*$ into the code

$$\frac{d\Gamma^{(2)}}{d\Phi_N}\mathcal{O}(\Phi_N) = \int_0^{\hat{m}_X^{2\text{cut}}} d\hat{m}_X^2 \, \frac{d\Gamma^{\text{N}^3\text{LL}}}{d\Phi_N \, d\hat{m}_X^2}\bigg|_{\mathcal{O}(\alpha_s^2)} \mathcal{O}(\Phi_N) + \int_{\hat{m}_X^{2\text{cut}}}^{\hat{m}_X^{2\text{max}}} d\hat{m}_X^2 \, \frac{d\Phi_{N+X}}{d\Phi_N}\frac{d\Gamma^{(1)}_{N+1}}{d\Phi_{N+X}}\mathcal{O}(\Phi_{N+X})$$
$$+ \int_{\hat{m}_X^{2\text{cut}}}^{\hat{m}_X^{2\text{max}}} d\hat{m}_X^2 \, \frac{d\Phi_{N+X}}{d\Phi_N}\frac{d\Gamma^{(0)}_{N+X}}{d\Phi_{N+2}}\mathcal{O}(\Phi_{N+X}) + O\left(\frac{1}{m_b}\right)$$

with $\mathcal{O}$ a $b \to uW^*$ obs (total rate, diff. moments, ...), $d\Phi_{N+X}$ is the LO (+ jets) PS and
$\hat{m}_X^2 = \dfrac{(p_u + p_X)^2}{m_b^2}$ is a **slicing parameter** that measures the jet invariant mass (partonic invariant mass)

Fits to $\mathcal{R}_i$: Double Differential Distribution

Bypass the analytic calculation of the $\mathcal{R}_i$ functions:

- ⇒ Calculate numerically suitable $b \to uW^*$ observables within the Slicing Method
- ⇒ Fit a parametrisation for the $\mathcal{R}_i$

We numerically calculate the double differential distributions

$$\frac{d\Gamma}{d\hat{m}_X^2 d\hat{q}^2} \sim \sqrt{\hat{q}_0^2 - \hat{q}^2}\left\{\hat{q}^2\, W_1 + \frac{1}{3}(\hat{q}_0^2 - \hat{q}^2)W_2\right\}$$

$$\frac{dM_1}{d\hat{m}_X^2 d\hat{q}^2} \sim \sqrt{\hat{q}_0^2 - \hat{q}^2}\left\{\hat{q}^2\hat{q}_0\, W_1 + \frac{1}{3}\hat{q}_0(\hat{q}_0^2 - \hat{q}^2)W_2 + \frac{1}{3}\hat{q}^2(\hat{q}_0^2 - \hat{q}^2)W_3\right\}$$

$$\frac{dM_2}{d\hat{m}_X^2 d\hat{q}^2} \sim \sqrt{\hat{q}_0^2 - \hat{q}^2}\left\{\frac{\hat{q}^2}{2}(4\hat{q}_0^2 - \hat{q}^2)\, W_1 + \frac{6\hat{q}_0^2 - \hat{q}^2}{10}(\hat{q}_0^2 - \hat{q}^2)W_2 + \hat{q}_0\hat{q}^2(\hat{q}_0^2 - \hat{q}^2)W_3\right\}$$

- ⇒ M_1, M_2 are the first two $\hat{E}_\ell$ moments of the $\hat{m}_X^2$, $\hat{q}^2$ distribution
- ⇒ These distributions are particularly sensitive to the structure of the form factors, providing key insights for the fitting process

Fits to $\mathcal{R}_i$: Double Differential Distribution

$\Rightarrow$ These distributions are calculated in bins of $\hat{m_X}^2$ and fixed $\hat{q}^2$

$$\left\langle \frac{d^2 M_{n=0,1,2}}{d\hat{m}_X^2 d\hat{q}^2} \right\rangle_{[\hat{m}_X^2]_i, \hat{q}_j^2} = \int_{\hat{m}_{X,i}^2}^{\hat{m}_{X,i+1}^2} d\hat{m}_X^2 \, \frac{d^2 M_{n=0,1,2}}{d\hat{m}_X^2 d\hat{q}^2} \bigg|_{\hat{q}^2 = \hat{q}_j^2}$$

with $[\hat{m}_X^2]_i = \left[\hat{m}_{X,i}^2, \hat{m}_{X,i+1}^2\right]$, and $\hat{m}_{X,i}^2 < \hat{m}_{X,i+1}^2$

Fits to $\mathcal{R}_i$: Singular Structures and Parametrising $\mathcal{R}_i$ at NLO and NNLO

Singular structures known exactly at NLO and NNLO from the SCET factorisation formula

$$W^{\mu\nu} = \sum_{i,j=1}^{3} H_{ij}(\bar{n}\cdot p)\,\mathrm{tr}\left(\bar{\Gamma}_j^{\mu}\,\frac{\not{p}_-}{2}\,\Gamma_i^{\nu}\,\frac{1+\not{v}}{2}\right)\int d\omega\, J(p_\omega^2)S(\omega)\ +\ \text{power corrections},$$

S. W. Bosch, B. O. Lange, M. Neubert, G. Paz; hep-ph/0402094
H. M. Asatrian, C. Greub, B. D. Pecjak; arXiv:0810.0987
T. Becher, M. Neubert; hep-ph/0603140

⇒ **Hard functions** H_{ij}, **jet function** J, and **soft function** S are all known at NNLO precision.

⇒ Incorporate these structures into our model for $W_i^{(j)}$

We parametrise the **remaining regular parts** $\mathcal{R}_i$ using a basis of $\hat{m}_X^2$ integrable functions

$$\mathcal{R}_i^{(j)}(\hat{q}_0,\hat{q}^2) = \sum_{j=1}^{n_f} \alpha_k^{W_i} f_k^{W_i}(\hat{q}_0,\hat{q}^2)$$

⇒ Schematically our parametrisation for the form factors

$$W_i^{(j)}(\hat{q}_0,\hat{q}^2) = \text{singular parts} + \sum_{j=1}^{n_f} \alpha_k^{W_i} f_k^{W_i}(\hat{q}_0,\hat{q}^2)$$

Preprocessing Integrals for $\hat{m}_X^2$ and $\hat{q}^2$ Distributions Fits

A given model for $W_i^{(j)}$ yields the following contribution to the $\hat{m_X}^2$, $\hat{q}^2$ distributions

$$\left\langle \frac{d^2 M_n}{d\hat{m}_X^2 d\hat{q}^2} \right\rangle_{[\hat{m}_X^2]_i, \hat{q}_j^2} = \frac{G_F^2 m_b^5 |V_{ub}|^2}{16\pi^3} \times \Bigg\{ \text{singular contributions} + $$

$$+ \sum_{k=1}^{n_f} \left(\alpha_k^{W_1} \left\langle \beta_k^{W_1,n} \right\rangle_{[\hat{m}_X^2]_i, \hat{q}_j^2} + \alpha_k^{W_2} \left\langle \beta_k^{W_2,n} \right\rangle_{[\hat{m}_X^2]_i, \hat{q}_j^2} + \alpha_k^{W_3} \left\langle \beta_k^{W_3,n} \right\rangle_{[\hat{m}_X^2]_i, \hat{q}_j^2} \right) \Bigg\}$$

$\Rightarrow$ In terms of **preprocessing integrals** $\left\langle \beta_k^{W_i} \right\rangle_{[\hat{m}_X^2]_k, \hat{q}_j^2}$

$$\left\langle \beta_k^{W_i,n} \right\rangle_{[\hat{m}_X^2]_k, \hat{q}_j^2} = \int_{\hat{q}_0^{\min}}^{\hat{q}_0^{\max}} 2\, d\hat{q}_0 \int_{\hat{E}_\ell^{\min}}^{\hat{E}_\ell^{\max}} d\hat{E}_\ell\, \hat{E}_\ell^n \left\{ \begin{array}{c} \hat{q}^2 \\ -\left(2\hat{E}_\ell^2 - 2\hat{E}_\ell \hat{q}_0 + \frac{\hat{q}^2}{2} \right) \\ \hat{q}^2 (2\hat{E}_\ell - \hat{q}_0) \end{array} \right\} f_k^{W_i}(\hat{q}_0, \hat{q}^2) \Big|_{\hat{q}^2 = \hat{q}_j^2}.$$

$\Rightarrow$ Fully vectorised implementation of the preprocessing integrals calculation with `JAX`

Default Fit Configuration: Moments, Bins, and Total Rate

Our **default fit** configuration includes

- ⇒ zeroth, first and second lepton energy moments of the $\hat{m_X}^2$, $\hat{q}^2$ distribution
- ⇒ 200 bins in $\hat{m_X}^2$: $\hat{m_X}^{2\,\text{min}} = 0$ and $\hat{m_X}^{2\,\text{min}} = 1$
- ⇒ However, first $\hat{m}_X^2$ bin is **excluded from the fit**, as it is dominated by singular structures that are already modeled exactly
- ⇒ 26 $\hat{q}^2$ points: evenly spaced from $\hat{q}^{2\,\text{min}} = 10^{-4}$ to $\hat{q}^{2\,\text{max}} = 0.8075$
- ⇒ Default NLO fit includes analytic **total rate** $\Gamma_{B\to X_u \ell\nu}$ (with arbitrarily small error)

Statistical Model and Fit Analysis

Construct a χ^2 statistic, incorporating both model and data errors

$$\chi^2(\theta_j) = \sum_{i=1}^{n_{\rm obs}} \frac{(y_i - \lambda_i(\theta_j))^2}{\sigma_{y_i}^2 + \sigma_{\lambda_i}^2(\theta_j)},$$

⇒ y_i represents the numerical data

⇒ σ_{y_i} denotes the errors on that data

⇒ λ_i represents our model, with θ_j parameters

⇒ σ_{λ_i} model errors from the numerical calculation of preprocessing integrals

Type of fits analysed

⇒ **NLO**, where we know the analytic structure, to test the framework

⇒ Then, we moved on to fits at **NNLO**

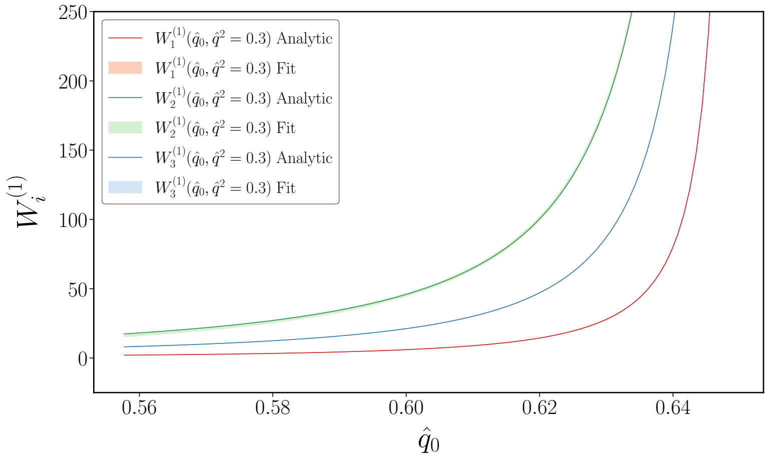
Model for NLO Fits

```
w = 1 - q2h
uh = 1 + q2h - 2 * q0h
lambdab = 4 * (q0h**2 - q2h)
I1 = jnp.log((1 + uh - q2h + jnp.sqrt(lambdab)) / (1 + uh - q2h - jnp.sqrt(lambdab))) / jnp.sqrt(lambdab)
B_func = (jnp.log(uh / ((1 - q2h)**2)) + (1 - q2h) * I1) / ((1 - q2h) * uh)

model_vec = jnp.array([
    jnp.ones_like(uh),
    uh,
    uh**2,
    uh**3,
    w,
    w * uh,
    w * uh**2,
    w * uh**3,
    w**2,
    w**2 * uh,
    w**2 * uh**2,
    w**2 * uh**3,
    jnp.log(uh / w**2),
    w * jnp.log(uh / w**2),
    jnp.log(w) * jnp.log(uh),
    jnp.log(w) * jnp.log(uh) * uh,
    jnp.sqrt(lambdab) * jnp.log(uh),
    uh * w * jnp.log(uh),
    w**2 * jnp.log(uh / w**2),
    w**3 * jnp.log(uh),
    B_func,
    w * B_func,
    w**2 * B_func,
    w**3 * B_func,
    w**4 * B_func
])
```

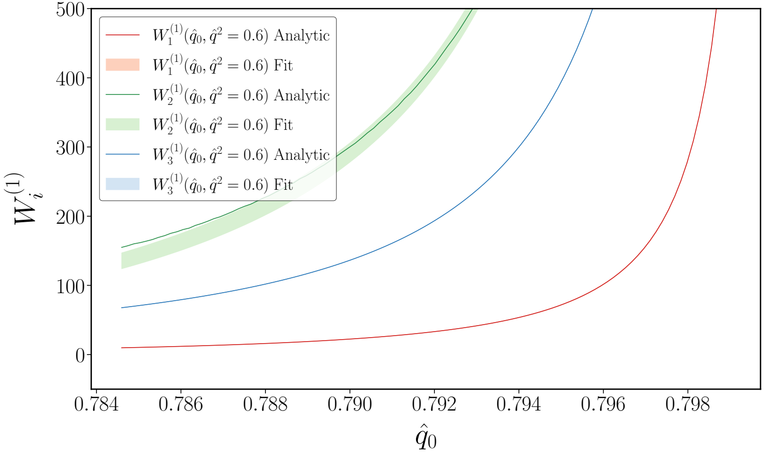
Results from NLO Fits: Form Factor Extraction

- ⇒ Results from the **NLO fits**
- ⇒ W_2 prefactors in $M_{0,1,2}$ are $(\hat{q}_0^2 - \hat{q}^2)$, $\hat{q}_0(\hat{q}_0^2 - \hat{q}^2)$, $\frac{6\hat{q}_0^2 - \hat{q}^2}{10}(\hat{q}_0^2 - \hat{q}^2)$
- ⇒ The fit is insensitive to W_2 at $\hat{q}_0^{\text{min}} = \sqrt{\hat{q}^2}$. Does **not affect** the calculation of observables



Results from NLO Fits: Form Factor Extraction

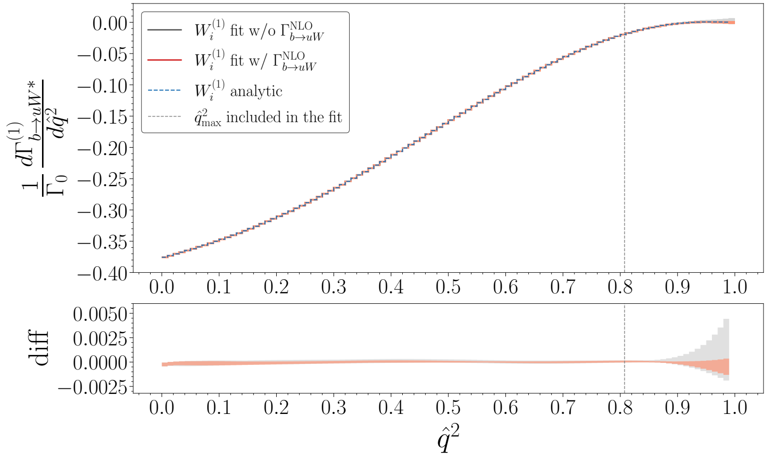
- $\Rightarrow$ Results from the **NLO fits**
- $\Rightarrow$ W_2 prefactors in $M_{0,1,2}$ are $(\hat{q}_0^2 - \hat{q}^2)$, $\hat{q}_0(\hat{q}_0^2 - \hat{q}^2)$, $\frac{6\hat{q}_0^2 - \hat{q}^2}{10}(\hat{q}_0^2 - \hat{q}^2)$
- $\Rightarrow$ The fit is insensitive to W_2 at $\hat{q}_0^{\text{min}} = \sqrt{\hat{q}^2}$. Does **not affect** the calculation of observables



NLO $\hat{q}^2$ Spectrum: Comparison of Fit and Analytic Results

⇒ **Calculation of the $\hat{q}^2$ spectrum** from the fit results (including and excluding the total rate in the fit) and its comparison with the analytic expression

⇒ diff = analytic − fit. Max diff $\sim 0.5\%$

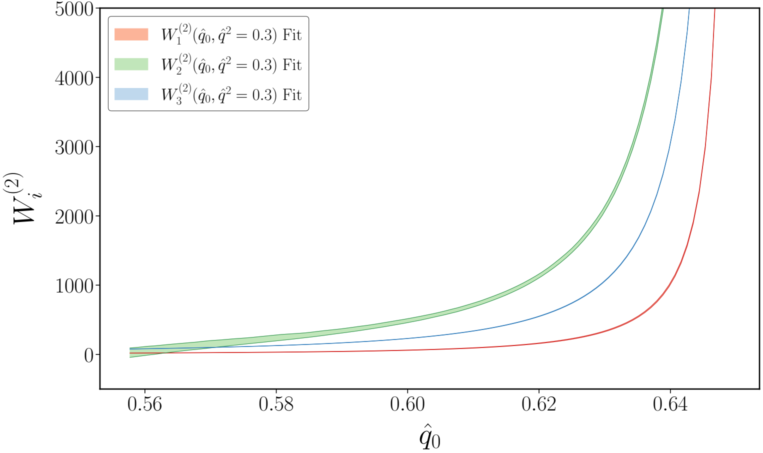


Model for NNLO Fits

```
model_vec = jnp.array([
    jnp.ones_like(uh),
    uh,
    uh**2,
    uh**3,
    uh**4,
    q2h,
    q2h * uh,
    q2h * uh**2,
    q2h * uh**3,
    q2h * uh**4,
    q2h**2,
    q2h**2 * uh,
    q2h**2 * uh**2,
    q2h**2 * uh**3,
    q2h**2 * uh**4,
    q2h**3,
    q2h**3 * uh,
    q2h**3 * uh**2,
    q2h**3 * uh**3,
    q2h**3 * uh**4,
    jnp.log(1 - q2h),
    q2h * jnp.log(1 - q2h),
    q2h**2 * jnp.log(1 - q2h),
    uh * jnp.log(1 - q2h),
    uh**2 * jnp.log(1 - q2h),
    uh**3 * jnp.log(1 - q2h),
    jnp.log(uh),
    q2h * jnp.log(uh),
    q2h**2 * jnp.log(uh),
    jnp.log(1 - q2h) * jnp.log(uh),
    jnp.log(uh)**2,
    q2h * jnp.log(uh)**2,
    q2h**2 * jnp.log(uh)**2,
    jnp.log(1 - q2h) * jnp.log(uh)**2,
    jnp.log(uh)**3,
    q2h * jnp.log(uh)**3,
    q2h**2 * jnp.log(uh)**3,
    jnp.log(1 - q2h) * jnp.log(uh)**3,
])
```

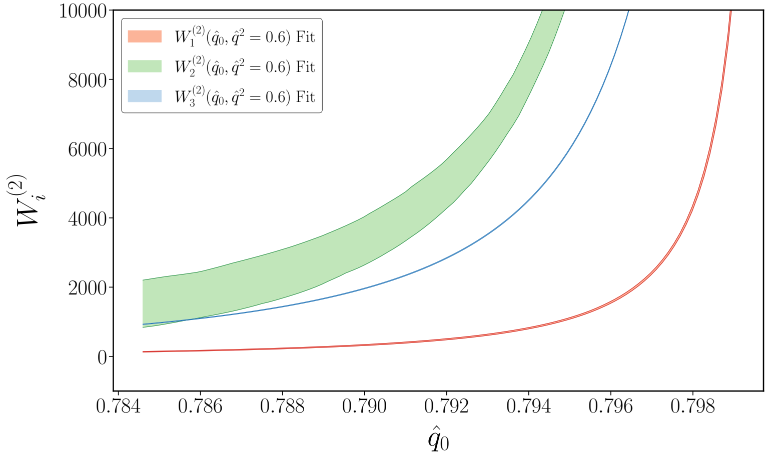
Results from NNLO Fits: Form Factor Extraction

⇒ Results from the **NNLO fits**



Results from NNLO Fits: Form Factor Extraction

⇒ Results from the **NNLO fits**



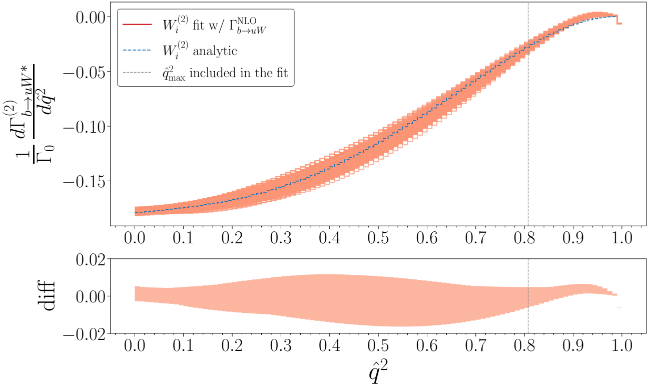
NNLO $\hat{q}^2$ Spectrum: Comparison of Fit and Analytic Results

⇒ **Calculation of the total rate at NNLO** from the fit results (excluding the total rate in the fit) and the known analytic result

L. Chen, H. T. Li, J. Wang, Y. Wang; arXiv:2212.06341

$$\Gamma^{(2)}_{B\to X_u\ell\nu} = -21.2955 \text{ (analytic)} \qquad \Gamma^{(2)}_{B\to X_u\ell\nu} = -21.4865 \pm 0.5730 \text{ (fit)}$$

⇒ We also compare the $\hat{q}^2$ **spectrum at NNLO** between the fit and the analytic results



Summary

⇒ We have developed a robust numerical approach to extract the $b \to uW^*$ form factors at NLO and NNLO, overcoming the challenges posed by missing analytic structures

⇒ Our fits at NLO show excellent agreement with known analytic structures, providing confidence in the reliability of our methods

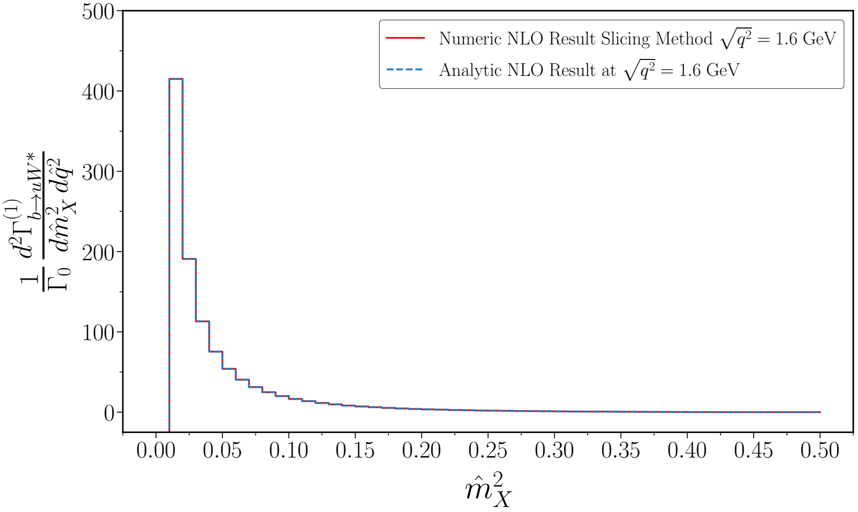
⇒ At NNLO, we have successfully extracted form factors and calculated the total rate and $\hat{q}^2$ spectrum, comparing them with the analytic predictions

⇒ These results pave the way for further improvements in our models, especially in refining the NNLO fits and including more sophisticated theoretical constraints

⇒ Future work will focus on extending the NNLO analysis, improving numerical precision

Thank You!

Numerical Tests at NLO: Slicing Method vs Analytic Results



Numerical Tests at NLO: Slicing Method vs Analytic Results