Inclusive semileptonic decays from Lattice QCD

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In collaboration with
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High Energy Accelerator Research Organization (KEK)

Challenges in Semileptonic B Decays, September 25th, 2024

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Today's agenda

- Quick review on lattice formulation of inclusive decays

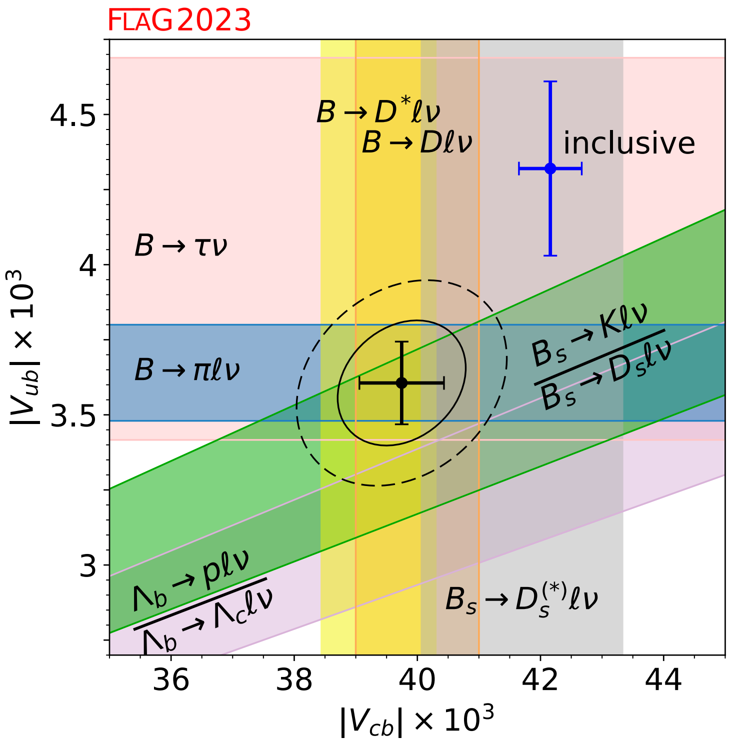
- Systematic errors in the analysis
  1. Finite-volume effects
  2. Finite polynomial approximation

- Extension to more observables

- Summary & Outlook

Introduction

Current landscape



[Y. Aoki et al., arXiv:2111.09849]

$\sim 3\sigma$ discrepancy between exclusive and inclusive determination

Many determinations for exclusive channels from lattice QCD and Experiment

Inclusive determination has relied on OPE

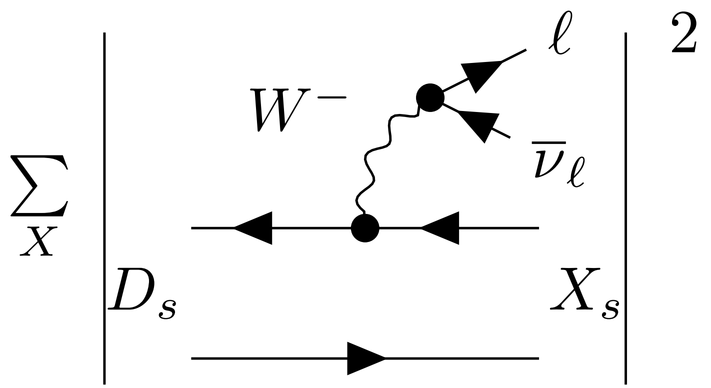
- Lattice QCD might be able to provide input

  enables fully nonperturbative theoretical treatment of QCD

Today:

Current status on the analyis of inclusive decays of charmed and bottomed mesons from lattice QCD

On-going analysis of inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$



$$\frac{d\Gamma}{dq^2 dq_0^2 dE_l} = \frac{G_F^2 |V_{cs}|^2}{8\pi^2} L_{\mu\nu} W^{\mu\nu}$$

$L_{\mu\nu}$: Leptonic tensor (analytically known)

$W^{\mu\nu}$: Hadronic tensor (nonperturbative QCD)

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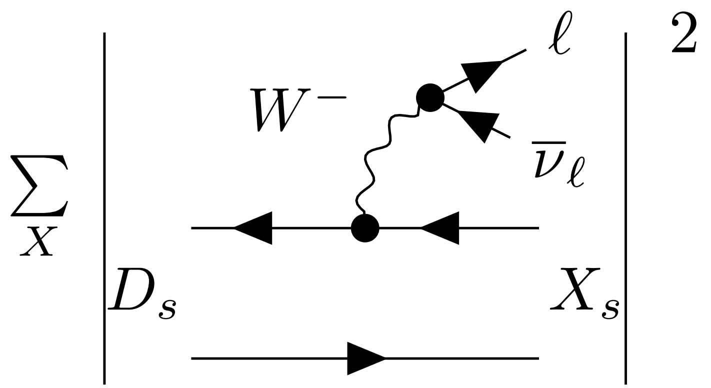

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Challenges on the lattice

- Require external states
  - ➢ Long time separations

4pt correlator

$$C_{\mu\nu}(t) \sim \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(\boldsymbol{q}) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\mathbf{0}) \rangle$$

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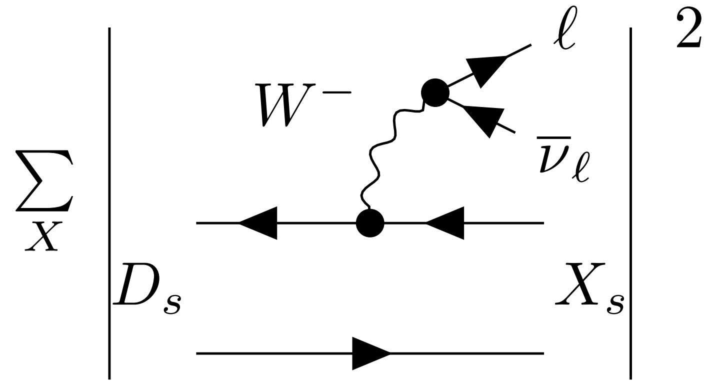

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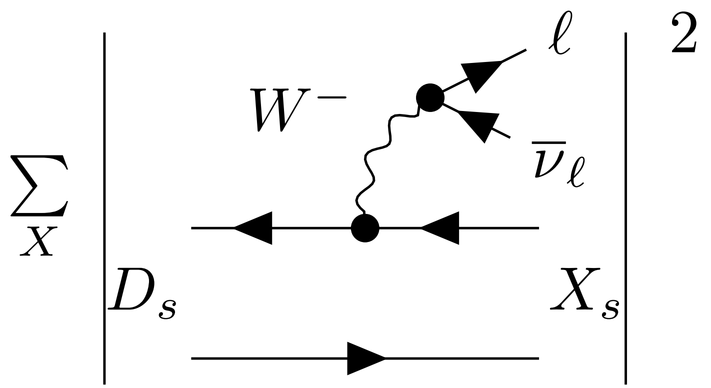
Challenges on the lattice

- Require external states
  - ➢ Long time separations
- Large number of states
  - ➢ Identify all of them?

$$C_{\mu\nu}(t) \sim \sum_{X_s} \langle D_s | \tilde{J}_\mu^\dagger(\boldsymbol{q}) | X_s \rangle \langle X_s | \tilde{J}_\nu(\boldsymbol{q}) | D_s \rangle e^{-E_{X_s} t}$$

On-going analysis of inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$



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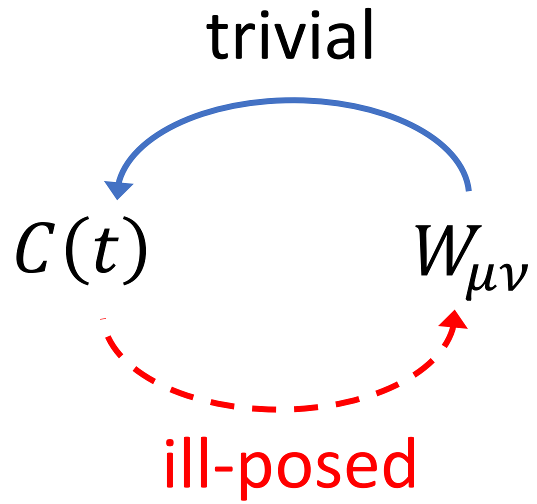
Challenges on the lattice

- Require external states
  - Long time separations
- Large number of states
  - Identify all of them?
- Extraction of $W_{\mu\nu}$ from correlator ill-posed problem (**inverse problem**)



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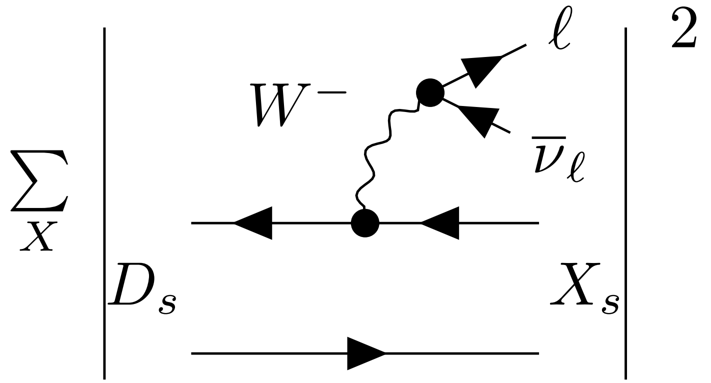

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Idea [P. Gambino & S. Hashimoto, 2005.13730]

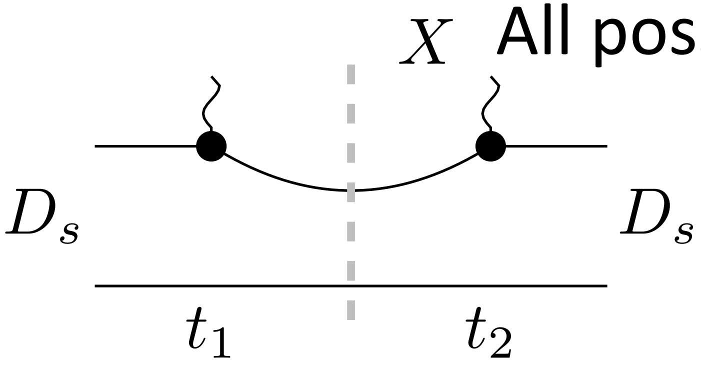
Smeared spectral density

$\rho_s(\omega)$

Smearing ≙ phase space integral

Approximation using **4Pt function** correlation function

All possible states



$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, [\,]_{\mathrm{Lattice}}$$

Inclusive Decays - Continuum

Total decay rate [2211.16830, 2305.14092]

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}(\boldsymbol{q}^2)$$

$\bar{X}^{(l)}(\boldsymbol{q}^2)$ integral over energy of hadronic final states

Kernel function

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) K^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)$$

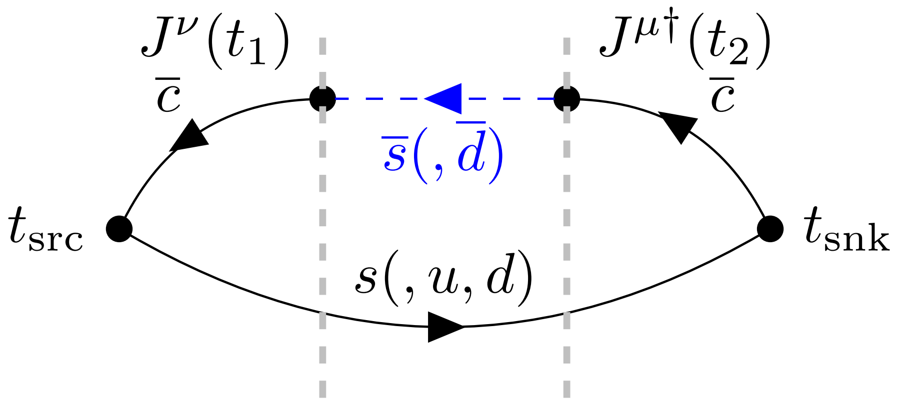
$$k^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)$$

Analytically known

l-th power of ω and $\boldsymbol{q}^2$

Step function

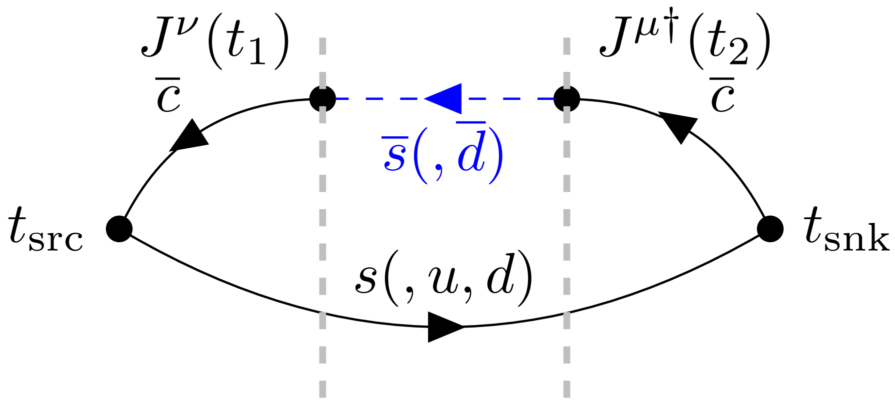
Inclusive decays – Lattice



- t_{src}, t_2, t_{snk} fixed
- $t_{src} \le t_1 \le t_2$
- $t = t_2 - t_1$

$$C_{\mu\nu}(\boldsymbol{q}, t) = \int_0^\infty d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega)\, e^{-\omega t}$$

Inclusive decays – Lattice



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Continuum expression

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^\infty d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) K_{\mu\nu}^{(l)}(\boldsymbol{q}, \omega)$$

Approximate Kernel in polynomials of $e^{-\omega}$

$$K(\omega, \boldsymbol{q}) \simeq k_0 + k_1 e^{-\omega} + \ldots + k_N e^{-N\omega}$$

C(0) C(N)

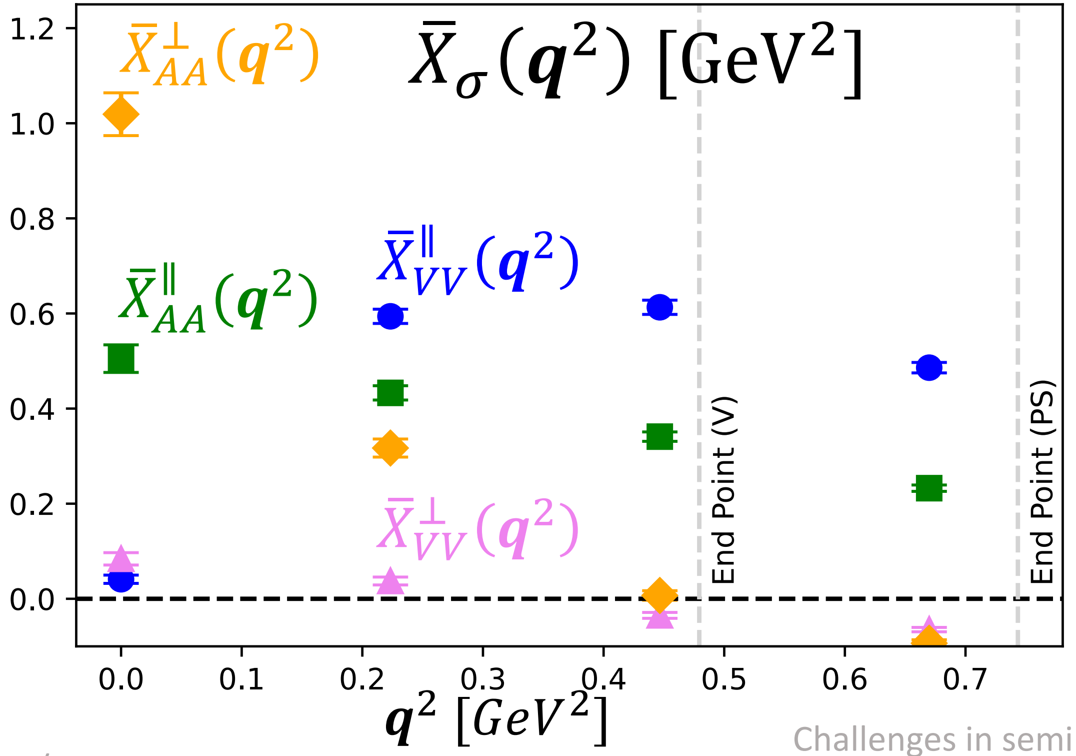
$$\bar{X}^{(l)}(\boldsymbol{q}^2) \sim k_0 \int_{\omega_0}^\infty d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) + \cdots + k_N \int_{\omega_0}^\infty d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) e^{-N\omega}$$

Numerical Results

The differential rate $\bar{X} \sim \frac{d\Gamma}{d\boldsymbol{q}^2}$

$$\bar{X}_\sigma(\boldsymbol{q}^2) = \sum_{l=0}^{2} \left\langle D_s(\boldsymbol{0}) \middle| \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K_\sigma^{(l)}(\hat{H}, \boldsymbol{q}^2) \tilde{J}_\nu(\boldsymbol{q}) \middle| D_s(\boldsymbol{0}) \right\rangle$$

$N = 10, \sigma = 0.1$



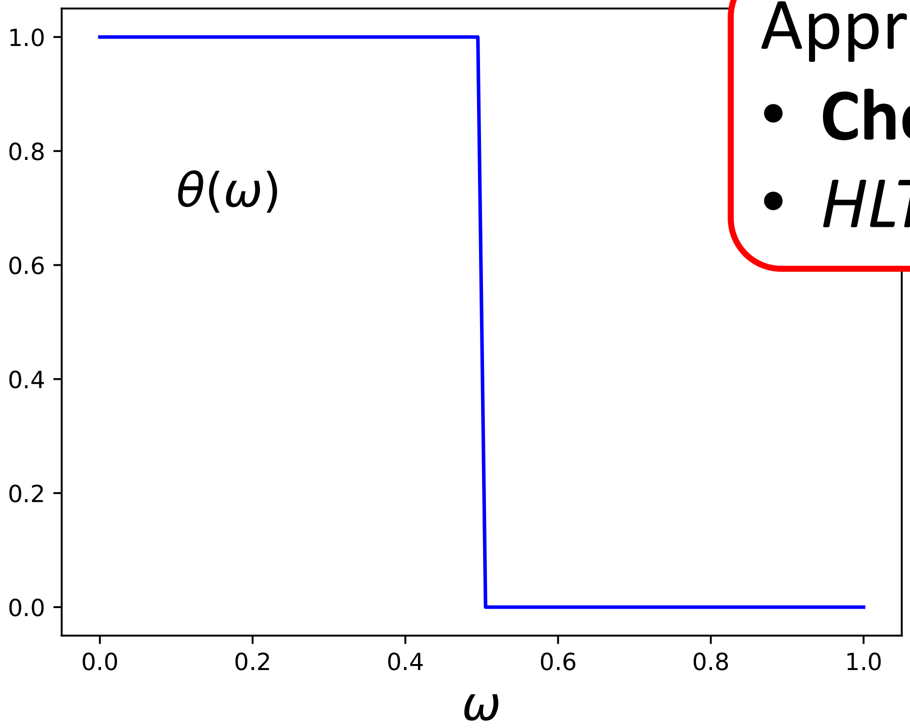
Decomposed $\bar{X}_\sigma(\boldsymbol{q}^2)$:

- Vector (*VV*) & Axial-vector (*AA*)
- ∥ and ⊥ polarization with respect to $\boldsymbol{q}$

Decomposition allows for comparison with ground state limit

Systematic error – kernel approximation

Upper limit of the energy integral

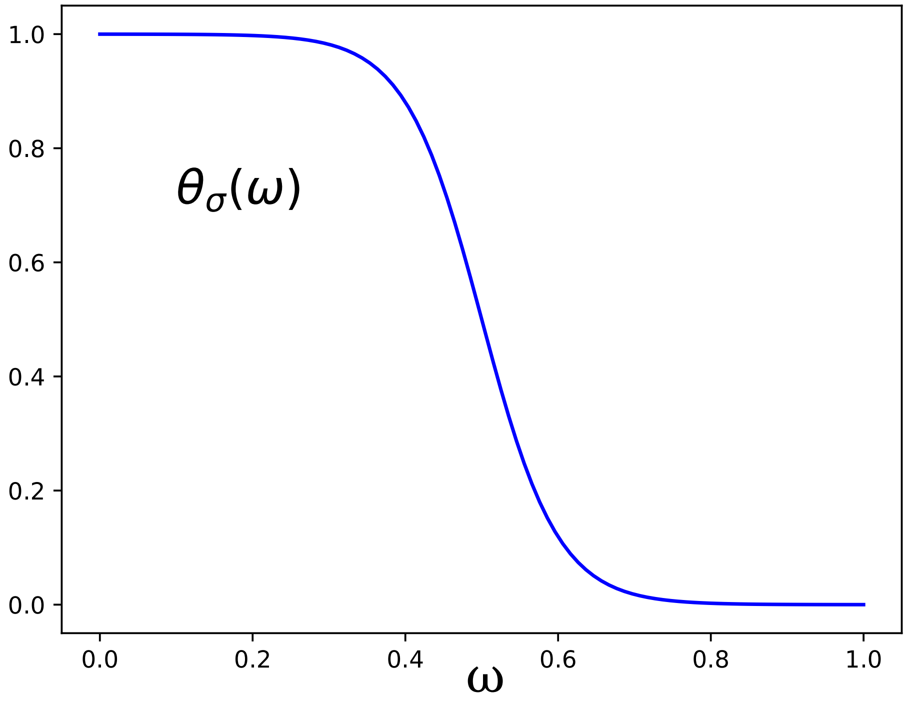


Approximation strategies [2305.14092, 1903.06476] :

- **Chebyshev approximation**
- *HLT approach*

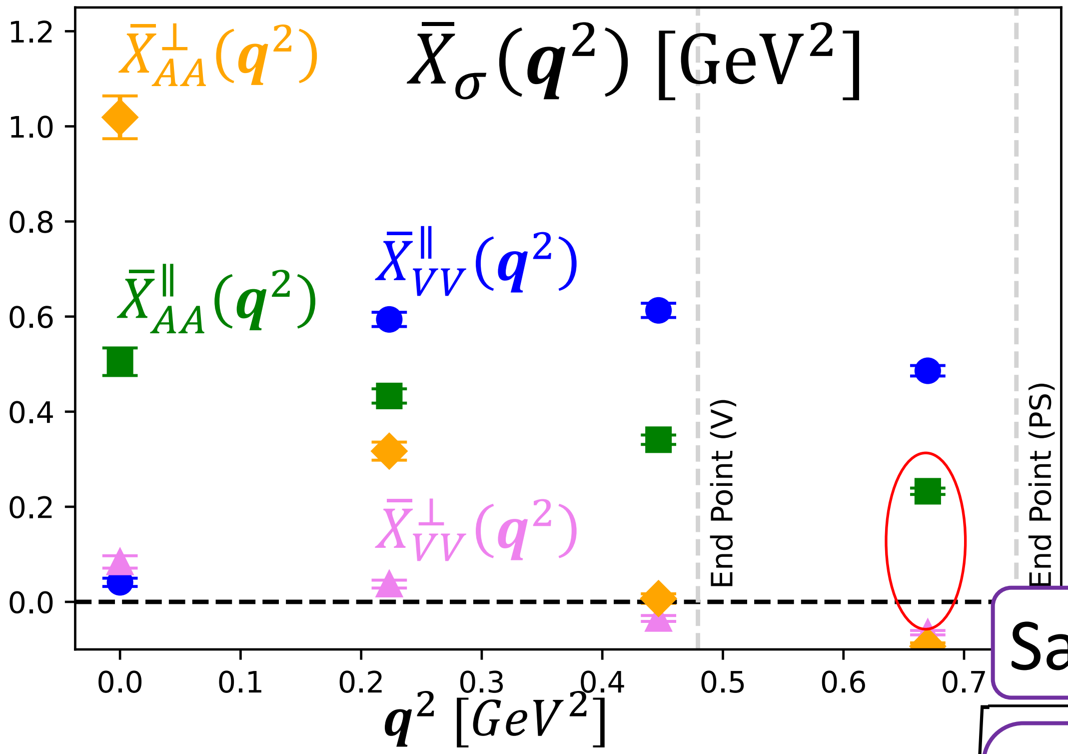
Direct approximation with $e^{-\omega(t_2-t_1)}$ not possible

Apply smearing



Systematic error - Approximation

$N = 10, \sigma = 0.1$



Create estimate [2211.16830]

- $N \to \infty$; frequency component
- $\sigma \to 0$; width

$$\sigma = \frac{1}{N}$$

Property of Chebyshev polynomials

$$\left|\langle \tilde{T}_k(\omega) \rangle\right| \leq 1$$

Sample size: 1000

Estimate error from

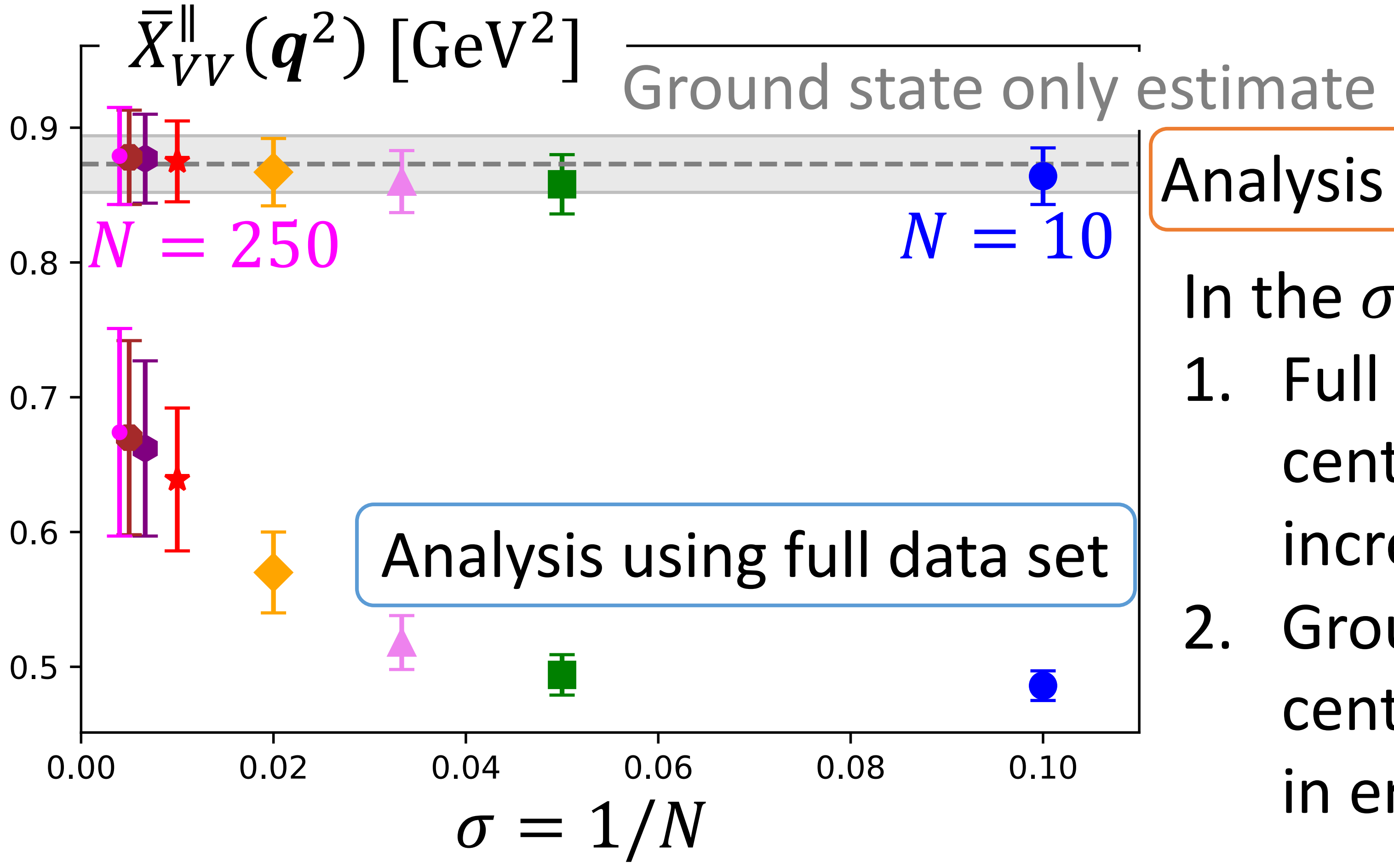
$$\sqrt{\mathrm{var}\left(\sum_{j=N_{\mathrm{cut}}}^{N} \tilde{c}_j^{(l)} \tilde{T}_j\right)}$$

Analytically known

Random variable taken from uniform distribution in $[-1; 1]$

Systematic error - Approximation

Application for $\bar{X}_{VV}^{\parallel}(\boldsymbol{q}^2)$ for $\boldsymbol{q} = (1,1,1)$



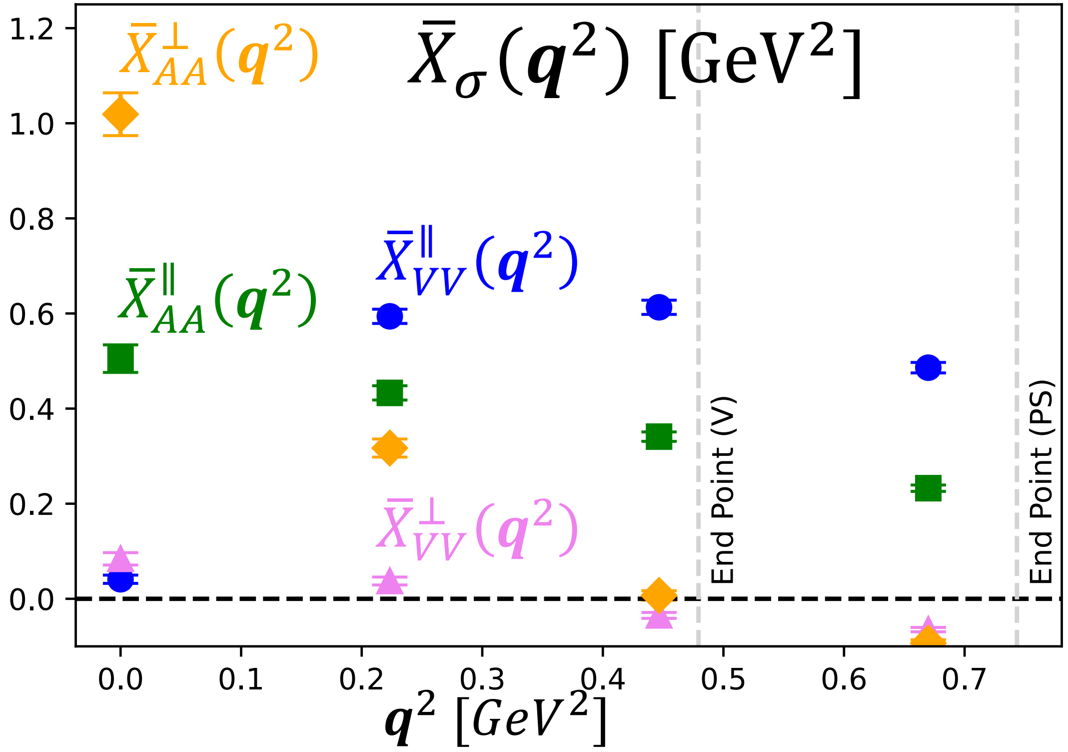
Analysis assuming ground state exact

In the $\sigma \to 0$ limit:

1. Full data set: Major shifts in central values + substantial increase in errors
2. Ground state exact: stable central values + small increase in errors

Systematic error - Finite volume

$N = 10, \sigma = 0.1$



Infinite volume limit? [2312.16442]

- In finite volume spectral density is a sum of delta peaks

Computing $\bar{X}_{\sigma}(\boldsymbol{q}^2)$ requires ordered limits

$$\lim_{\sigma \to 0} \lim_{V \to \infty} \bar{X}_{\sigma}(\boldsymbol{q}^2)$$

Necessary data not available

➡ Estimate finite-volume effects using a model (non-interacting two-body states)

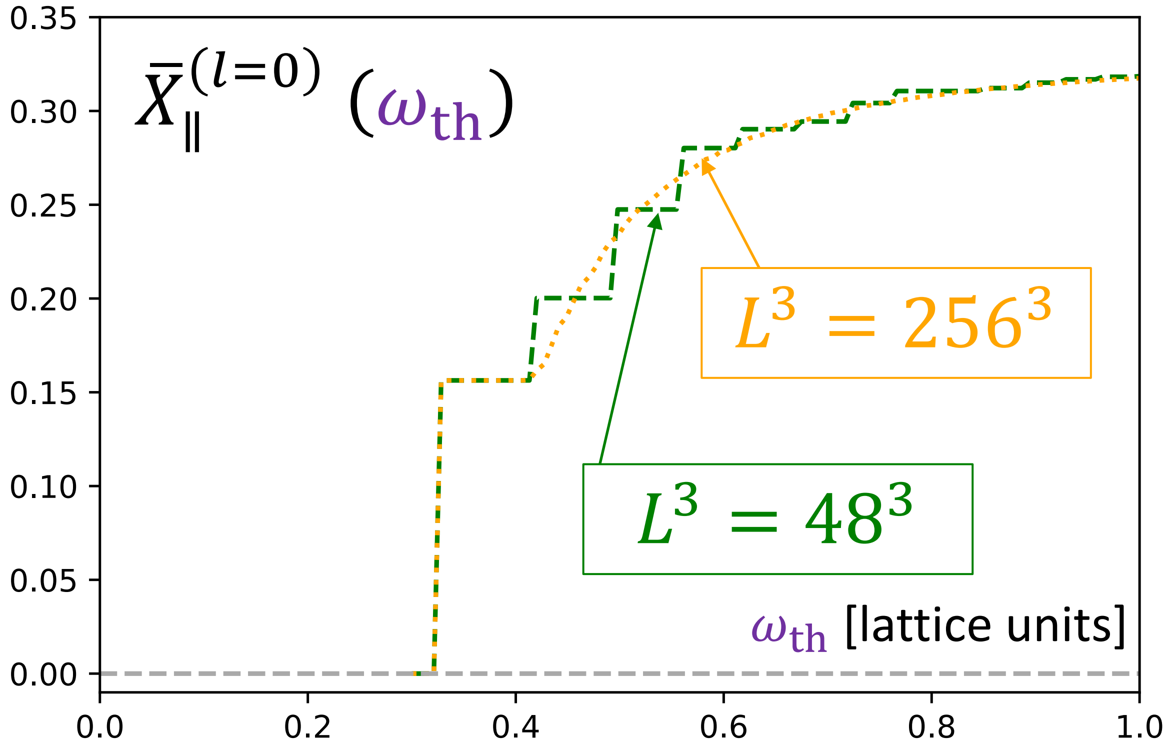
Finite volume – Model analysis

$\bar{X}^{\parallel}_{AA}(\boldsymbol{q}^2)$ for $\boldsymbol{q} = (0,0,0)$

Model-based results

$$\bar{X}^{(l)}(\omega_{\mathrm{th}}) \sim \int_{\omega_0}^{\infty} d\omega\, \rho(\omega) k_{\sigma}^{(l)}(\boldsymbol{q},\omega)\theta(\omega_{\mathrm{th}} - \omega)$$

Test by (artificially) varying the upper limit of the integral



- Heaviside function
  - Slight volume dependence

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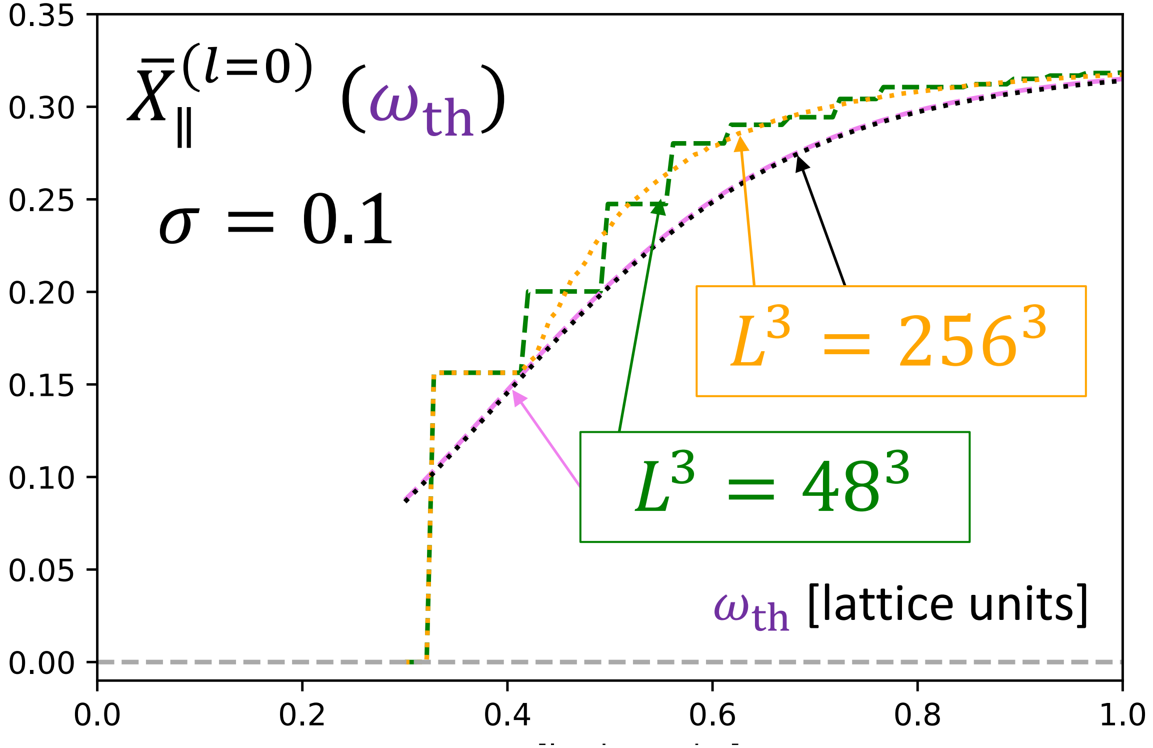

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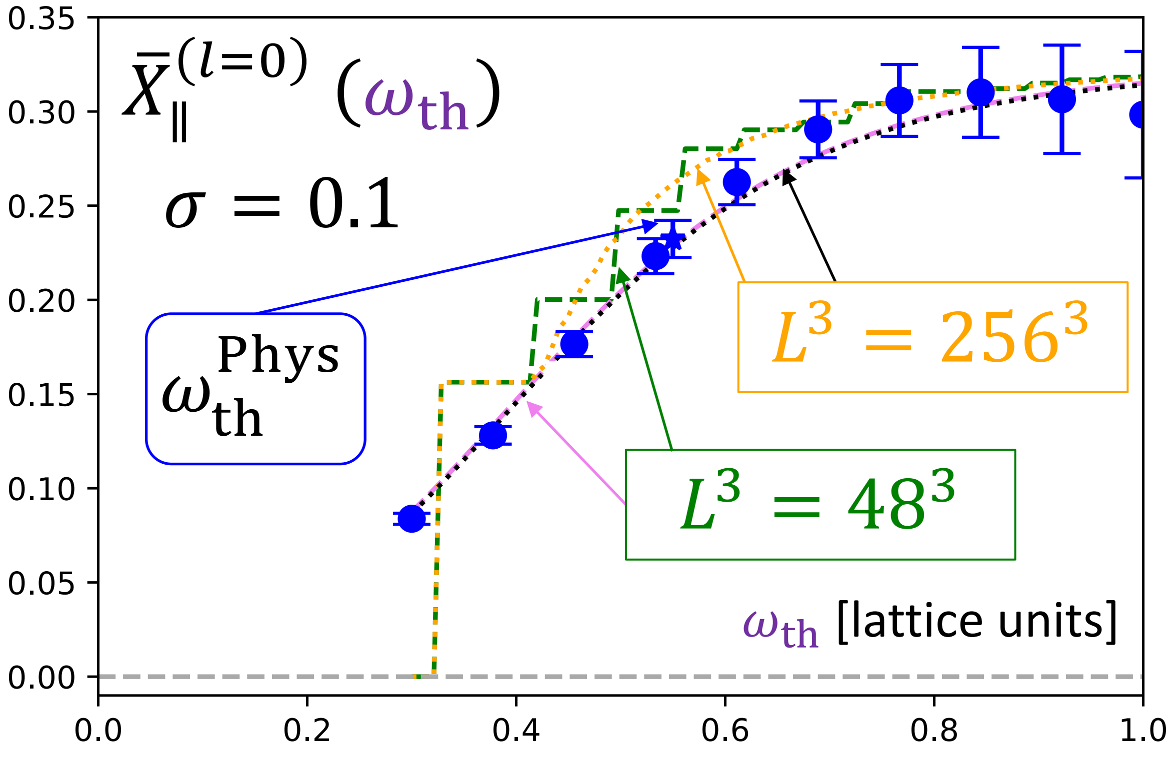

- Heaviside function
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\+ apply smearing
  - ➢ Volume dependence washes out

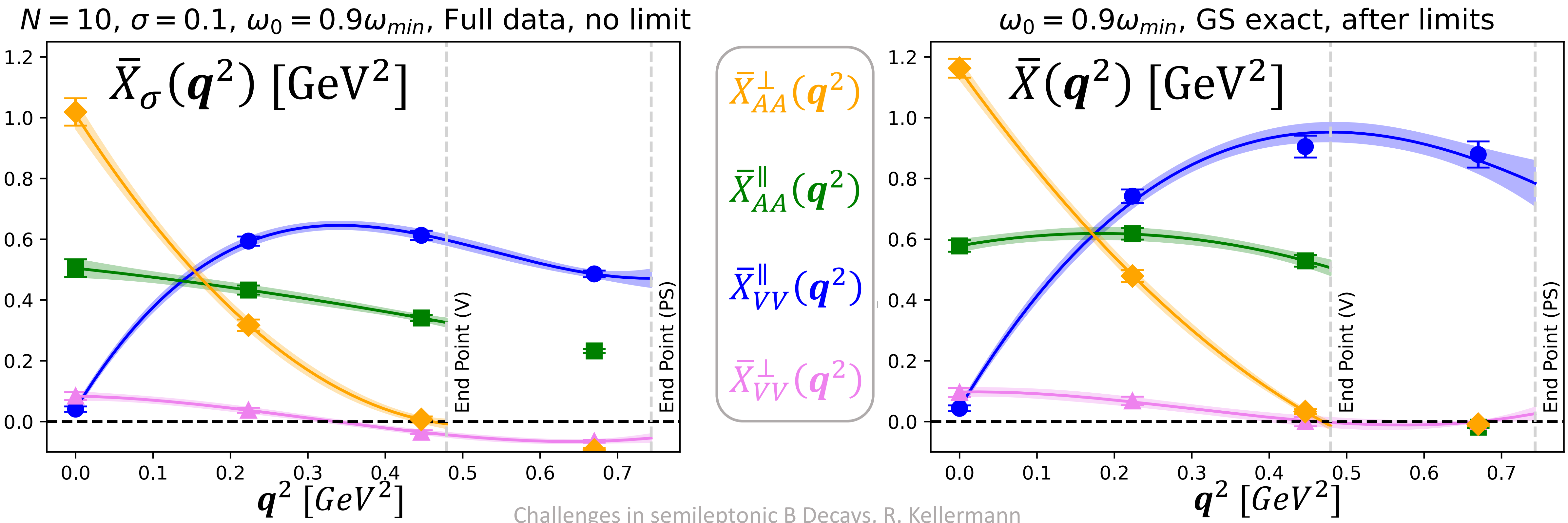
\+ include lattice data
  - ➢ Nicely follows model prediction

Estimating the systematic corrections

Channels:

1. AA: infinite-volume limit
2. VV: finite-volume corrections expected small; only $\sigma \to 0$ limit

\+ subtr. Ground state from correlator and assume as exact



Future Prospects

Extension: Moments [in collaboration with Matteo Fael]

Consider other observables; q^2 kinematical moments

$$Q_n(q^2_{\mathrm{cut}}) = \int_{q^2_{\mathrm{cut}}}^{q^2_{\mathrm{max}}} (q^2)^n \left[\frac{d\Gamma}{dq^2 dq_0 dE_\ell}\right] dq^2 dq_0 dE_\ell$$

Or: centralized moments $q_n(q^2_{\mathrm{cut}})$ of differential distributions

- Higher sensitivity to power corrections
- Independent of CKM elements

$$q_1(q^2_{\mathrm{cut}}) = \langle q^2 \rangle_{q^2 \geq q^2_{\mathrm{cut}}}, \qquad n = 1$$

$$q_n(q^2_{\mathrm{cut}}) = \langle (q^2 - \langle q^2 \rangle)^n \rangle_{q^2 \geq q^2_{\mathrm{cut}}}, \qquad n \geq 2$$

$$\langle (q^2)^n \rangle_{q^2 \geq q^2_{\mathrm{cut}}} = \frac{Q_n}{Q_0}$$

Moments – Lattice and Continuum

Adjust analysis of the decay rate

$$Q_n(q^2_{\text{cut}}) = \int_{\boldsymbol{q}^2_{\text{cut}}}^{\boldsymbol{q}^2_{\text{max}}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}_{Q_n}(\boldsymbol{q}^2) \qquad \bar{X}_{Q_n}(\boldsymbol{q}^2) = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega\, k_{Q_n,\mu\nu} \times W^{\mu\nu}$$

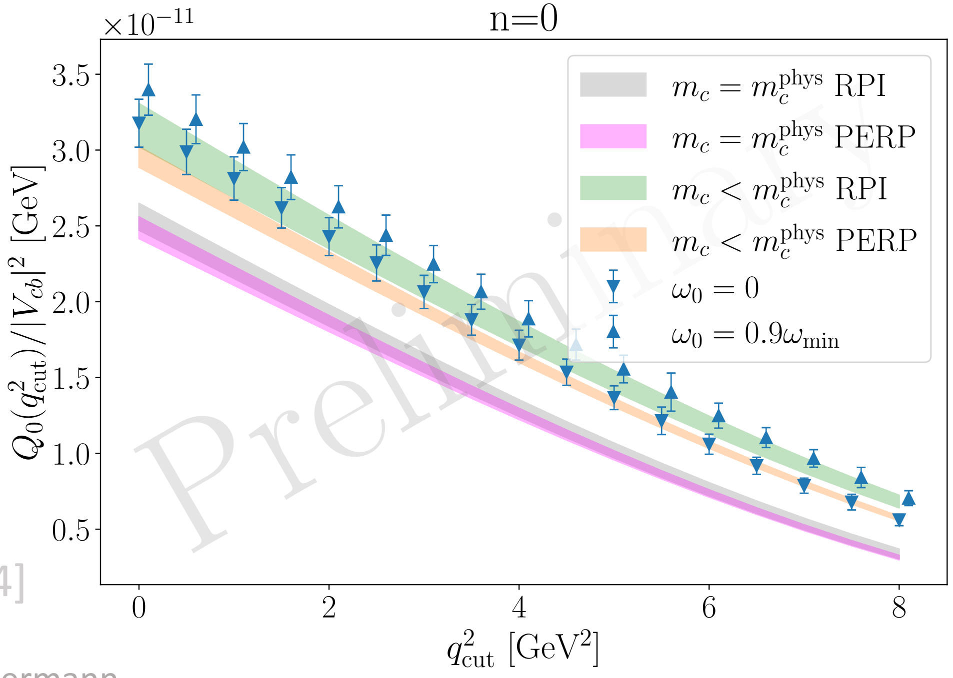
Rescale charm mass in continumm prediction to match lattice data

HQET relations between heavy mesons and quarks

$$M_{D_s} = m_c + \bar{\Lambda} + \frac{\mu_\pi^2 - d_H/2\mu_G^2}{2m_c} + \mathcal{O}\left(\frac{1}{m_c^2}\right)$$

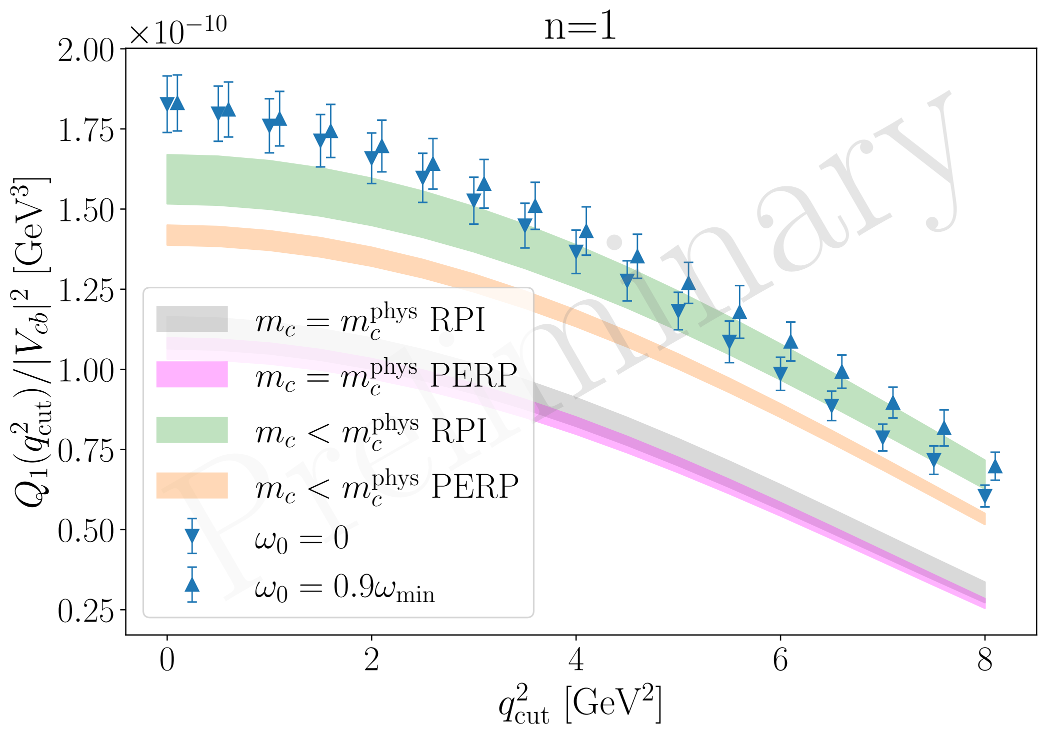
HQET Parameters

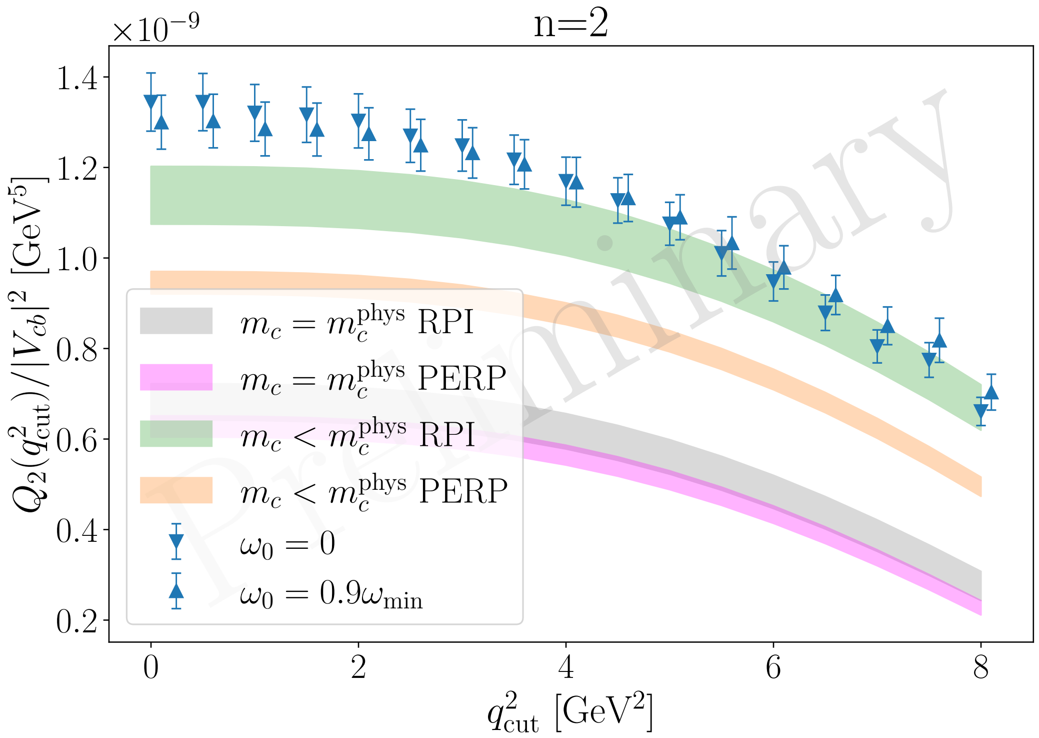
- RPI [Bernlocher et al., 2205.10274]
- PERP [Finauri & Gambino, 2310.20324]



Moments – Lattice and Continuum

Increasing disagreement for higher n between RPI/PERP and lattice

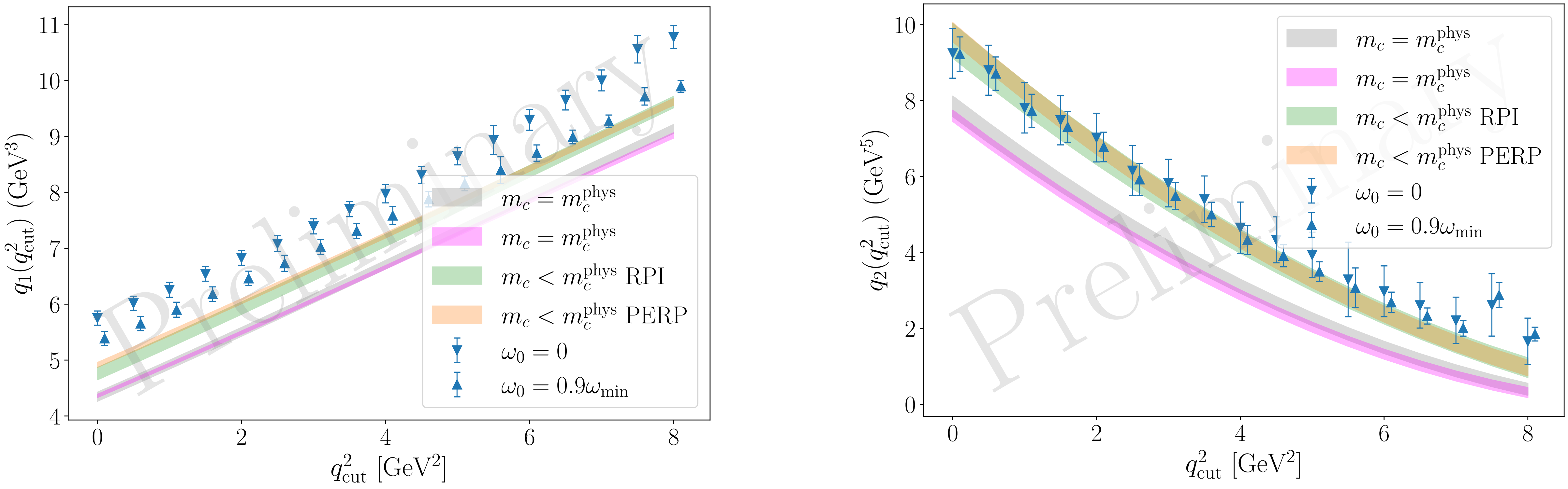




Note: better agreement is expected on the tails;
Small q^2_{cut} $\widehat{=}$ large $\boldsymbol{q}^2$ $\rightarrow$ larger cut-off effects

Centralized Moments – Lattice and Continuum

Feasibility study



After extrapolation to the physical world:

Lattice data can be used to extract HQET parameters for the OPE

Summary & Outlook

Summary

- Study into systematic effects in the inclusive analysis of semileptonic decays on the lattice
  - Error from Chebyshev polynomial approximation
    - Obtained a better estimate following the first idea
  - Finite volume corrections
    - Work out further details; supplement with data
- Publication in work (hopefully this year)

Outlook

- Discretization effects & continuum limit need to be addressed
- Extend towards a full analysis in the bottom sector
- Extend to different observables, e.g. moments
  - Increase pool for comparison to experiment and continuum theory predictions, e.g. OPE
- P-wave form factors from inclusive lattice simulation

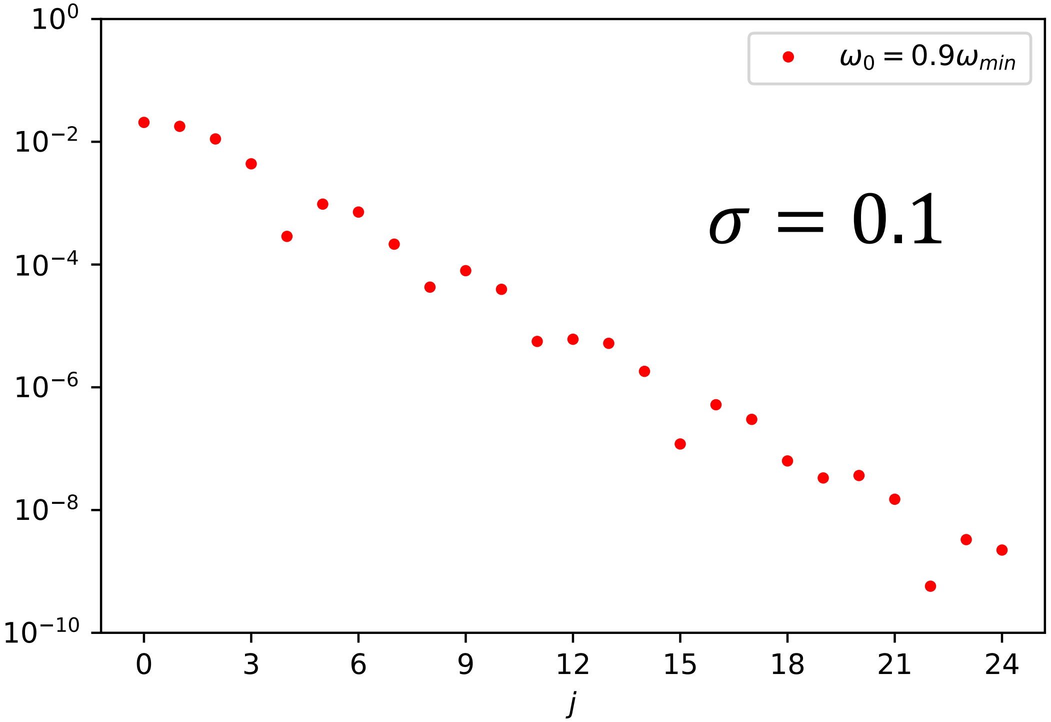
Back-up

Systematic errors - Approximation

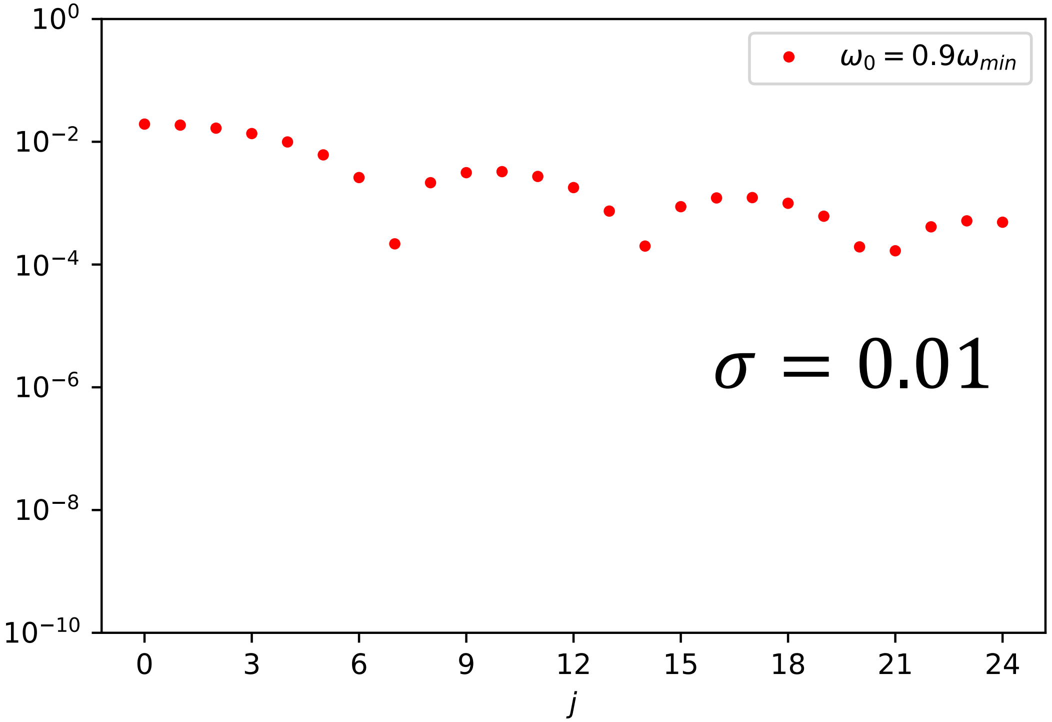
$\boldsymbol{q}^2 = 0.66\ \mathrm{GeV}^2$ $\omega_0 = 0.9\omega_{\mathrm{min}},$

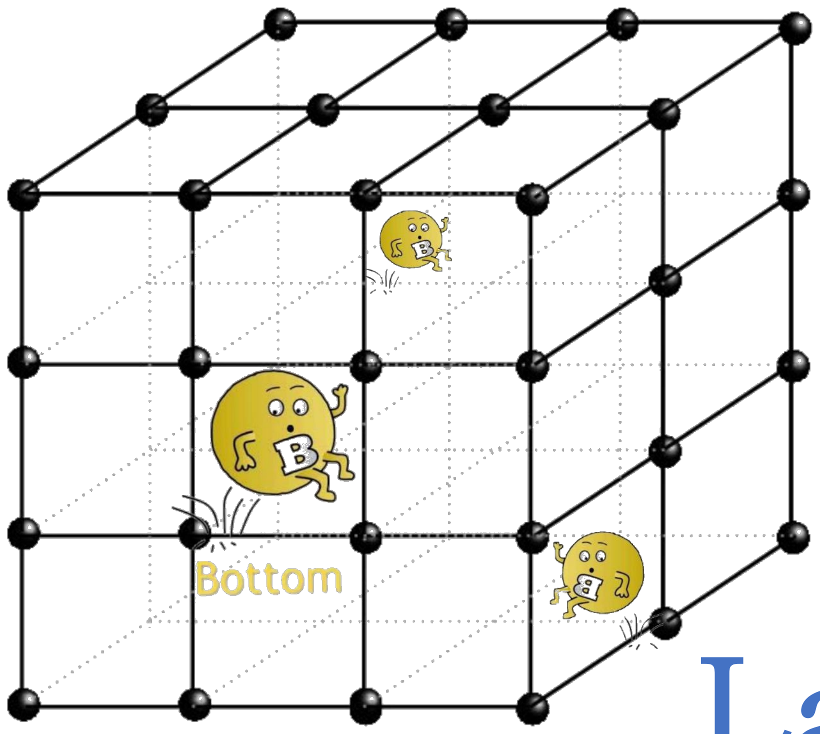
Coefficients for kernel with $l = 0$

$\left|\tilde{c}_{j,0.1}^{(0)}\right|$



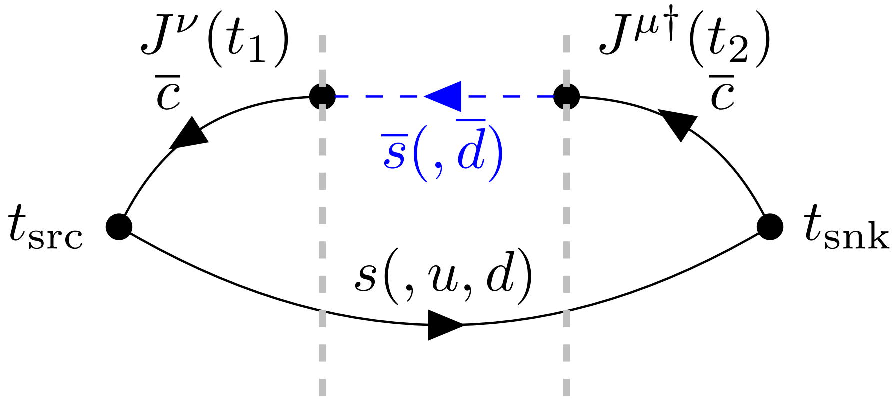
$\left|\tilde{c}_{j,0.01}^{(0)}\right|$





Lattice Setup

Simulations conducted on Fugaku using Grid [P. Boyle et al., https://github.com/paboyle/Grid] and Hadrons [A. Portelli et al., https://github.com/aportelli/Hadrons] software packages



Lattice setup:

- Lattice size: $48^3 \times 96$
- Lattice Spacing: $a = 0.055$ fm
- $M_\pi \simeq 300$ MeV

Simulation:

- 2+1 Möbius domain-wall fermions
- s, c quarks simulated at near-physical values
- Cover whole kinematical region $\boldsymbol{q} = (0,0,0) \rightarrow (1,1,1)$