The sign problem and real-time simulations

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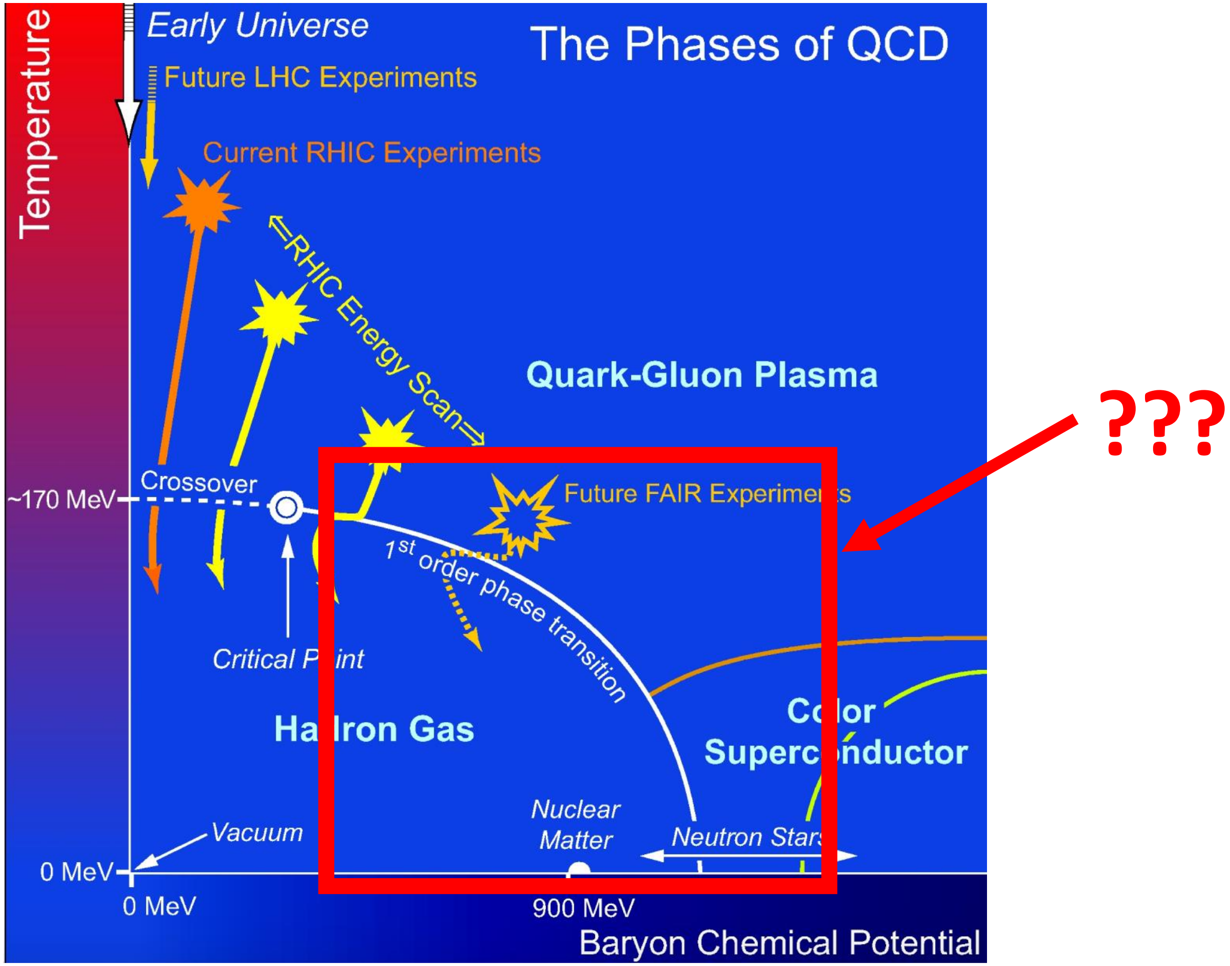
Phase diagram of QCD

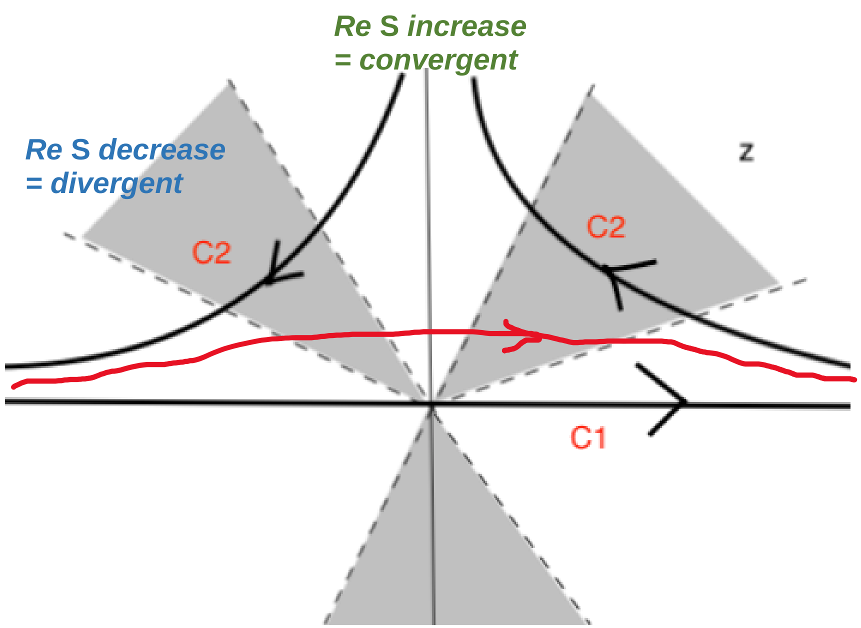


No small parameter for dense cool nuclear matter (everything is of order of strong scale Λ ~ 1 GeV)

Contour deformation methods

$$\langle \mathcal{O} \rangle \equiv \frac{\int e^{-S}\mathcal{O}}{\int e^{-S}} = \frac{\int \mathcal{O}e^{-S} / \int e^{-\mathrm{Re}\,S}}{\int e^{-S} / \int e^{-\mathrm{Re}\,S}} \equiv \frac{\langle \mathcal{O}e^{-i\mathrm{Im}\,S} \rangle_Q}{\langle e^{-i\mathrm{Im}\,S} \rangle_Q}$$

Average sign



The goal is to find a *complex contour* which *maximizes the sign*.

1) **Guess :)**
2) **Optimization over *learnifolds* (manifolds parameterized by neural networks)**
3) **Continuous deformations such as *holomorphic flows***

$$\frac{dz}{dt} = \overline{\frac{\partial S}{\partial z}}$$

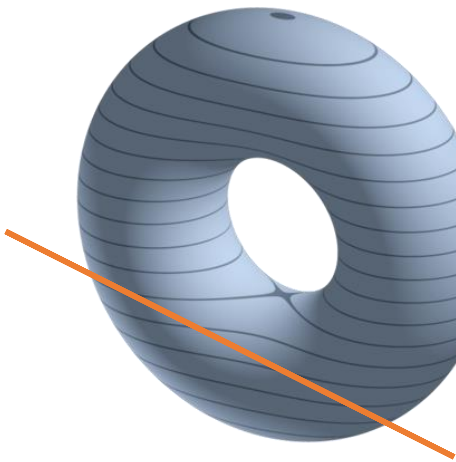
Cauchy theorem: $C_1 + C_2 + C_3 = 0$

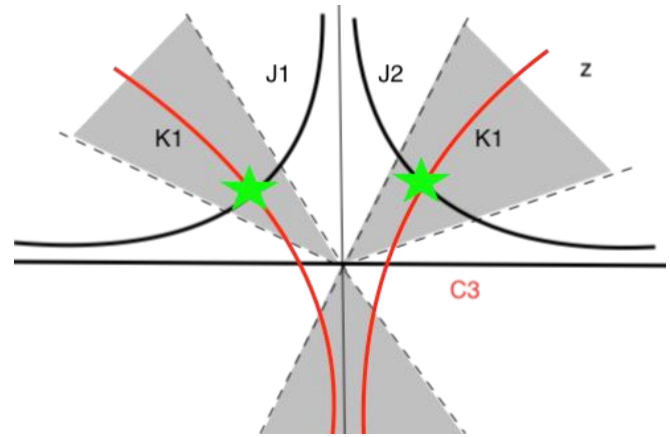
$$C_1 \neq C_2$$

Morse theory and Lefschetz thimbles

Real manifolds

Complex manifolds





Critical points of Morse function describe the topology of the manifold

Thimble: $\mathcal{J}_\sigma : \frac{dz(t)}{dt} = -\frac{\delta \bar{S}(z)}{\delta \bar{z}}$

Anti-thimble: $\mathcal{K}_\sigma : \frac{dz(t)}{dt} = +\frac{\delta \bar{S}(z)}{\delta \bar{z}}$

$\mathrm{Re} S$ **is Morse function in space of fields**

$$\mathcal{Z} = \sum_m n_m \mathcal{Z}_m$$

$\mathrm{Im} S = \mathrm{const}$ **on a thimble**

Non-relativistic nuclear matter

$$H = \sum_{i=1}^{A} \frac{\mathbf{p}_i^2}{2M_N} + \sum_{j>i=1}^{A} v_{ij}$$

$$v_{ij}(\boldsymbol{r}_{ij}) = \sum_p v_p(|\boldsymbol{r}_{ij}|)\, O_{ij}^p(\boldsymbol{r}_{ij})$$

Non-analytic potential

$$\begin{aligned} O_{ij}^1 &= 1, & O_{ij}^2 &= \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \\ O_{ij}^3 &= \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, & O_{ij}^4 &= (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \\ O_{ij}^5 &= S_{ij}(\boldsymbol{r}_{ij}), & O_{ij}^6 &= (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)\, S_{ij}(\boldsymbol{r}_{ij}) \end{aligned}$$

AV6P, AV18 parameterizations

Interesting questions:

1) **Fermion sign problem!**
2) **Real-time dynamics of many nuclei**
3) **NN, NNN, NNNN, … at very high densities?!**

Non-holomorphic actions

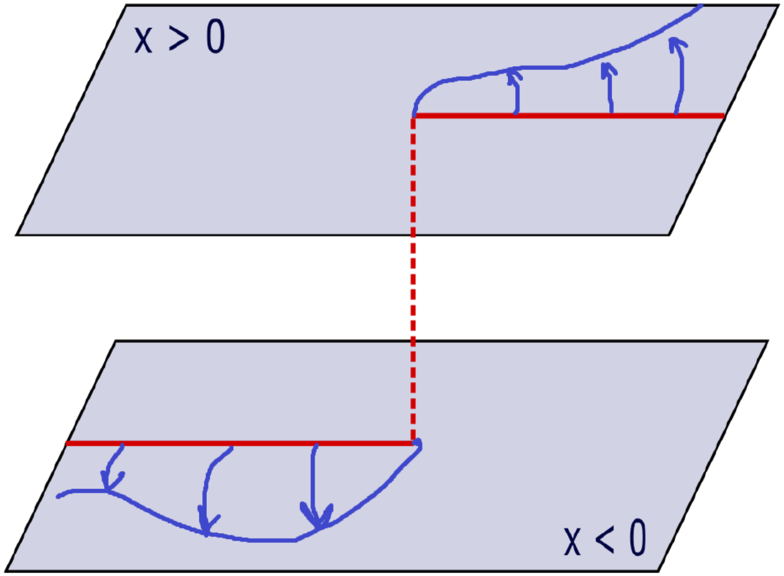
$$S(x) = \alpha|x|$$

non-analytic if viewed as function of complex argument

We only care about restriction to real domain

$$S_{\mathbb{R}}(x) = \begin{cases} \alpha x & x \geq 0 \\ -\alpha x & x < 0 \end{cases}$$

This allows us to analytically continue on *multiple* Riemann sheets



Consider branch structure of $\sqrt{(x-a)(x+a)}$ **when** $a \to 0$

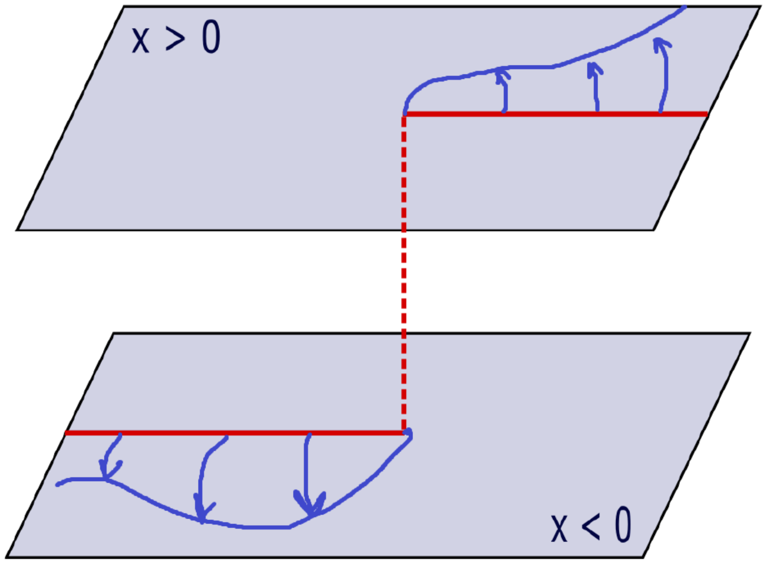
Modified holomorphic flow

Make sure that the contour is continuous; branch points must remain fixed under the flow

$$\frac{dx}{dT} = s(x)\overline{\frac{\partial S}{\partial x}}$$

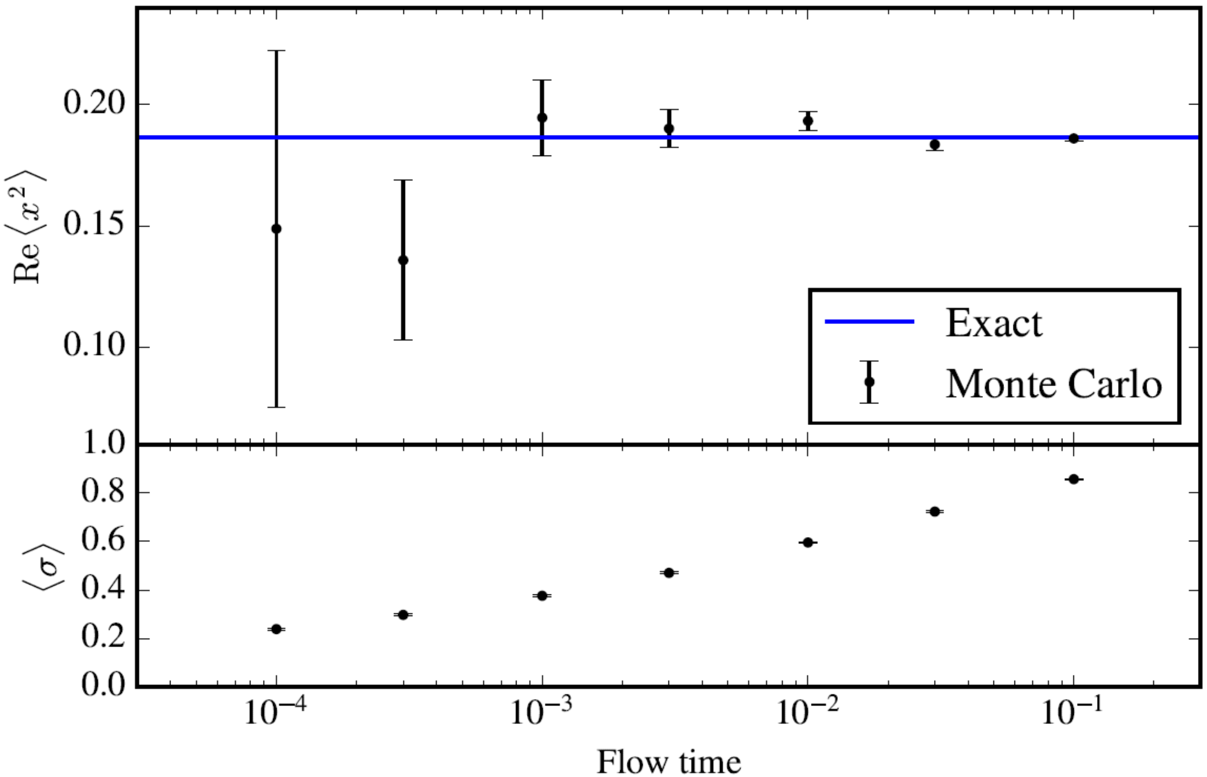
Scaling function s(x) must *vanish* sufficiently quickly *at singularities*

Not unique!



Simple demonstration

$$S(x) = i|x|^3$$

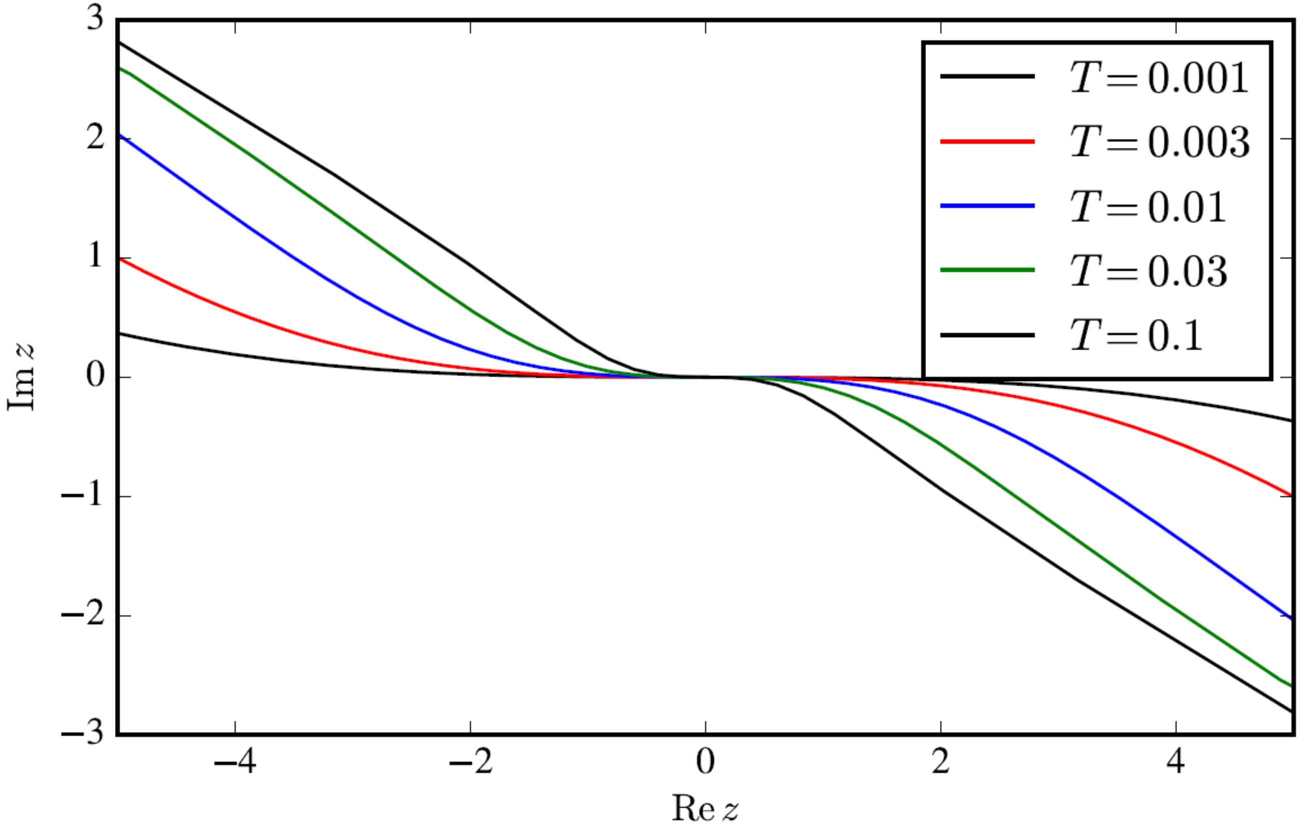


The sign: $\langle\sigma\rangle = \left\langle e^{-i\mathrm{Im}S}\right\rangle_Q$

Scaling function: $s(x) = |x|$

Simple demonstration

$$S(x) = i|x|^3$$



Scaling function: $s(x) = |x|$

Not so simple demonstration

Quantum mechanics in one dimension

Initial state:

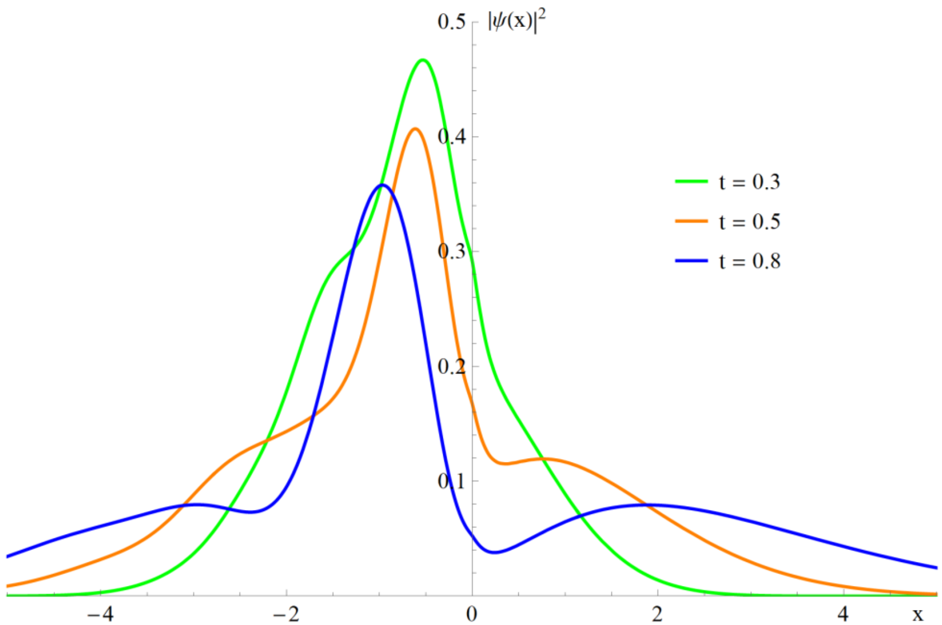
$$\Psi_i(x) \propto e^{-\frac{(x-x_0)^2}{2\sigma^2} - ip_0 x}$$

$$\sigma = 0.2 \quad p_0 = 1 \quad x_0 = -1$$

"Yukawa" potential:

$$V(x) = g^2 |x| e^{-m|x|}$$

$$g^2 = 150 \quad m = 10$$



Solution of Schrodinger's equation

We are interested in scattering

In particular, we compute the probability of tunneling

Not so simple demonstration

Observable:

$$F(t) \equiv \langle \Psi_i | e^{iHt} \mathcal{O} e^{-iHt} | \Psi_i \rangle$$

2nd order Suzuki-Trotter:

$$e^{-iHt} \approx \left(e^{-iV(x)\frac{\delta}{2}} e^{-i\frac{p^2}{2M}\delta} e^{-iV(x)\frac{\delta}{2}} \right)^{t/\delta}$$

Identity resolution:

$$I \propto \int dx \, |x\rangle\langle x|$$

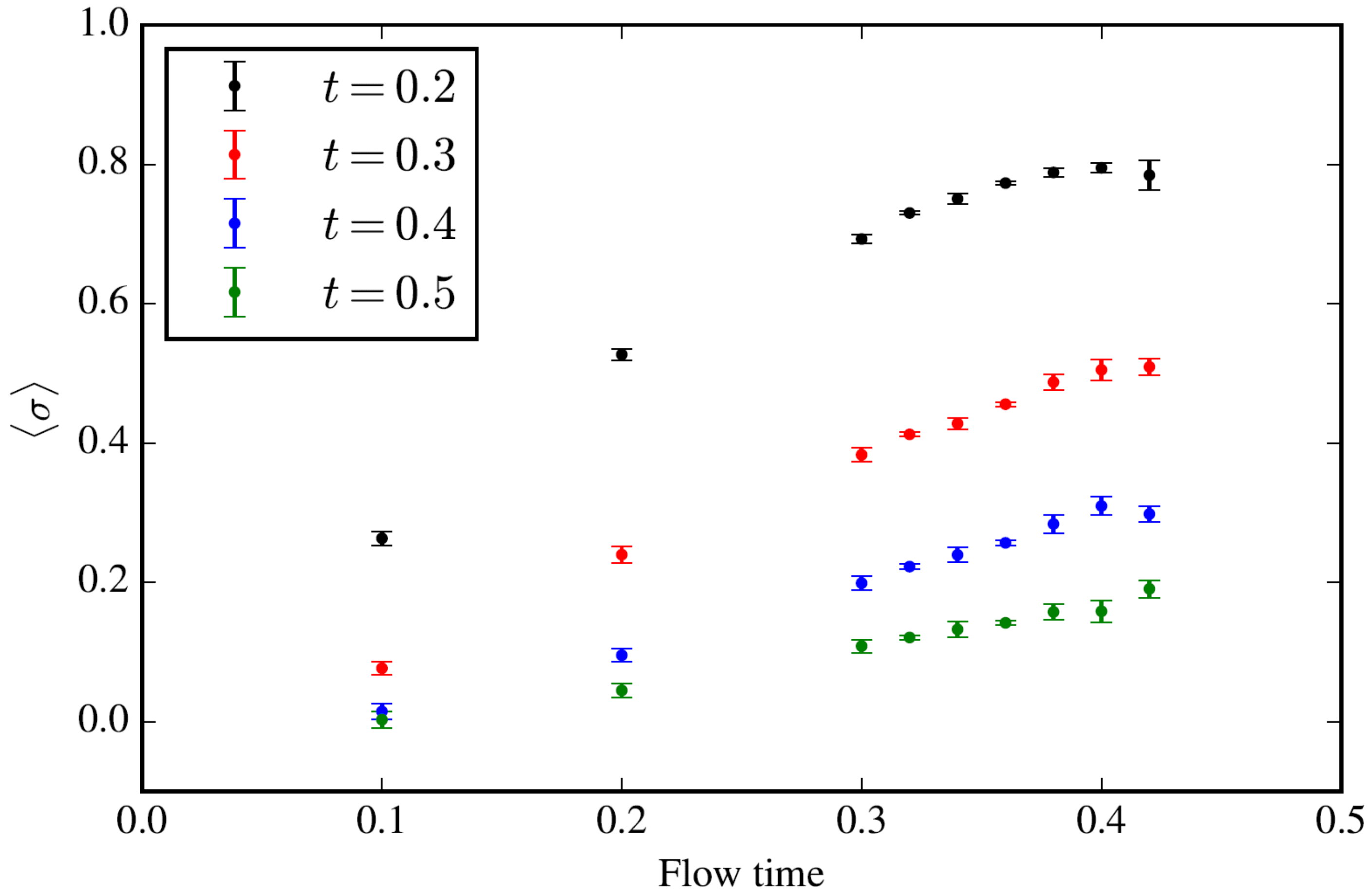
Resulting path integral:

$$F(t) = \frac{\int \left[\prod_{n=0}^{2N} dx_n\right] e^{-S(x)} \mathcal{O}(x_N)}{\int \left[\prod_{n=0}^{2N} dx_n\right] e^{-S(x)}}$$

… and action:

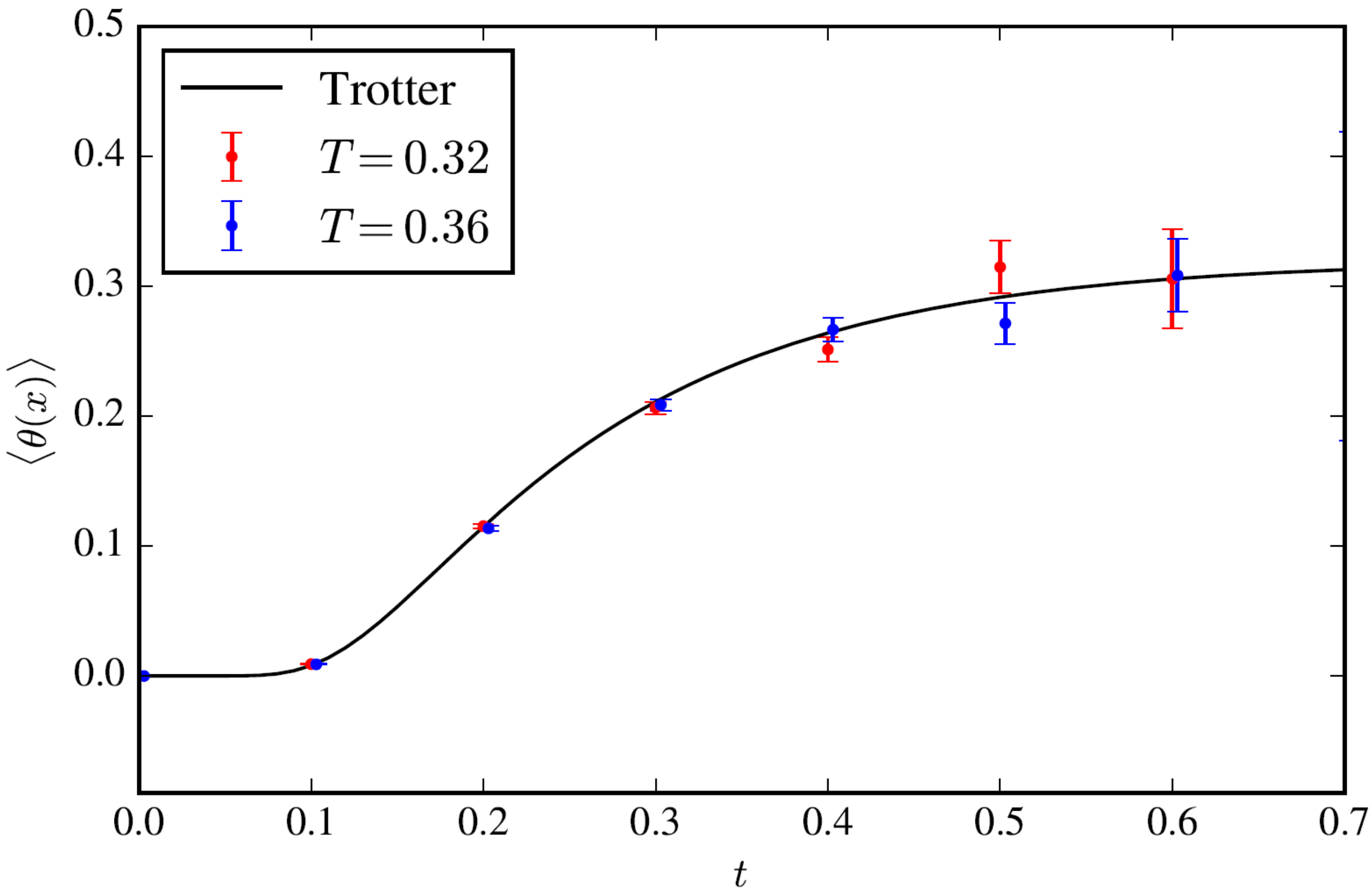
$$S(x) = -\log \Psi_i(x_0) - \log \Psi_i^\dagger(x_{2N}) - i \sum_{n=0}^{N-1} \left[\frac{(x_n - x_{n+1})^2}{\delta} + \delta_n V(x_n) \right] + i \sum_{n=N+1}^{2N} \left[\frac{(x_{n-1} - x_n)^2}{\delta} + \delta_n V(x_n) \right]$$

Results



The sign: $\langle\sigma\rangle = \left\langle e^{-i\mathrm{Im}S}\right\rangle_Q$ **Scaling function:** $s(x) = e^{-\frac{|x|}{\sqrt{m}}} I_1\left(\frac{|x|}{\sqrt{m}}\right)$

Results



Probability of tunneling

Conclusions

1) Contour deformations and holomorphic flows can be applied to certain non-holomorphic actions

2) However, holomorphic flow is not quite practical. Learnifolds are perhaps better

3) Study analytic structure of thimbles to find a better choice of contour?

Thank you!

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