4D-TExS: A new 4D lattice-QCD equation of state with extended density coverage

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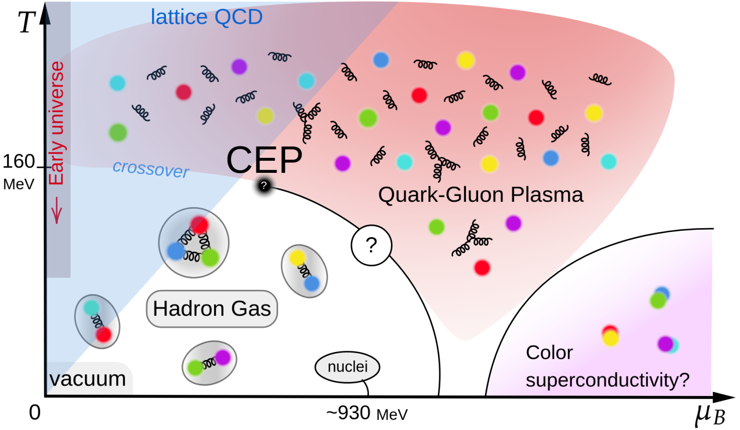
Phase diagram of nuclear matter

Understanding thermodynamics of nuclear matter, or how matter in atomic nuclei behaves under extreme conditions like in the early Universe ⇔ studying the phase diagram of nuclear matter.

- crossover between hadronic gas and QGP predicted by lattice QCD at low baryonic density ($T_c \sim 160$ MeV)

- 1[st] order phase transition
+ critical endpoint (CEP) predicted by by extrapolation from the chiral limit and several models (PNJL, fRG, holography...) → see Joaquin's talk

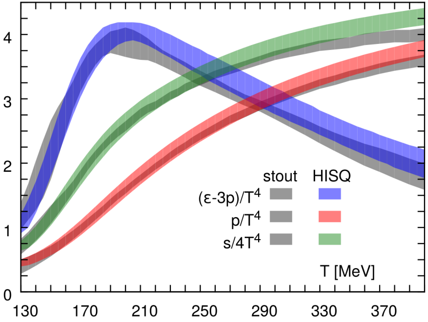
- colour superconductivity expected at low-T / high-μ_B



Determining the thermodynamics of nuclear matter

Most of the simulation tools used to model heavy-ion collisions (HIC), as well as neutron star (NS) mergers are based on microscopic transport and hydrodynamics models that **require** an equation of state (EoS).

Among the different ways to calculate the EoS of nuclear matter, lattice QCD is the most accurate way to get thermodynamics directly from QCD first principles.



Bazavov *et al.*, PRD 90 (2014), 094503

Starting from the grand-canonical partition function $\mathscr{Z}$, one can derive the following quantities:

- Pressure: $P = -T\dfrac{\partial \ln(\mathscr{Z})}{\partial V}$
- Entropy density: $s = \left(\dfrac{\partial P}{\partial T}\right)_{\mu_i}$
- Charge densities: $n_i = \left(\dfrac{\partial P}{\partial \mu_i}\right)_{T,\mu_{j\neq i}}$
- Energy density: $\varepsilon = sT - P + \sum_i \mu_i . n_i$

Computing at finite density from lattice QCD

However, simulations on the lattice suffer the so-called **fermion sign problem** which only allows to run simulations at $\mu_i = 0$, or at purely imaginary μ_i.

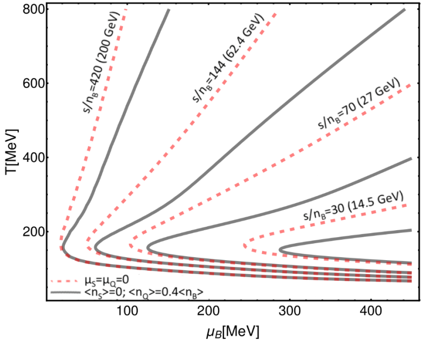
To reach finite density, one can expand using Taylor series:

$$\frac{P(T,\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BSQ}(T)\, \hat{\mu}_B^i\, \hat{\mu}_Q^j\, \hat{\mu}_S^k \qquad \left(\text{with } \hat{\mu}_i = \frac{\mu}{T}\right)$$

with Taylor coefficients χ_{ijk}^{BQS} (susceptibilities), obtained from pressure (for order $i+j+k = 2n$ with $n \in \mathbb{N}$) computed on the lattice at zero chemical potentials:

$$\chi_{ijk}^{BQS}(T) = \left.\frac{\partial^{i+j+k}(P/T^4)}{\partial\hat{\mu}_B^i\,\hat{\mu}_S^j\,\hat{\mu}_Q^k}\right|_{\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q=0}$$

4D-Taylor EoS built from continuum extrapolated diagonal + off-diagonal $\chi_{2/4}^{BQS}(T)$



Noronha-Hostler *et al.*, PRC 100 (2019) 6, 064910

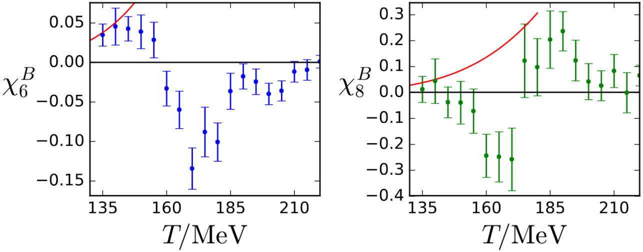
Limitations of the Taylor expansion

Recently, those coefficients have been obtained up to 6th and 8th orders for baryon number. [1]

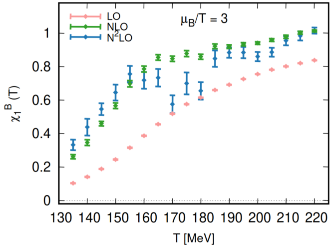
Although it allows to push the Taylor expansion to higher orders, notably in the μ_B plane:

- still limited to $\mu_i/T \lesssim 2.5$ for all conserved charges
- lack of convergence arises from large errors on high order terms which dominates at high $\hat{\mu}$
- expansion achieved at $T =$const, missing out the curved behaviour of pseudo-critical line

Borsányi *et al.*, JHEP 10 (2018) 205

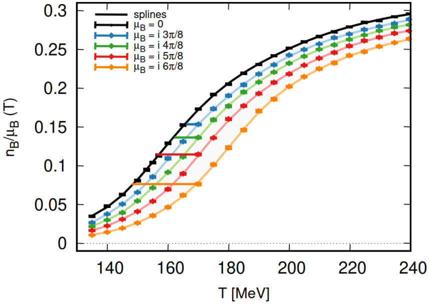


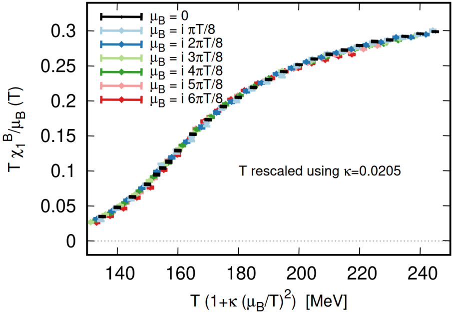
Borsányi *et al.*, PRL 126 (2021) 23, 232001



[1] D'Elia *et al.*, PRD 95, 094503 (2017) / Bazavov *et al.*, PRD 101, 074502 (2020) / Borsányi *et al.*, arXiv:2312.07528

A novel expansion scheme for lattice QCD EoS at finite μ_B





Simulations at $\mathrm{Im}(\hat{\mu}_B)$: T-dependence of normalised baryon density $\left(\chi_1^B = n_B/T^3\right)$ at finite $\hat{\mu}_B$ appears to be shifted from the value at $\hat{\mu}_B = 0$.

For the 0/0 limit, we have: $\frac{\chi_1^B(T,\hat{\mu}_B)\to 0}{\hat{\mu}_B \to 0} \to \frac{\partial \chi_1^B}{\partial \hat{\mu}_B} = \chi_2^B$

Borsányi *et al.*, PRL 126 (2021) 23, 232001

Main identity: $$\frac{\chi_1^B(T,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)$$

with $\mathbf{T'(T,\hat{\mu}_B) = T\left(1 + \kappa_2\,.\,\hat{\mu}_B^2 + \kappa_4\,.\,\hat{\mu}_B^4 + \dots\right)}$

captures the finite $\hat{\mu}_B$ dependence of the expansion

2D equation of state from T'-Expansion Scheme

Borsányi *et al.*, PRL 126 (2021) 23, 232001

New *TExS EoS* based on coefficients $\kappa_{2/4}^{BB}(T)$ evaluated directly from lattice QCD simulations at $\mu_B = 0$

$$T'(T,\mu_B) = T\left(1 + \kappa_2^{BB}(T)\hat{\mu}_B^2 + \kappa_4^{BB}(T)\hat{\mu}_B^4 \dots\right)$$

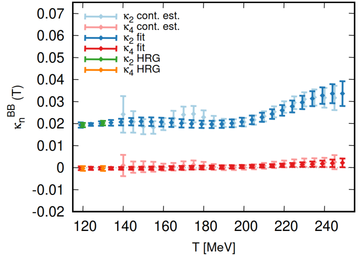
with coefficients $\kappa_i^{BB}(T)$ connected to Taylor coefficients $\chi_i^B(T)$:

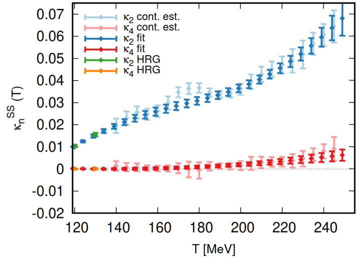
- $\kappa_2^{BB}(T,0) = \frac{1}{6T}\frac{\chi_4^B(T)}{\chi_2'^B(T)}$ with $\chi'(T) = \frac{\partial\chi(T)}{\partial T}$
- $\kappa_4^{BB}(T,0) = \frac{1}{360T \times \chi_2'^B(T)^3}\left(3\chi_2'^B(T)\times\chi_6^B(T) - 5\chi_2''^B(T)\times\chi_4^B(T)^2\right)$

⇒ Clear separation of scales between $\kappa_2^{BB}(T)$ and $\kappa_4(T)$

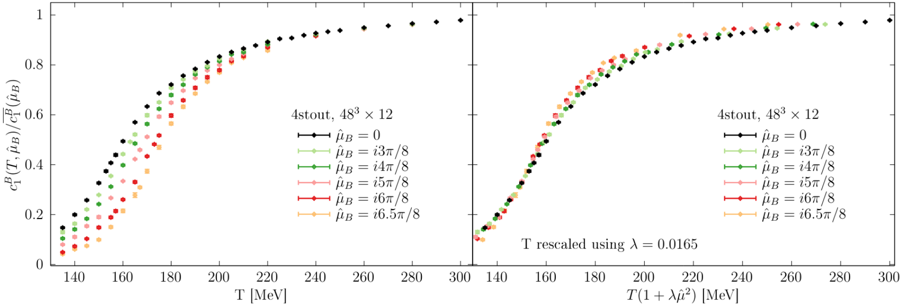
⇒ $\kappa_4(T)$ is almost 0 → faster convergence

⇒ $\kappa_{2/4}(T)$ has a smooth T-dependence





Applying Stefan-Boltzmann limit normalisation



To ensure that our main identity holds when $T \to \infty$, needs to normalise by Stefan-Boltzmann limits $\bar\chi_1^B(\hat\mu_B)$ and $\bar\chi_2^B(0)$:

$$\boxed{\frac{\chi_1^B(T,\hat\mu_B)}{\bar\chi_1^B(\hat\mu_B)} = \frac{\chi_2^B(T'(T,\hat\mu_B),0)}{\bar\chi_2^B(0)}}$$

This leads to redefine:

$$T'(T,\mu_B) = T\left(1+\lambda_2^{BB}(T)\hat\mu_B^2+\dots\right)$$

with the new expansion coef. embedding the S.B. limit:

$$\lambda_2^{BB}(T) = \frac{1}{6T\chi_2'^B(T)} \times \left(\chi_4^B(T) - \frac{\bar\chi_4^B(0)}{\bar\chi_2^B(0)}\chi_2^B(T)\right)$$

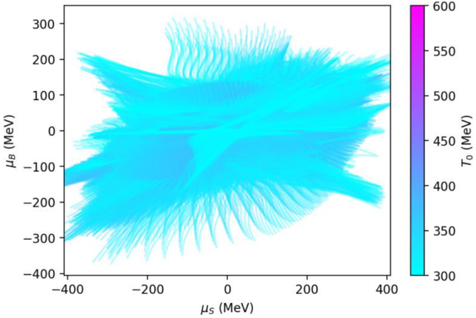
Borsányi *et al.*, PRD 105 (2022) 11, 114504

Why do we need a 4D EoS with extended coverage?

- **Hydrodynamics simulation for HIC becomes more accurate and realistic:**
  need to go beyond usual criteria of strangeness neutrality ($\langle n_S \rangle = 0$) and global charge conservation ($n_Q = 0.4 n_B$)

  $\rightarrow$ offer an EoS with 3 independent (μ_B, μ_Q, μ_S) which goes beyond the limit of Taylor ($\hat{\mu}_i \lesssim 2.5$) and is better suited for simulations at lower collision energies

Almaalol, talk at QM 2023



- **Entering a new era for astrophysics with the observation of NS mergers:**
  merger simulations also employs hydrodynamics which need an EoS going to finite μ_B and finite μ_I (related to μ_Q)

$\Rightarrow$ **Why not generalising the T'-Expansion Scheme to several conserved charges..?**

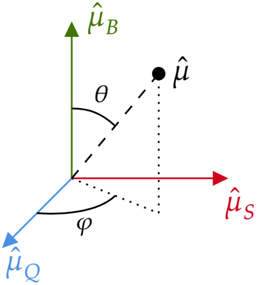
Construction of the new scheme - Basics

One can chose to project the $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ Cartesian coordinate system into a spherical one using $(\hat{\mu}, \theta, \varphi)$, following the relations:

$$\hat{\mu}_B = \hat{\mu}\,.\cos(\theta) \qquad \hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}$$

$$\hat{\mu}_Q = \hat{\mu}\,.\sin(\theta)\cos(\varphi) \quad \Longleftrightarrow \quad \varphi = \arccos\left(\frac{\hat{\mu}_Q}{\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}}\right)$$

$$\hat{\mu}_S = \hat{\mu}\,.\sin(\theta)\sin(\varphi) \qquad \theta = \arccos\left(\frac{\hat{\mu}_B}{\hat{\mu}}\right)$$



Simple way to transform the problem from 4D to 2D: a single $\hat{\mu}$ projected along a given direction in the 3D $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ space.

→ All previous equations from the 2D-TExS can be used as is!

Redefinition of the lattice-based Taylor coefficient

We introduce then X_2, a "generalised 2nd order susceptibility" at $\hat{\mu} = 0$:

$$X_2^{\theta,\varphi}(T) = \left.\frac{\partial^2 P/T^4}{\partial \hat{\mu}^2}\right|_{\hat{\mu}=0}$$
$$= c_\theta^2 \cdot \chi_2^B(T) + s_\theta^2 c_\varphi^2 \cdot \chi_2^Q(T) + s_\theta^2 s_\varphi^2 \cdot \chi_2^S(T)$$
$$+ 2c_\theta s_\theta c_\varphi \cdot \chi_{11}^{BQ}(T) + 2c_\theta s_\theta s_\varphi \cdot \chi_{11}^{BS}(T) + 2s_\theta^2 c_\varphi s_\varphi \cdot \chi_{11}^{QS}(T)$$

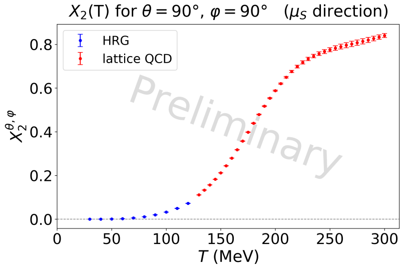
as a combination of the usual susceptibilities $\chi_{11/2}^{BQS}(T)$ at $\hat{\mu}_B = \hat{\mu}_S = \hat{\mu}_Q = 0$ computed from HRG (at low T) + lattice QCD.

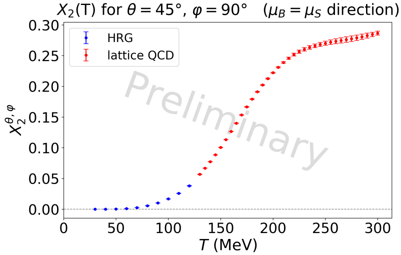
Examples:

- for $(\theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \hat{\mu}_S \quad \leftrightarrow \quad X_2 = \chi_2^S$
- for $(\theta = \frac{\pi}{4}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2} \quad \leftrightarrow \quad X_2 = \frac{\chi_2^B}{2} + \frac{\chi_2^S}{2} + \chi_{11}^{BS}$

The same way, one obtains:

$$X_4^{\theta,\varphi}(T) = c_\theta^4 \cdot \chi_4^B(T) + s_\theta^4 c_\varphi^4 \cdot \chi_4^Q(T) + s_\theta^4 s_\varphi^4 \cdot \chi_4^S(T) + \dots$$





The expansion coefficient $\lambda_2^{\theta,\varphi}(T)$

From there, we can build the generalised 2^{nd} order expansion coefficient λ_2:

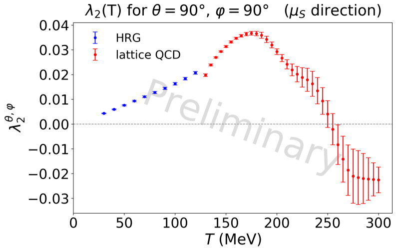
$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T}\frac{1}{X_2'^{\theta,\varphi}(T)} \times \left(X_4^{\theta,\varphi}(T) - \frac{\overline{X}_4^{\theta,\varphi}(0)}{\overline{X}_2^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T) \right)$$

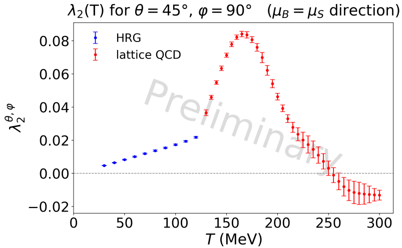
embedding the S.B. limit correction $\left(\overline{\lambda}_2 = \lim_{T\to\infty}(\lambda_2) = 0\right)$, with $\overline{X}_{2/4}^{\theta,\varphi}(0)$ being the S.B. limits for $X_{2/4}^{\theta,\varphi}(T)$ at $\hat{\mu} = 0$.

We employ here the latest $\chi_{2/4}^{BQS}$ data from the WB collaboration.

Examples:

- *for* $(\theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \hat{\mu}_S$
- *for* $(\theta = \frac{\pi}{4}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2}$





From shifted temperature T' to generalised charge density X_1

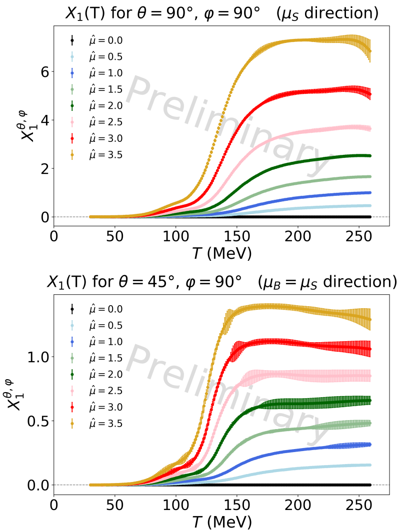
Using the previously obtained expansion coefficient $\lambda_2^{\theta,\varphi}(T)$, one can build the shifted temperature expansion $T'^{\theta,\varphi}(T,\hat{\mu})$:

$$T'^{\theta,\varphi}(T,\hat{\mu}) = T\left(1+\lambda_2^{\theta,\varphi}(T)\hat{\mu}_B^2\right)$$

Then, using the TExS main identity, one can express the generalised charge density $X_1^{\theta,\varphi}$ at finite $\hat{\mu}$:

$$X_1^{\theta,\varphi}(T,\hat{\mu}) = \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times X_2^{\theta,\varphi}\left(T'^{\theta,\varphi}(T,\hat{\mu}),0\right)$$

where we compute $X_2^{\theta,\varphi}(T',0)$ using χ_2^{BQS} data from the Wuppertal-Budapest collaboration. [a, b, c]



[a] Borsányi *et al.*, JHEP 01 (2012) 138
[b] Bellwied *et al.*, PRD 101 (2020) 3, 034506
[c] Borsányi *et al.*, PRL 126 (2021) 23, 232001

Pressure

We integrate $X_1^{\theta,\varphi}(T,\hat{\mu})$ to compute the pressure:

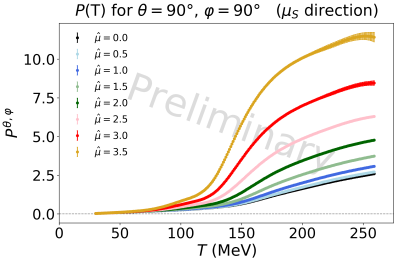
$$P^{\theta,\varphi}(T,\hat{\mu}) = P(T,0) + \int_0^{\hat{\mu}} X_1^{\theta,\varphi}(T,\hat{\mu}')d\hat{\mu}'$$

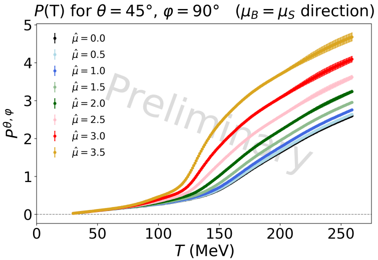
$$= P(T,\hat{\mu}_B,\hat{\mu}_Q,\hat{\mu}_S)$$

using lattice results for $P(T,0)$ with recent precision improvement from the Wuppertal-Budapest collaboration.[a]

Examples:

- for $(\theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \hat{\mu}_S$
- for $(\theta = \frac{\pi}{4}, \varphi = \frac{\pi}{2})$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2}$





[a] Parotto, talk at QM 2023

Charge densities & entropy density

The expression for number density n_i of any conserved charge $i = B, Q, S$ is then:

$$n_i = \frac{\partial P}{\partial \mu_i}\bigg|_T = \frac{\partial}{\partial \mu_i}\left[\int_0^{\mu'} X_1^{\theta,\varphi}(T,\hat{\mu})d\mu'\right]_T$$
$$= \frac{\partial}{\partial \mu_i}\left[\int_0^{\mu'} \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times X_2^{\theta,\varphi}(T'^{\theta,\varphi}(T,\hat{\mu}),0)d\mu'\right]_T \quad ,$$

and entropy density s is obtained through:

$$s = \frac{\partial P}{\partial T}\bigg|_\mu = \frac{\partial}{\partial T}\left[\int_0^{\mu'} X_1^{\theta,\varphi}(T,\hat{\mu})d\mu'\right]_\mu$$
$$= s(T,0) + \int_0^{\mu'} \frac{\partial}{\partial T}\left[\frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)}\right]_\mu \times X_2^{\theta,\varphi}(T',0)d\mu' + \int_0^{\mu'} \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times \frac{\partial T'}{\partial T} \times \frac{\partial X_2^{\theta,\varphi}(T',0)}{\partial T'}d\mu' \quad .$$

Summary

We present a new 4D lattice-based EoS going to finite (T, μ_{BQS}), using the T'-Expansion Scheme to extend the coverage from the 4D Taylor expansion ($\hat{\mu} \lesssim 2.5$) up to $\hat{\mu} \sim 3.5$.

4D-TExS EoS

Status:

- ☑ computing generalised charged density at finite $\hat{\mu}$ from lattice data at $\hat{\mu} = 0 \rightarrow$ **proof of concept**
- ⊡ computing all thermodynamics quantities at finite T and $\hat{\mu}_B$, $\hat{\mu}_Q$ and $\hat{\mu}_S$ + higher order derivatives (pressure P, charge densities $n_{B/Q/S}$, entropy density s, energy density ε, speed of sound c_s^2...)
- ⊡ scan the whole parameters space to constrain the limit of applicability
- ☐ Disclaimer: error shown in the preliminary results of this talk are underestimated
  $\rightarrow$ need to complete the analysis of error consistently

Outlook:

- Micheal's talk: including a critical point from the 3D Ising universality class into a 2D-TExS[1]
  $\rightarrow$ what about the 4D-TExS: critical point? line? plane?

[1] M. Kahangirwe, S. Bass, E. Bratkovskaya, J.J., P. Moreau, P. Parotto, D. Price, C. Ratti, O. Soloveva & M. Stephanov (arXiv:2402.08636)

Additional material

Lattice QCD datasets

To compute $\lambda_2^{\theta,\varphi}(T)$: continuum extrapolated $\chi_{2/4}^{BQS}$ from the latest LT=2 (small volume) 4HEX WB data.

To compute $X_1^{\theta,\varphi}(T,\hat{\mu})$: continuum extrapolated $\chi_{2/4}^{BQS}$ from the LT=4 ($\approx \infty$ volume) 4stout WB data.

Why do we mix 2 datasets? *($\approx$ lattice phenomenology)*

$\rightarrow$ 4HEX data don't have a sufficient coverage in T yet
$\rightarrow$ 4stout has too big errors in the transition region

BUT because of the small volume used for 4HEX (LT=2), more configurations simulated

$\rightarrow$ smaller errors

Checking the convergence of the 2D-TExS

Computed thermodynamics quantities in the 2D-TExS adding $\kappa_4^{BB}(T)$ (NLO in T' expansion).

$\rightarrow$ adding κ_4 only increases the errors but we see no change in the result overall