AXIALLY SYMMETRIC RELATIVISTIC THIN DISKS AND SPHEROIDAL HALOS WITH MAGNETICALLY POLARIZED MATTER

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A static axisymmetric spacetime with a thin disk in the surface $z = 0$,

$$g_{ab}(r, z) = g_{ab}(r, -z) \,, \tag{1}$$

$$g_{ab,z}(r, z) = -g_{ab,z}(r, -z) \,. \tag{2}$$

The metric tensor is continuous through the disk,

$$[g_{ab}] = g_{ab}|_{z=0^+} - g_{ab}|_{z=0^-} = 0 \,, \tag{3}$$

with a finite discontinuity in its first normal derivative,

$$\gamma_{ab} = [g_{ab,z}] = 2 g_{ab,z}|_{z=0^+} \,. \tag{4}$$

A four potential A_a with reflection symmetry,

$$A_a(r,z) = A_a(r,-z)\,, \tag{5}$$

$$A_{a,z}(r,z) = -A_{a,z}(r,-z)\,. \tag{6}$$

The electromagnetic four potential is continuous through the disk,

$$[A_a] = 0\,, \tag{7}$$

with a finite discontinuity in its first normal derivative, expressed as

$$[A_{a,z}] = 2A_{a,z}|_{z=0^+}\,. \tag{8}$$

The Einstein-Maxwell equations for continuum media,

$$G_{ab} = T^M_{ab} + T^F_{ab} + T^{FM}_{ab} \,, \tag{9}$$

$$F^{ab}{}_{;b} = M^{ab}{}_{;b} \,, \tag{10}$$

where

$$F_{ab} = A_{b,a} - A_{a,b} \,, \tag{11}$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} \,, \tag{12}$$

$$M^{ab} = \varepsilon^{abcd} M_c u_d \,. \tag{13}$$

The energy-momentum tensor has a component due to the matter

$$T^{M}_{ab} = \rho u_a u_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \tag{14}$$

with $\{u_a, x_a, y_a, z_a\}$ an orthonormal tetrad, one due to the electromagnetic fields

$$T^{F}_{ab} = F_{ac}F_b{}^{c} - \frac{1}{4} g_{ab} F_{cd}F^{cd}\,, \tag{15}$$

and one due to the electromagnetic interaction with the polarized matter

$$T^{FM}_{ab} = F_{ac}M^{c}{}_{b}\,. \tag{16}$$

Now, we write the metric tensor as

$$g_{ab} = g^{+}_{ab}\theta(z) + g^{-}_{ab}\left\{1 - \theta(z)\right\} \ , \tag{17}$$

with $\theta(z)$ the Heaviside distribution, and the Einstein tensor as

$$G_{ab} = G^{+}_{ab}\theta(z) + G^{-}_{ab}\left\{1 - \theta(z)\right\} + Q_{ab}\delta(z)\,, \tag{18}$$

where

$$G^{\pm}_{ab} \quad = \quad R^{\pm}_{ab} - \frac{1}{2}g_{ab}R^{\pm}, \tag{19}$$

$$Q_{ab} \quad = \quad H_{ab} - \frac{1}{2}g_{ab}H\,, \tag{20}$$

$$H_{ab} \quad = \quad \frac{1}{2}\left(\gamma^{z}{}_{a}\delta^{z}{}_{b} + \gamma^{z}{}_{b}\delta^{z}{}_{a} - \gamma^{\mu}{}_{\mu}\delta^{z}{}_{a}\delta^{z}{}_{b} - g^{zz}\gamma_{ab}\right) \ , \tag{21}$$

and $H = g^{ab}H_{ab}$.

We also write the matter energy-momentum tensor as,

$$T^{ab}_M = T^{ab}_M{}^{+}\theta(z) + T^{ab}_M{}^{-}\{1-\theta(z)\} + \tau^{ab}_M\delta(z)\,, \tag{22}$$

where τ^{ab}_M stands for the surface energy-momentum tensor of the disk that, in an orthonormal tetrad, can be written as

$$\tau^{ab}_M = \sigma u_a u_b + \pi_1 x_a x_b + \pi_2 y_a y_b, \tag{23}$$

where σ, π_1 and π_2 are, respectively, the surface energy density and the stresses of the disk.

Now, the electromagnetic polarization tensor is written as,

$$M^{ab} = M^{ab^{+}}\theta(z) + M^{ab^{-}}\{1-\theta(z)\} + \Pi^{ab}\delta(z)\,, \tag{24}$$

with Π^{ab} the polarization tensor of the disk, and the magnetization vector of the disk is given by

$$\mathcal{M}_a = \frac{1}{2} u^b \varepsilon_{bacd} \Pi^{cd} \,. \tag{25}$$

The electromagnetic energy-momentum tensor and the electromagnetic interaction with the polarized matter are written as,

$$T_F^{ab} = {T_F^{ab}}^+ \theta(z) + {T_F^{ab}}^- \{1 - \theta(z)\} \,, \tag{26}$$

$$T_{FM}^{ab} = {T_{FM}^{ab}}^+ \theta(z) + {T_{FM}^{ab}}^- \{1 - \theta(z)\} + \tau_{FM}^{ab} \delta(z) \,, \tag{27}$$

with τ_{FM}^{ab} the electromagnetic interaction in the disk, given by

$$\tau^{ab}_{FM} = \bar{F}^{a}{}_{c}\Pi^{cb} \,, \tag{28}$$

where $\bar{F}^{a}{}_{c}$ is the average electromagnetic tensor through the disk,

$$\bar{F}^{a}{}_{c} = \frac{F^{a}{}_{c}{}^{+} + F^{a}{}_{c}{}^{-}}{2} \,. \tag{29}$$

Then, the field equations leads to

$$G^{\pm}_{ab} \;\; = \;\; \left(T^{M}_{ab}\right)^{\pm} + \left(T^{F}_{ab}\right)^{\pm} + \left(T^{FM}_{ab}\right)^{\pm} \,, \tag{30}$$

$$F^{ab}_{\pm;b} \;\; = \;\; M^{ab}_{\pm;b} \,, \tag{31}$$

for $z > 0$ and $z < 0$, and

$$Q_{ab} = \tau^{M}_{ab} + \tau^{FM}_{ab} \,, \tag{32}$$

$$\Pi^{ab}{}_{;b} = [F^{az}] - [M^{az}] \,, \tag{33}$$

$$\Pi^{az} = 0 \,, \tag{34}$$

for $z = 0$, which are the field equations on the disk.

Now, in an axially symmetric conformastatic spacetime

$$ds^2 = - \ e^{2\psi} dt^2 + e^{-2\psi}(dr^2 + r^2 d\varphi^2 + dz^2), \tag{35}$$

where ψ only depends on r and z, we take the potential as

$$A_a = (0, 0, A, 0), \tag{36}$$

where A also depends only on r and z. The Einstein equations imply that

$$M_{ab} = \xi F_{ab}, \tag{37}$$

with ξ a constant related to the magnetic susceptibility, and the Maxwell equations leads to the equation

$$r[(r^{-1}A_{,r})_{,r} + (r^{-1}A_{,z})_{,z}] + 2\nabla A \cdot \nabla\psi = 0, \tag{38}$$

where ∇ is the usual differential operator in cylindrical coordinates.

We consider the solution

$$A_{,r} = \quad 2re^{-2\psi}\psi_{,z}, \tag{39}$$

$$A_{,z} = -2re^{-2\psi}\psi_{,r}, \tag{40}$$

with the integrability of this overdetermined system granted by the equation

$$\nabla^2\psi = 2\nabla\psi \cdot \nabla\psi. \tag{41}$$

Now, by using (39), (40) and (41) in the Einstein equations, and taking $\xi = 1/2$, we obtain

$$\rho = (3e^{2\psi} - 2)\nabla\psi \cdot \nabla\psi\,, \tag{42}$$

$$p_1 = p_2 = e^{2\psi}\nabla\psi \cdot \nabla\psi\,, \tag{43}$$

$$p_3 = (2 - e^{2\psi})\nabla\psi \cdot \nabla\psi\,, \tag{44}$$

$$\sigma = 4e^{4\psi}\psi_{,z}, \tag{45}$$

$$\pi_1 = \pi_2 = 0\,, \tag{46}$$

$$\mathcal{M}_z = 2\alpha e^{-\psi}\,, \tag{47}$$

with α an arbitrary constant.

Finally, the components of the magnetic field are given by

$$B_{\widehat{r}} = \frac{e^{2\psi} A_{,z}}{r}, \tag{48}$$

$$B_{\widehat{z}} = -\frac{e^{2\psi} A_{,r}}{r}, \tag{49}$$

so the field lines can be obtained by solving the differential equation

$$\frac{dz}{B_{\widehat{z}}} = \frac{dr}{B_{\widehat{r}}}, \tag{50}$$

which, using the equations (48) and (49), can be written as

$$dA = A_{,r} dr + A_{,z} dz = 0. \tag{51}$$

Now we write equation (41) as

$$\nabla^2(e^{-2\psi}) = 0, \tag{52}$$

and the metric function ψ through the relation

$$e^{-2\psi} = 1 + U, \tag{53}$$

where

$$\nabla^2 U = 0, \tag{54}$$

and we only consider solutions that vanish at infinity in order that the spacetime will be asymptotically flat.

We consider a family of solutions obtained by taking

$$U(r,z) = \sum_{\ell=0}^{n} \frac{C_\ell}{R^{\ell+1}} P_\ell \left(\frac{z}{R} \right), \tag{55}$$

with $n \geq 0$, $R^2 = r^2 + z^2$ and $P_\ell(z/R)$ the Legendre Polynomials. For the electromagnetic potential we obtain the family of solutions

$$A(r,z) = \sum_{\ell=0}^{n} (-1)^{\ell+1} \frac{C_\ell}{\ell!} \frac{\partial^\ell}{\partial z^\ell} \left[\frac{z}{R} \right], \tag{56}$$

where we take the integration constant as equal to zero.

Then, we make the transformation $z \to |z| + a$ in order to obtain everywhere continuous functions but with their first z-derivatives discontinuous at $z = 0$. The result will be a solution with a singularity of the delta function type on $z = 0$ that can be interpreted as an infinite thin disk.

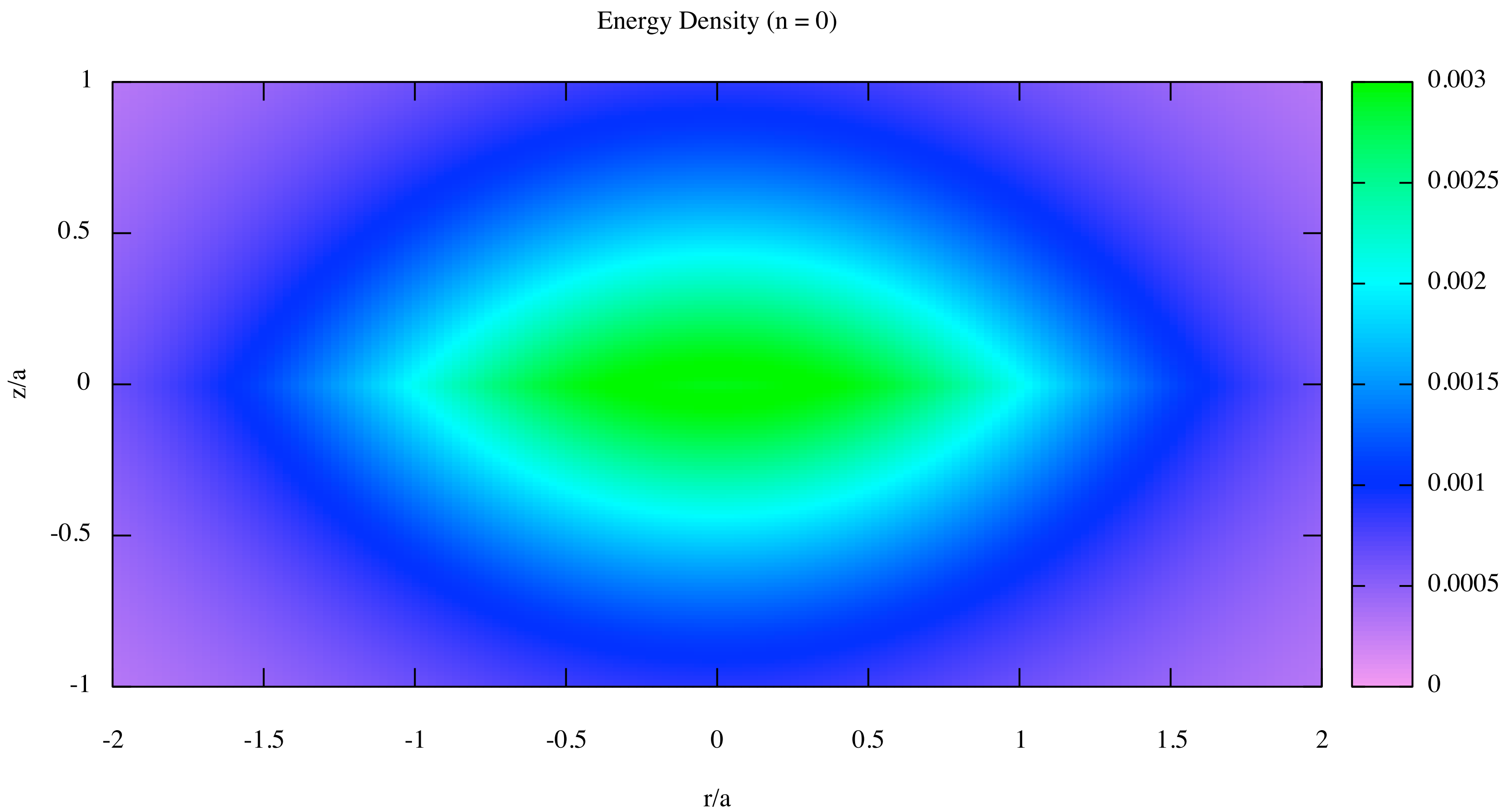
As an example, we consider the model $n = 0$, in which

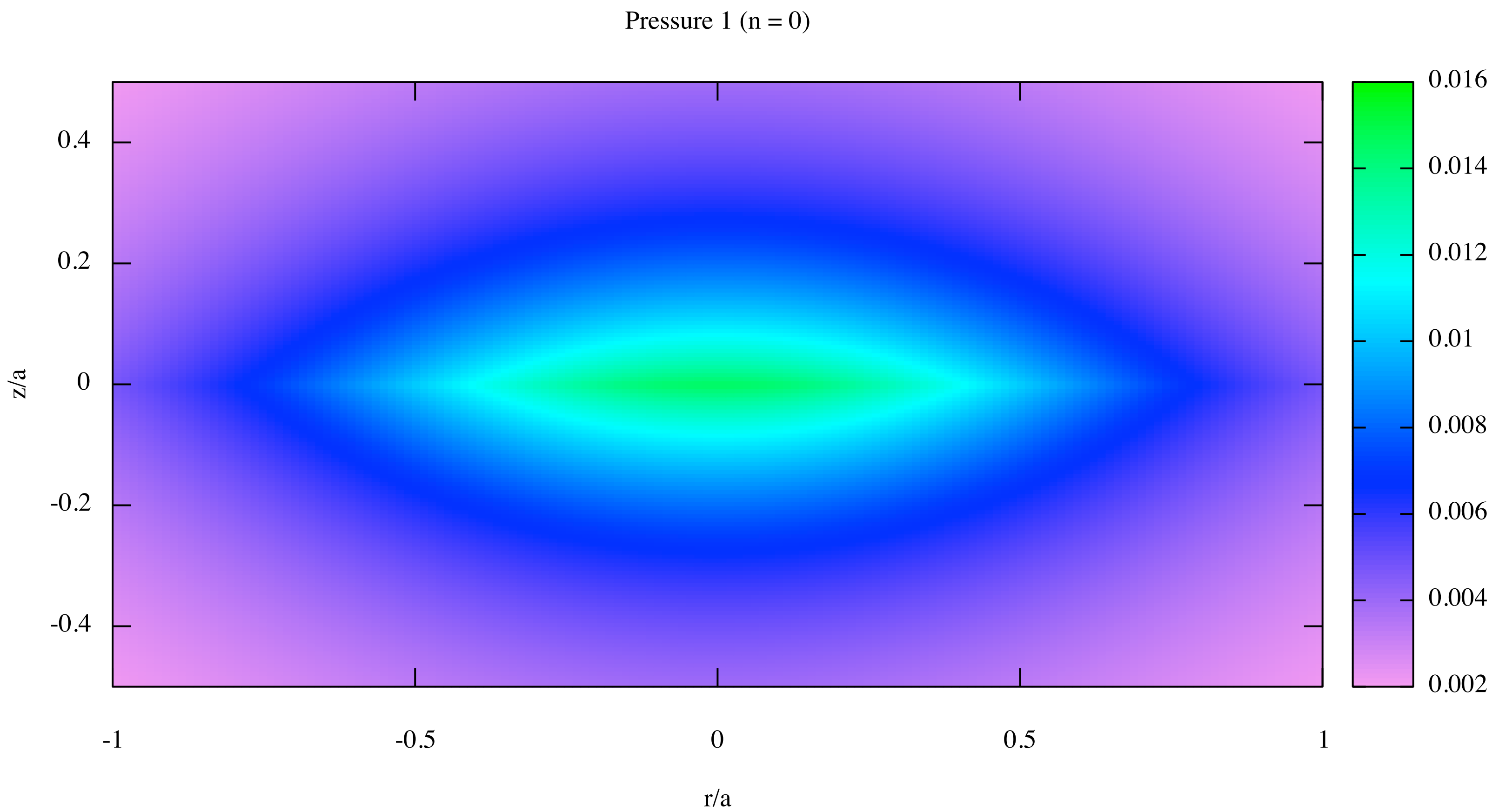
$$U = \frac{\widetilde{C}_0}{\sqrt{\widetilde{r}^2 + (1 + |\widetilde{z}|)^2}}, \tag{57}$$

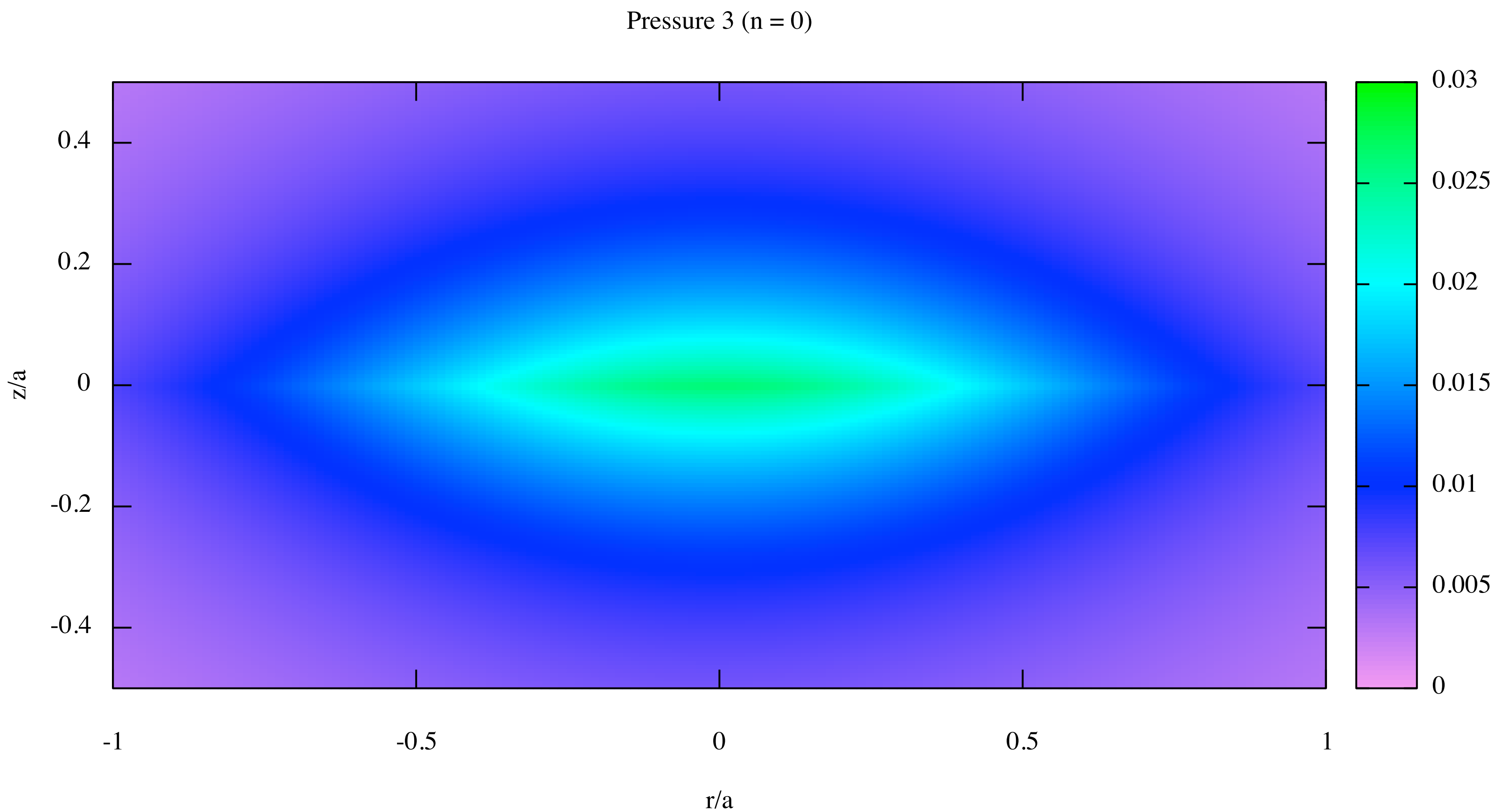
$$\widetilde{A} = -\frac{\widetilde{C}_0(1 + |\widetilde{z}|)}{\sqrt{\widetilde{r}^2 + (1 + |\widetilde{z}|)^2}}, \tag{58}$$

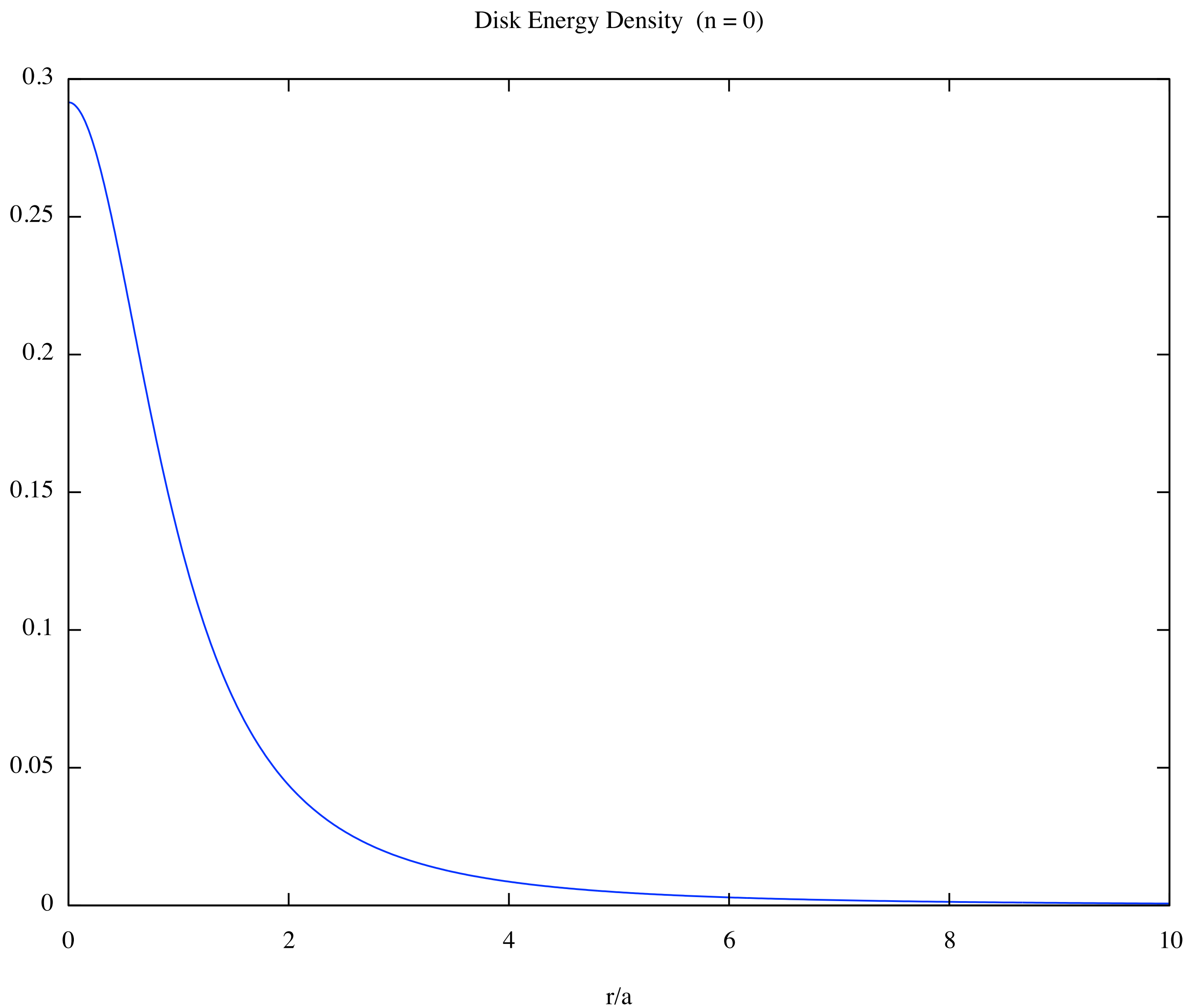
where $\widetilde{f} = f/a$.

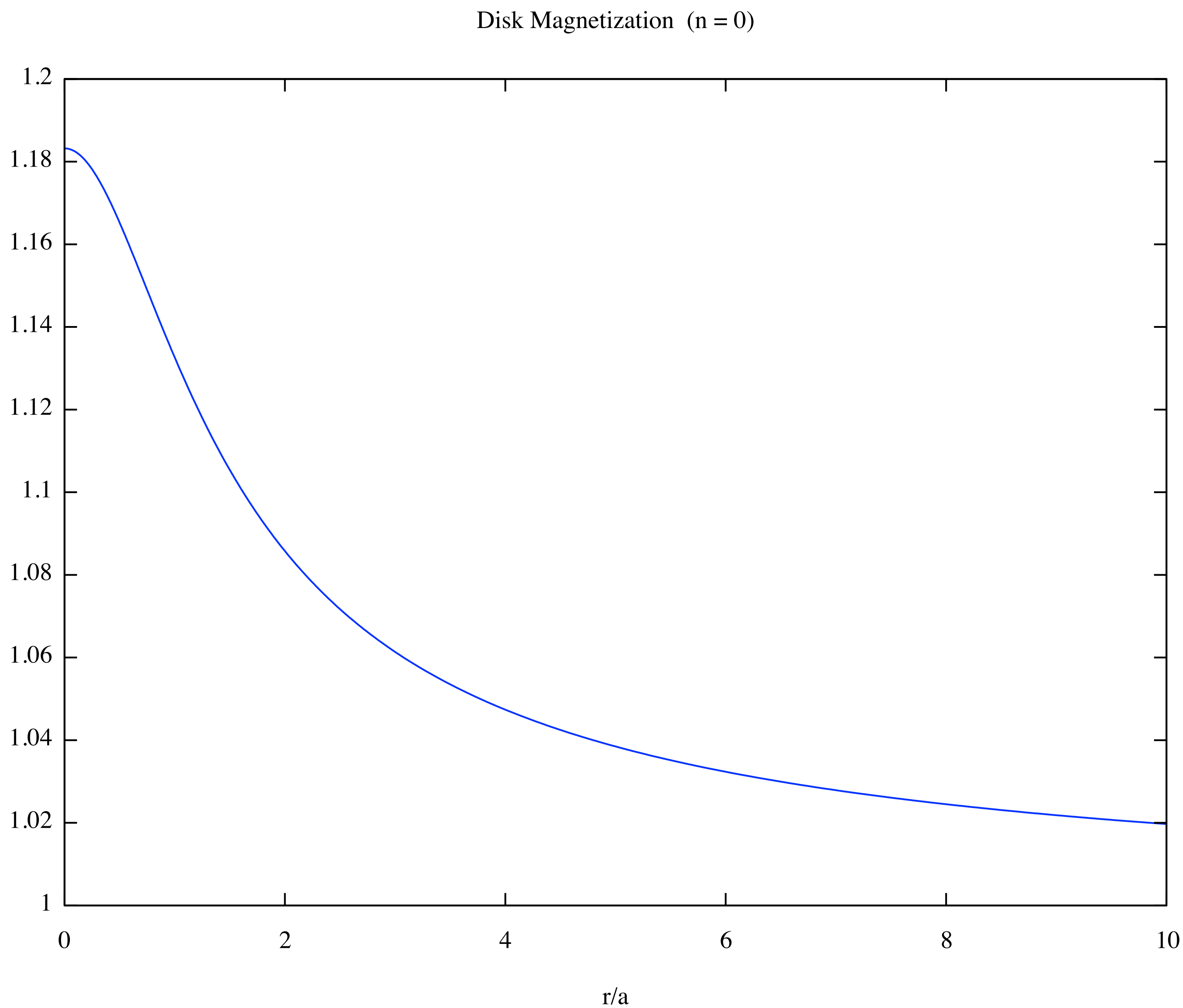
In the following figures we show the behavior of the halo energy density ρ, the halo pressures p_1 and p_3, the disk energy density σ, the disk magnetization $\mathcal{M}_z$ and the magnetic field lines by taking $\widetilde{C}_0 = \frac{2}{5}$ and $\alpha = \frac{1}{2}$.

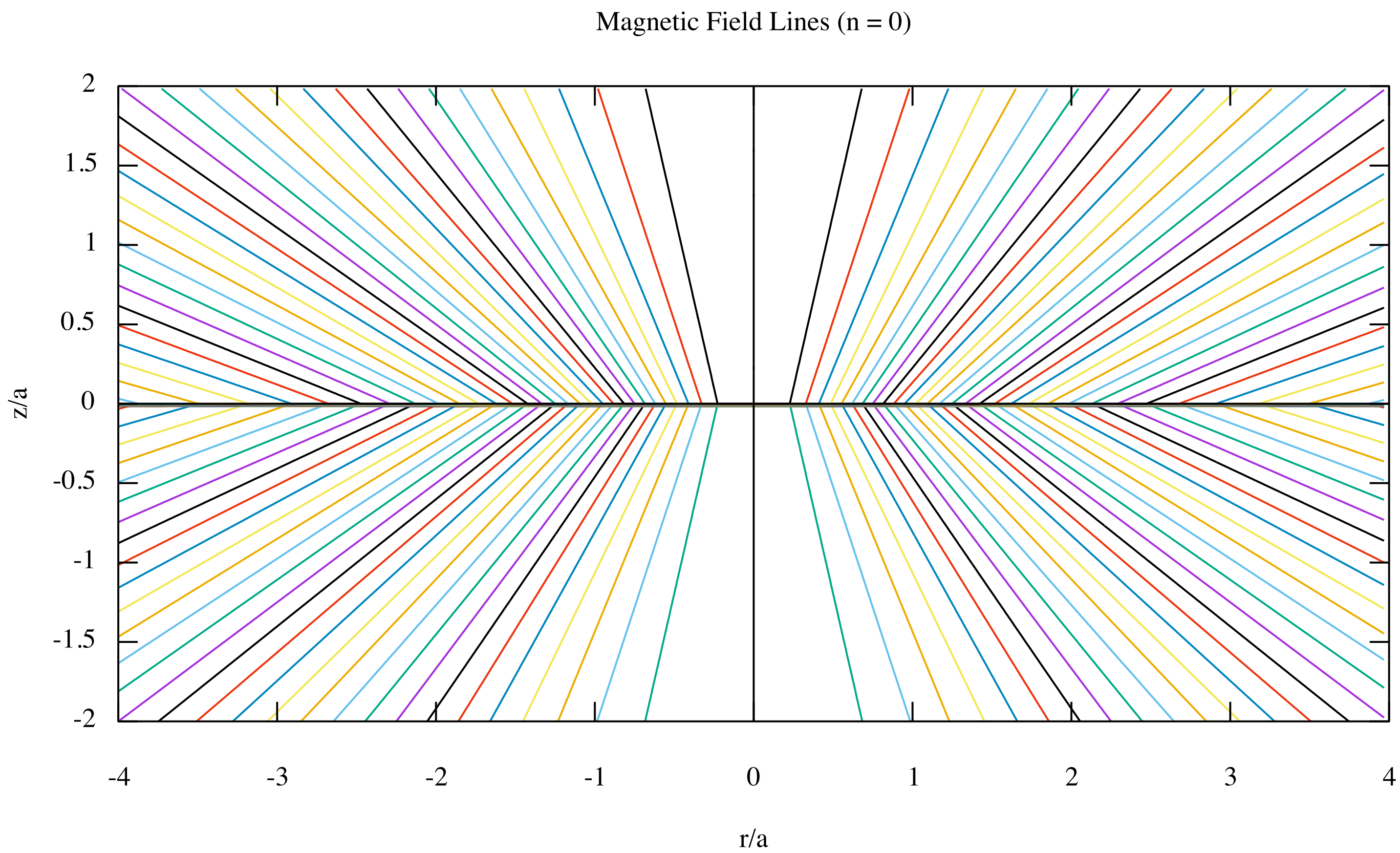

Energy Density (n = 0)
1
0.5
0
-0.5
-1
z/a
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
r/a
0.003
0.0025
0.002
0.0015
0.001
0.0005
0


Pressure 1 (n = 0)
0.4
0.2
0
-0.2
-0.4
z/a
-1
-0.5
0
0.5
1
r/a
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002


Pressure 3 (n = 0)
0.4
0.2
0
-0.2
-0.4
z/a
-1
-0.5
0
0.5
1
r/a
0.03
0.025
0.02
0.015
0.01
0.005
0


Disk Energy Density (n = 0)
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
r/a


Disk Magnetization (n = 0)
1.2
1.18
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0
2
4
6
8
10
r/a


Magnetic Field Lines (n = 0)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
z/a
-4
-3
-2
-1
0
1
2
3
4
r/a