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Understanding parton evolution in matter from renormalization group analysis

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Largely based on the following paper 2301.11940 [hep-ph]





Outline of the talk

- Introduction and motivation
- Renormalization group analysis of parton shower evolution in matter. Application to DIS





i) Thanks to organizers for the opportunity to give this talk

ii) Credit for the work presented goes to my collaborators – for this work W. Ke

R. Abdul-Khalek et al. (2021)



EIC Yellow Report



Radiative corrections in matter



Parton showers in the vacuum and in nuclear matter





Our goal is to achieve an accurate, systematically improvable description of hadron, heavy flavor and jet production

In the description of high energy processes significant effort has been devoted to logs, legs and loops

- Log ratios of mass and energy scales, phase space, cuts. Goal is to resum
- Legs the formation of parton shower, branchings, evolution
- Loops virtual corrections. Goal is to include, find automated way to do some of the loops

Similar challenges exist in heavy-ion physics and similar theoretical approaches can be adapted to reactions with nuclei

EFTs for parton showers in matter





Double Born

- Evaluated using EFT approaches - SCET_G, SCET_{M,G}
- Cross checked using light cone wavefunction approach
- Factorize from the hard part
- Gauge invariant
- Contain non-local quantum coherence effects (LPM)
- Depend on the properties of the nuclear medium

G. Ovanesyan et al. (2011)

Compute analogues of the Altarelli-Parisi splitting functions Enter higher order and resumed calculations

$$egin{aligned} m{A}_{\perp} = m{k}_{\perp}, \ m{B}_{\perp} = m{k}_{\perp} + x m{q}_{\perp}, \ m{C}_{\perp} = m{k}_{\perp} - (1-x) m{q}_{\perp}, \ m{D}_{\perp} = m{k}_{\perp} - m{L}_{\perp} = m{L}_{\perp} - m{L}_{\perp}, \ m{L}_{\perp} = m{L}_{\perp} = m{L}_{\perp} + m{L}_{\perp}, \ m{L}_{\perp} = m{L}_{\perp} - m{L}_{\perp} = m{L}_{\perp} = m{L}_{\perp} - m{L}_{\perp} = m{L}_{\perp} = m{L}_{\perp} - m{L}_{\perp} = m{L}_{\perp}$$

Kinematic variables and mass dependence

 $\begin{aligned}
\nu &= m & (g \to Q\bar{Q}), \\
\nu &= xm & (Q \to Qg), \\
\nu &= (1-x)m & (Q \to gQ),
\end{aligned}$

Quark to quark splitting function example

$$\begin{split} & \left(\frac{dN^{\text{med}}}{dxd^{2}k_{\perp}}\right)_{Q \to Qg} = \frac{\alpha_{s}}{2\pi^{2}}C_{F}\int \frac{d\Delta z}{\lambda_{g}(z)}\int d^{2}q_{\perp}\frac{1}{\sigma_{el}}\frac{d\sigma_{el}^{\text{med}}}{d^{2}q_{\perp}}\left\{\left(\frac{1+(1-x)^{2}}{x}\right)\left[\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}\right.\right.\\ & \left.\times\left(\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}-\frac{C_{\perp}}{C_{\perp}^{2}+\nu^{2}}\right)\left(1-\cos[(\Omega_{1}-\Omega_{2})\Delta z]\right)+\frac{C_{\perp}}{C_{\perp}^{2}+\nu^{2}}\cdot\left(2\frac{C_{\perp}}{C_{\perp}^{2}+\nu^{2}}-\frac{A_{\perp}}{A_{\perp}^{2}+\nu^{2}}\right)\\ & \left.-\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}\right)\left(1-\cos[(\Omega_{1}-\Omega_{3})\Delta z]\right)+\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}\cdot\frac{C_{\perp}}{C_{\perp}^{2}+\nu^{2}}\left(1-\cos[(\Omega_{2}-\Omega_{3})\Delta z]\right)\\ & \left.+\frac{A_{\perp}}{A_{\perp}^{2}+\nu^{2}}\cdot\left(\frac{D_{\perp}}{D_{\perp}^{2}+\nu^{2}}-\frac{A_{\perp}}{A_{\perp}^{2}+\nu^{2}}\right)\left(1-\cos[\Omega_{4}\Delta z]\right)-\frac{A_{\perp}}{A_{\perp}^{2}+\nu^{2}}\cdot\frac{D_{\perp}}{D_{\perp}^{2}+\nu^{2}}\left(1-\cos[\Omega_{5}\Delta z]\right)\\ & \left.+\frac{1}{N_{c}^{2}}\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}\cdot\left(\frac{A_{\perp}}{A_{\perp}^{2}+\nu^{2}}-\frac{B_{\perp}}{B_{\perp}^{2}+\nu^{2}}\right)\left(1-\cos[(\Omega_{1}-\Omega_{2})\Delta z]\right)\right]\\ & \left.+x^{3}m^{2}\left[\frac{1}{B_{\perp}^{2}+\nu^{2}}\cdot\left(\frac{1}{B_{\perp}^{2}+\nu^{2}}-\frac{1}{C_{\perp}^{2}+\nu^{2}}\right)\left(1-\cos[(\Omega_{1}-\Omega_{2})\Delta z]\right)+\ldots\right]\right\} \end{split}$$

Z. Kang et al. (2016)

M. Sievert et al. (2019)

Properties of in-medium showers

Longitudinal (x) distribution



- Enhancement of wide-angle radiation, implications for reconstructed jets and jet substructure
- Limited to specific kinematic regions
- Medium-induced scaling violations, new contributions to the jet function

- In-medium parton showers are softer and broader than the ones in the vacuum
- There is even more matter-induced soft gluon emission enhancement

Angular (k_{τ}) distribution – relative to vacuum



B. Yoon et al. (2019)

Same behavior in cold nuclear matter

In-medium evolution of fragmentation functions

Medium-induced splitting functions provide correction to vacuum showers and correspondingly modification to DGLAP evolution for FFs

$$\begin{aligned} \frac{\mathrm{d}D_q(z,Q)}{\mathrm{d}\ln Q} &= \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{\mathrm{d}z'}{z'} \left\{ P_{q \to qg}(z',Q) D_q\left(\frac{z}{z'},Q\right) + P_{q \to gq}(z',Q) D_g\left(\frac{z}{z'},Q\right) \right\}, \\ \frac{\mathrm{d}D_{\bar{q}}(z,Q)}{\mathrm{d}\ln Q} &= \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{\mathrm{d}z'}{z'} \left\{ P_{q \to qg}(z',Q) D_{\bar{q}}\left(\frac{z}{z'},Q\right) + P_{q \to gq}(z',Q) D_g\left(\frac{z}{z'},Q\right) \right\}, \\ \frac{\mathrm{d}D_g(z,Q)}{\mathrm{d}\ln Q} &= \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{\mathrm{d}z'}{z'} \left\{ P_{g \to gg}(z',Q) D_g\left(\frac{z}{z'},Q\right) + P_{\bar{q} \to gq}(z',Q) D_g\left(\frac{z}{z'},Q\right) \right\}, \\ + P_{g \to q\bar{q}}(z',Q) \left(D_q\left(\frac{z}{z'},Q\right) + f_{\bar{q}}\left(\frac{z}{z'},Q\right) \right) \right\}. \end{aligned}$$

N. Chang et al. (2014) Hybrid approach Z. Kang et al. (2014)

Y-T. Chien et al. (2015)





Predictions vs charged hadrons at in central Pb+Pb at the LHC

- Fully numerical implementation, including the determination of virtual corrections
- Phenomenologically successful, e.g. best predictions for hadron suppression in heavy ion collisions at the LHC

New RG approach in evolution in matter



The renormalization group

The theory of how to connect physics at different scales. Applicable to "a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects."

Origins can be traced to:

Ideas of scale transformations in QED M. Gell-Man, I. Low (1954) Handling infinities in field theories

R. Feynman, J. Schwinger, S. Tomonaga, Nobel Prize (1965)

- Hydrodynamics
- Social networks
- Low energy nuclear physics
- Small-x physics

- V. Yakhot et al. (1986)
- M. Newman et al. (1999)
- S. Bogner et al. (2007)
- J. Jalilian-Marian et al. (1998)

In particle and nuclear physics – a way of making sense of inherently divergent theories



K. Willson (1982) Nobel Prize speech

Scales in the in-medium parton shower problem

In-medium DGLAP does not tell us what kind of large logs are being resummed



We encounter many ratios of scales in DIS on nuclei. Will resum large logarithms of Q/Qo and E/ξ^2L

Let's revisit the calculation of semiinclusive hadron production

Consider differential hadron production in ep and eA



W. Ke et al. (2023)

Rather than evolving the fragmentation functions, we will evolve the parton shower / distribution of partons inside the shower

The invariant distribution of parton j in a shower initiated by i depends on 2 scales μ_1 and μ_2

- Evolution μ_1 in leads to standard vacuum DGLAP
- The bare F_{ij} needs to be renormalized by a medium term that only depends on μ_2 . At one loop determined to cancel the poles in the medium bare part

$$F_{ij}(z,\mu_1^2,\mu_2^2) \to F_{ik}(y,\mu_1^2,\mu_2^2) \otimes M_{kj}\left(\frac{z}{y},\mu_2^2\right) + \mathcal{F}(z). \quad M_{kj} = M_{kj}^{(0)} + M_{kj}^{(1)} + \cdots$$

Technical aspect one: the splitting functions

In cold nuclear matter (uniform density) we can analytically integrate over the path length. We can can significantly simplify the propagator and phase structure that arises form in-medium interactions

Up to color and kinematic factors, the splitting functions have the same universal form

$$\begin{split} & \text{Up to color and kinematic factors, the splitting} \\ & \text{functions have the same universal form} \\ & P_{ij}^{(1)}(x, E, \mu_2^2) = \frac{\alpha_s^{(0)} P_{ij}(x)}{2\pi^2} L \int \frac{\mu_2^{2\epsilon} d^{2-2\epsilon} \mathbf{k}}{(2\pi)^{-2\epsilon}} \frac{\Phi\left[\frac{\mathbf{k}^2 L}{2x(1-x)E}\right]}{\mathbf{k}^2} \\ & \times \sum_n \int \frac{\mu_2^{2\epsilon} d^{2-2\epsilon} \mathbf{q}}{(2\pi)^{-2\epsilon}} \frac{\rho_G \alpha_s^{(0)} C_n^{ij} \Delta_n^{ij}(x)}{\pi(\mathbf{q}^2 + \xi^2)^2} \frac{\mathbf{q} \cdot [\mathbf{k} + \Delta_n^{ij}(x)\mathbf{q}]}{[\mathbf{k} + \Delta_n^{ij}(x)\mathbf{q}]^2} \end{split}$$

The remaining integration over the momentum exchanges with the can be performed using dim. reg. and by expanding the integrand

Final result Slowly varying functions O(one/few) $P_{ij}^{(1)}(x,E,\mu_2^2) = \frac{\alpha_s^2(\mu_2^2)\rho_G L}{8E/L} \frac{P_{ij}(x)}{[x(1-x)]^{1+2\epsilon}} \left[\frac{\mu_2^2 L}{\chi(w)E}\right]^{2\epsilon} \int_0^w du \frac{4}{\pi} \frac{\Phi(u)}{u^{2+2\epsilon}} = B(w)[\chi(w)/2]^{-2\epsilon} + \mathcal{O}(\epsilon^2)$ $\times B(w) \sum_{n} C_n^{ij} [\Delta_n^{ij}(x)]^{2-2\epsilon} (1 + \mathcal{O}(\epsilon^2)) (1 + \mathcal{O}(v))$ $B(w) = \frac{4}{\pi} \int_{0}^{w} \Phi(x) \frac{dx}{x^{2}}, \quad \chi(w) = 2 \exp\left\{\frac{1}{B(w)} \frac{4}{\pi} \int_{0}^{w} \Phi(x) \ln(x) \frac{dx}{x^{2}}\right\}$

One important part here is the additional 1/x(1-x) divergence at the endpoints of the splitting function

Technical aspect two: the subtraction of divergences

Take the flavor non-singlet distribution for simplicity

$$\Delta F_{\rm NS}^{\rm med}(z) = \int_{z}^{1} \frac{dx}{x} F_{\rm NS}(\frac{z}{x}) P_{qq}^{\rm med(1)}(x) + \text{ virtual term.}$$
$$P_{qq}^{\rm med(1)}(x) = A(\alpha_{s}, \cdots) \cdot \frac{P_{qq}^{\rm vac(0)}(x)}{[x(1-x)]^{1+2\epsilon}} \cdot \left[\frac{\mu^{2}L}{\chi z\nu}\right]^{2\epsilon} \cdot C_{n}\Delta_{n}(x)$$

Define a generalized + prescription and a subtracted function so that the integral with endpoint divergences is finite

$$\begin{split} &\int_{0}^{1} \frac{G(x)}{x^{1+2\epsilon}(1-x)^{2+2\epsilon}} dx = \int_{0}^{1} \frac{\{G(x)\}_{qq}}{x(1-x)^{2}} dx & \{G(x)\}_{qq} = G(x) - (1-x)^{2} G(0) \\ &- \frac{G(0)}{2\epsilon} + \frac{G'(1)}{2\epsilon} - G(1) \left(\frac{1}{2\epsilon} + 2\right) + \mathcal{O}(\epsilon). & -x(2-x)G(1) - x(x-1)G'(1). \end{split}$$

The large medium induced logarithms that need to be resummed

The $1/\epsilon$ divergence and $M^{(1)}$ counter term that is determined to cancel it. It arises from the soft-collinear sector

$$\begin{split} \Delta F_{\rm NS} &= \frac{\alpha_s^2 B(w) \rho_G L}{8\nu/L} \left(\frac{1}{2\epsilon} + \ln \frac{\mu_2^2 L}{\chi z \nu} \right) 2C_F \left(\frac{2C_A + C_F}{z} - 2C_A \frac{d}{dz} \right) F_{\rm NS}(z) + \int_0^1 \frac{dy}{y} F_{\rm NS}(y) M_{qq}^{(1)} \left(\frac{z}{y}, \mu_2, z\nu \right) \\ &+ \frac{\alpha_s^2 B(w) \rho_G L}{8\nu/L} C_F \left[\int_0^1 \frac{\left\{ \sum_n C_n^{qq} [\Delta_n^{qq}(x)]^2 (1 + x^2) \left[\frac{x}{z} F_{\rm NS} \left(\frac{z}{x} \right) - \frac{F_{\rm NS}(z)}{z} \right] \right\}_{qq} dx + \frac{(4C_A - C_F) F_{\rm NS}(z)}{z} \right] . \end{split}$$

Fixed order contribution - free of divergences, no large log enhancement

Emergent analytic understanding of the in-medium shower

 Derived a full set of RG evolution equations. The NS distribution has a very elegant traveling wave solution

Suitable change of variables. Also captures the density, path length and energy dependence

$$\tau(\mu^2) = \frac{\rho_G L^2}{\nu} \frac{\pi B}{2\beta_0} \left[\alpha_s(\mu^2) - \alpha_s\left(\chi \frac{z\nu}{L}\right) \right]$$

Flavor non-singlet (NS = q-qbar)

$$\frac{\partial F_{\rm NS}(\tau, z)}{\partial \tau} = \left(4C_F C_A \frac{\partial}{\partial z} - \frac{4C_F C_A + 2C_F^2}{z}\right) F_{\rm NS}$$

Flavor singlet (f = q+qbar, g)

$$\frac{\partial F_f}{\partial \tau} = \left(4C_F C_A \frac{\partial}{\partial z} - \frac{4C_F C_A + 2C_F^2}{z}\right) F_f + 2C_F T_F \frac{F_g}{z},$$

$$\frac{\partial F_g}{\partial \tau} = \left(4C_A^2 \frac{\partial}{\partial z} - \frac{2N_f C_F}{z}\right) F_g + 2C_F^2 \sum_f \frac{F_f}{z}.$$

 $F_{\rm NS}(\tau, z) = \frac{F_{\rm NS}(0, z + 4C_F C_A \tau)}{(1 + 4C_F C_A \tau/z)^{1 + C_F/(2C_A)}}$



Can directly identify parton energy loss, the nuclear size dependence of the modification, etc

Phenomenological applications of the new RG analysis to HERMES

Revisiting the HERMES data



Observable chosen to eliminate initialstate effects



- RG evolution gives a good description of the data at small to intermediate z_h .
- Fixed order corrections improve the agreement at large z_h

Zh

RG evolution advantages

W. Ke et al. (2023)

- The method is systematically improvable both higher logarithmic accuracy and fixed order terms, if higher order splitting functions are available
- Numerically, it is much faster to implement and solve in comparison to in-• medium DGLAP evolution
- The proper in-medium scale separation increases predictive power
- At the level of cross sections one can identify the effects of "energy loss"

Demonstration of predictive power

Addressing EMC data

- EMC measurement for C, Cu, and Sn nuclei at similar x_B much higher Q² ~ 11 GeV²
- Same effective Glauber
 gluon density used

Predictions for the EIC

- The modifications to hadronization at EIC depends on kinematics x_B,Q²
- At large x_B and (forward rapidities) the modification can be very significant

W. Ke et al. (2023)



Fixed order (FO) + RG evolution compared to EMC data



Be careful about EIC matter effects

Not the topic of this talk at all – triggered by some discussion in the morning

 The name of the game is to go as far forward as possible (Y) and lower CM energy, subject to being able go measure the channel. It is an optimization problem

Constrained calculations are done for:

- light and heavy hadrons: 2007.10994
- Light/inclusive jets: 2010.05912
- Heavy flavor jets: 2108.07809

- Centrality dependence of hadron and jet dependence 2303.14201

Jets and hadrons – factors of 2-3, centrality can enhance effects, as well as differential measurements. Substructure effects generically hover around 10% even under in favorable kimematics



Analytic comparison of RG and in-medium DGLAP

We can use the insight from the RG evolution and revisit the in-medium DGLAP approach

For simplicity: fixed coupling, focus on the soft gluon emission region

• We can show that mDGLAP also resums matter-induced logarithms of the type $\ln[E/(L\mu_D^2)]$

A similar traveling wave solution

$$F_{NS}^{+}(z) = \frac{F_{NS}^{-}(z + 4C_F C_A \tau_{\text{fix}})}{1 + 4C_F C_A \tau_{\text{fix}}/z},$$

$$\tau_{\text{fix}} = A_0 \ln \frac{2\pi E}{\mu_D^2 L}$$

Let's write mDGLAP for the flavor non-singlet distribution

$$egin{aligned} &rac{\partial F_{
m NS}(z)}{\partial \ln \mu^2} = \int_0^1 \mathbf{k}^2 rac{d[P_{qq}(x,\mathbf{k}^2)+P_{qq}^{(1)}(x,\mathbf{k}^2)]}{dxd\mathbf{k}^2} \ & imes \left[F_{
m NS}\left(rac{z}{x}
ight)-F_{
m NS}(z)
ight]dx\,, \end{aligned}$$

$$\begin{aligned} \frac{\partial F_{\rm NS}}{\partial \ln \mu^2} &= 4C_F C_A A_0 \int_0^{1-\frac{\mu_D^2}{\mu^2}} \frac{4}{\pi} \frac{\Phi(u)}{u} \frac{\frac{x}{z} F_{\rm NS}(\frac{z}{x}) - \frac{F_{\rm NS}(z)}{z}}{(1-x)^2} dx \\ &\approx \frac{4}{\pi} \frac{\Phi(u)}{u} 4C_F C_A A_0 \left[\frac{\partial F_{\rm NS}}{\partial z} - \frac{F_{\rm NS}}{z} \right] \ln \frac{\mu^2}{\mu_D^2} \\ &\approx \delta \left(\mu^2 - \frac{2\pi E}{L} \right) 4C_F C_A A_0 \left[\frac{\partial F_{\rm NS}}{\partial z} - \frac{F_{\rm NS}}{z} \right] \ln \frac{\mu^2}{\mu_D^2}, \end{aligned}$$

$$A(\mu_2^2, E, w_{\max}) = \alpha_s^2(\mu_2^2)L^2B(w_{\max})\rho_G/(8E)$$

W. Ke et al., (2023)

Conclusions

- Resummation of large nuclear matter induced logarithms are essential to interpret the experimental results in reactions with nuclei
- The traditional approach to accomplish this task is through fully numerical solution to in-medium DGLAP. It has been phenomenologically very successful
- Analytic insights, however, have thus far been absent. We derived RG evolution approach that overcomes this limitation. It is fast, efficient, improvable, and represents the first rigorous development in this direction in nearly a decade
- Successfully applied to phenomenology. Helped understand the physics contained in in-medium DGLAP



Differences between AA and eA

 AA and eA collisions are very different. Due to the LPM effect the "energy loss" decreases rapidly. The kinematics to look for in-medium interactions / effects on hadronization very different



- Jets at any rapidity roughly in the co-moving plasma frame (Only~ transverse motion at any rapidity)
- Largest effects at midrapidity
- Higher C.M. energies correspond to larger plasma densities



- Jets are in the nuclear rest frame Longitudinal momentum matters
- Largest effects are at forward rapidities
- Smaller C.M. energies (larger only increase the rapidity gap)

Phenomenological results hadrons

 Differential hadronization cross sections normalized by the cross section for R=1 jet

$$R_{eA}^{h}(z) = \frac{\frac{N^{h}(p_{T},\eta,z)}{N^{\text{inc}}(p_{T},\eta)}\Big|_{eA}}{\frac{N^{h}(p_{T},\eta,z)}{N^{\text{inc}}(p_{T},\eta)}\Big|_{ep}}$$

- Modifications to hadronization grow form backward to forward rapidity
- Transition from enhancement to suppression for heavy flavor



Backward rapidity, large

Forward rapidity, small C.M. energy



Differential in p_T

Light pions



Heavy flavor



Differential in p_T and z

Radiative energy loss processes and jet quenching



RHIC (though not the first HI machine) has played a very important role in truly developing a new field – interaction of hard probes in matter. Motivated energy loss studies

 M. Gyulassy et al. (1993)
 M. Gyulassy et al. (2000)

 B. Zakharov (1995)
 X. Guo et al. (2001)

- R. Baier et al. (1997)
- X. Guo et al. (2001)

Very successful but with limitations: not systematically improvable, limited connection to established QCD techniques



Discovery of jet quenching

QCD in the medium remains a multiscale problem. As such, it is well suited to an EFT approach

A. Idilbi et al. (2008)

G. Ovanesyan et al. (2011)

Z. Kang et al. (2016)

Technical aspect one: the splitting functions

The first simplification for cold nuclear matter (uniform density) is to analytically integrate over the path length

$$P_{ij}^{(1)}(x) = \frac{\alpha_s}{2\pi^2} P_{ij}(x) L \int d^2 \mathbf{k} \sum_T \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{\rho_T}{d_A} \frac{4\pi \alpha_s C_T \times 4\pi \alpha_s}{(\mathbf{q}^2 + \xi^2)^2} W_{ij}(x, \mathbf{k}, \mathbf{q}, E/L)$$
$$\equiv \frac{\alpha_s}{2\pi^2} P_{ij}(x) L \int d^2 \mathbf{k} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{\rho_G \times 4\pi \alpha_s}{(\mathbf{q}^2 + \xi^2)^2} W_{ij}(x, \mathbf{k}, \mathbf{q}, E/L).$$

One can define an effective Glauber gluon density and obtain the transport coefficient in nuclear matter: consistent with mDGLAP

$$\rho_G \equiv \sum_T \rho_T \frac{4\pi \alpha_s^{\text{med}} C_T}{d_A} \,.$$
$$\hat{q}_F = \alpha_s(\xi^2) C_F \rho_G \int_0^{\nu_{\xi/2}} \frac{\mathbf{q}^2}{(\mathbf{q}^2 + \xi^2)^2} \frac{d^2 \mathbf{q}}{\pi} \approx 0.052 \text{ GeV}^2/\text{fm}$$

We have to simplify the W_{ii} , by keeping leading terms in the power counting of the EFT

One can define an effective Glauber gluon density and obtain the $P_{ij}^{(1)}(x, E, \mu_2^2) = \frac{\alpha_s^2(\mu_2^2)\rho_G L}{8E/L} \frac{P_{ij}(x)}{[x(1-x)]^{1+2\epsilon}} \left[\frac{\mu_2^2 L}{\chi(w)E}\right]^{2\epsilon} \text{ onsistent with mDGLAP}$ $\times B(w) \sum_n C_n^{ij} [\Delta_n^{ij}(x)]^{2-2\epsilon} (1 + \mathcal{O}(\epsilon^2))(1 + \mathcal{O}(v)) \,. \, (6)$