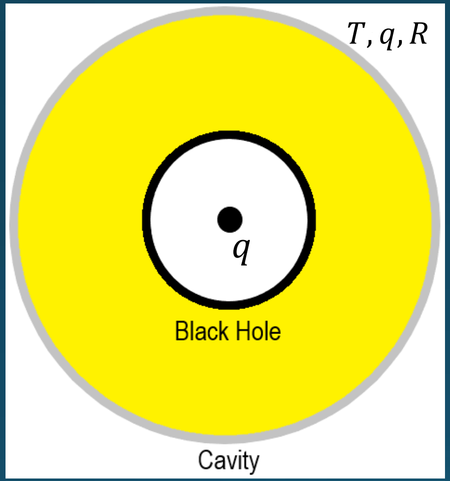
The **canonical** ensemble of a d-dimensional Reissner-Nordström black hole in a cavity



Tiago Vasques Fernandes
In collab. with: José P. S. Lemos

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Motivation

- Thermodynamic properties of black holes
  - As a semiclassical approximation of an underlying quantum theory of gravity

- Cavity [J. W. York, 1986] may allow for a stable black hole solution

- Grand canonical done in [Braden et al, 1990], generalization for higher dimensions done in [TF & Lemos, to be published].

- Generalization of [Lundgren, 2006] for higher dimensions
  - Connection to theories that require higher dimensions
  - Behaviour in“d”.

Partition Function

We consider a spacetime with electric charge in a cavity.

The partition function is

$$Z = \int Dg\, DF_{ab}\, e^{i\, I[g,\, F_{ab}]} \Rightarrow \int Dg_E\, DF_{ab}\, e^{-I_E[g_E,\, F_{ab}]}$$

Complex analytic extension
Wick Rotation

where g_E is periodic in time.

The boundary of the cavity is a heat reservoir with inverse temperature β and has a fixed electric flux (electric charge q).

The objective is to obtain the Helmoltz potential with $Z = e^{-\beta F}$

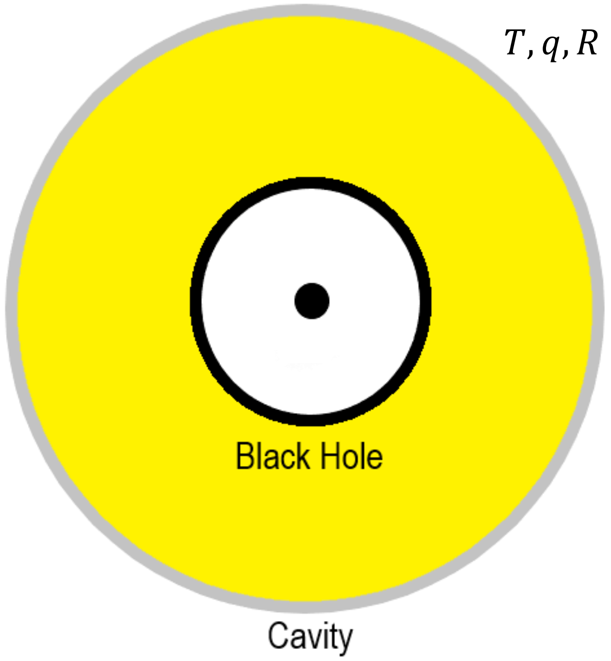
Action and metric

Action:

$$I_E = -\int_M \left(\frac{R}{16\,\pi} - \frac{(d-3)F_{ab}F^{ab}}{4\Omega}\right)\sqrt{g}\,d^d x - \frac{1}{8\pi}\int_C (K-K_0)\sqrt{\gamma}d^{d-1}x + \frac{d-3}{\Omega}\int_C F^{ab}A_a r_b\sqrt{\gamma}d^{d-1}x$$

with $F_{ab} = \partial_a A_b - \partial_b A_a$.

C is the thin shell at radius R.



Metric:

$$ds^2 = b(y)^2 d\tau^2 + \alpha(y)^2 dy^2 + r(y)^2 d\Omega^2 \quad , \tau \in [0,2\pi[\quad , y \in]0,1]$$

Boundary Conditions:

$y = 0$: Regularity ($\mathbb{R}^2 \times \mathbb{S}^{d-2}$), Zero electric potential, $r(0) = r_+$

$y = 1$: $\beta = 2\pi b(1)$, $q = \frac{1}{\Omega}\int_{y=1} F^{ab} dS_{ab}$

Zero Loop approximation

$$\mu = \frac{8\pi}{(d-2)\Omega}$$

$$f(R,q;r_+) = \left(1 - \frac{r_+^{d-3}}{R^{d-3}}\right)\left(1 - \frac{\mu\, q^2}{(r_+R)^{d-3}}\right)$$

Minimize the action in variations of b ⟶ Hamiltonian constraint

Minimize the action in variations of F ⟶ Gauss constraint

$$Z = \int Dr_+ \, e^{-I_E^*(\beta,q,R;r_+)}$$

Reduced action

$$I_E^*(\beta,q,R;r_+) = \frac{(d-2)\Omega R^{d-3}\beta}{8\pi}\left(1 - \sqrt{f(R,q;r_+)}\right) - \frac{\Omega\, r_+^{d-2}}{4}$$

$$\beta\, \mathrm{F}_{\mathrm{gen}} = \beta E - S$$

Solutions and stability

$$\mu = \frac{8\pi}{(d-2)\Omega}$$

$$f(R,q;r_+) = \left(1 - \frac{r_+^{d-3}}{R^{d-3}}\right)\left(1 - \frac{\mu\, q^2}{(r_+ R)^{d-3}}\right)$$

Minimize the action in variations of r_+

$$\beta = \frac{4\pi}{d-3}\,\frac{r_+^{2d-5}}{r_+^{2d-6} - \lambda q^2}\sqrt{f}$$

The partition function will then be given by
$Z = e^{-I_E^*(\beta,q,R;r_+(\beta,q,R))}$

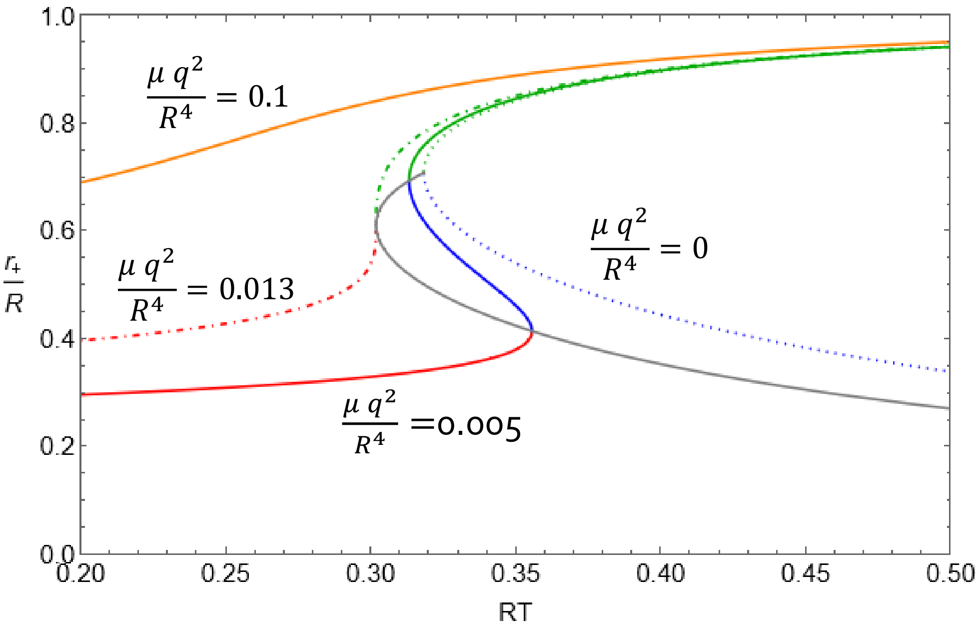
We want to find $r_+(\beta, q, R)$

Condition for stability:

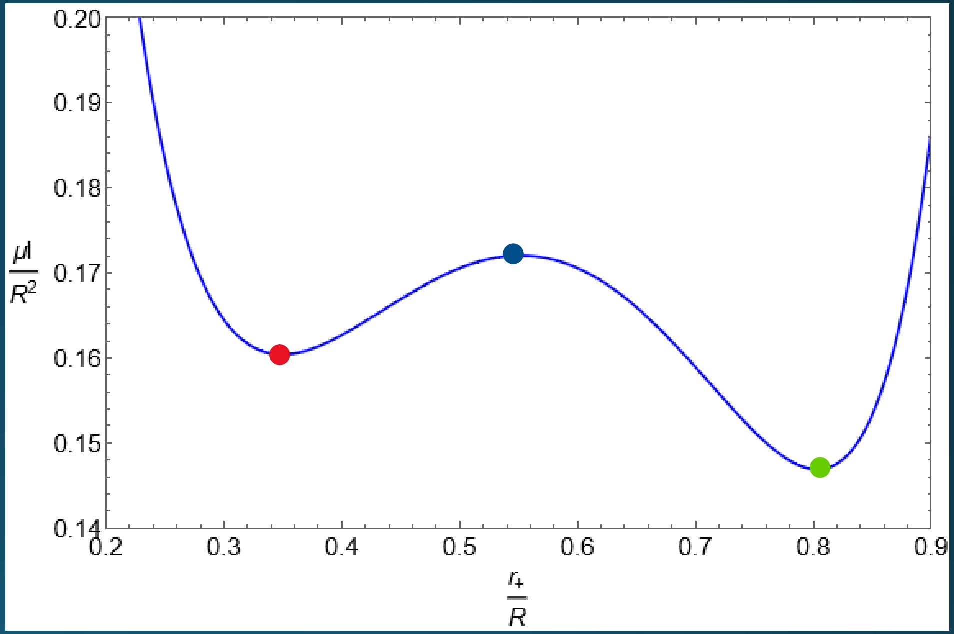
$$\frac{\partial^2 I^*}{\partial {r_+}^2} \geq 0 \Leftrightarrow \frac{\partial \beta}{\partial r_+} \leq 0$$

The critical points of β give the limits of stability

Solutions and stability ($d = 5$)



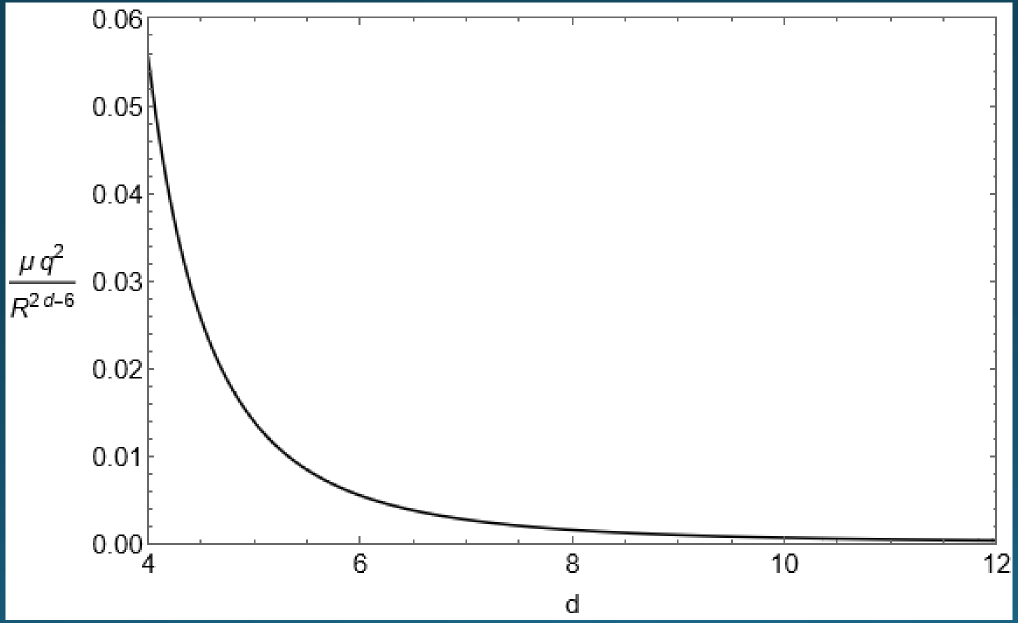
Solutions in red, orange and green are stable.
Solutions in blue are unstable.
Stability: increase in RT leads to increase in $\frac{r_+}{R}$



$\frac{\mu q^2}{R^4} = 0.005$, $RT = 0.33$

Higher dimensions

Critical charge $$\frac{\mu\, q_c^2}{R^{2d-6}} = \frac{\left[(d-1)(3d-7)(3d^2-16d+22) - 3\sqrt{3}(d-3)(d-2)^2\sqrt{(d-1)(3d-7)}\right]^2}{4(d-1)(2d-5)^3(3d-7)}$$



q_c dependence in d

Thermodynamics

$$\mu = \frac{8\pi}{(d-2)\Omega}$$

$$f(R,q;r_+) = \left(1 - \frac{r_+^{d-3}}{R^{d-3}}\right)\left(1 - \frac{\mu\, q^2}{(r_+R)^{d-3}}\right)$$

We have the correspondence $\beta F = I_E^*(\beta, q, R; r_+(\beta, q, R))$

$$E = \frac{(d-2)\Omega R^{d-3}}{8\pi}\left(1 - \sqrt{f}\right)$$

$$\Phi = \frac{q}{\sqrt{f}}\left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}}\right)$$

$$p = \frac{(d-3)}{16\pi R\sqrt{f}}\left(\left(1 - \sqrt{f}\right)^2 - \frac{\mu\, q^2}{R^{2d-6}}\right)$$

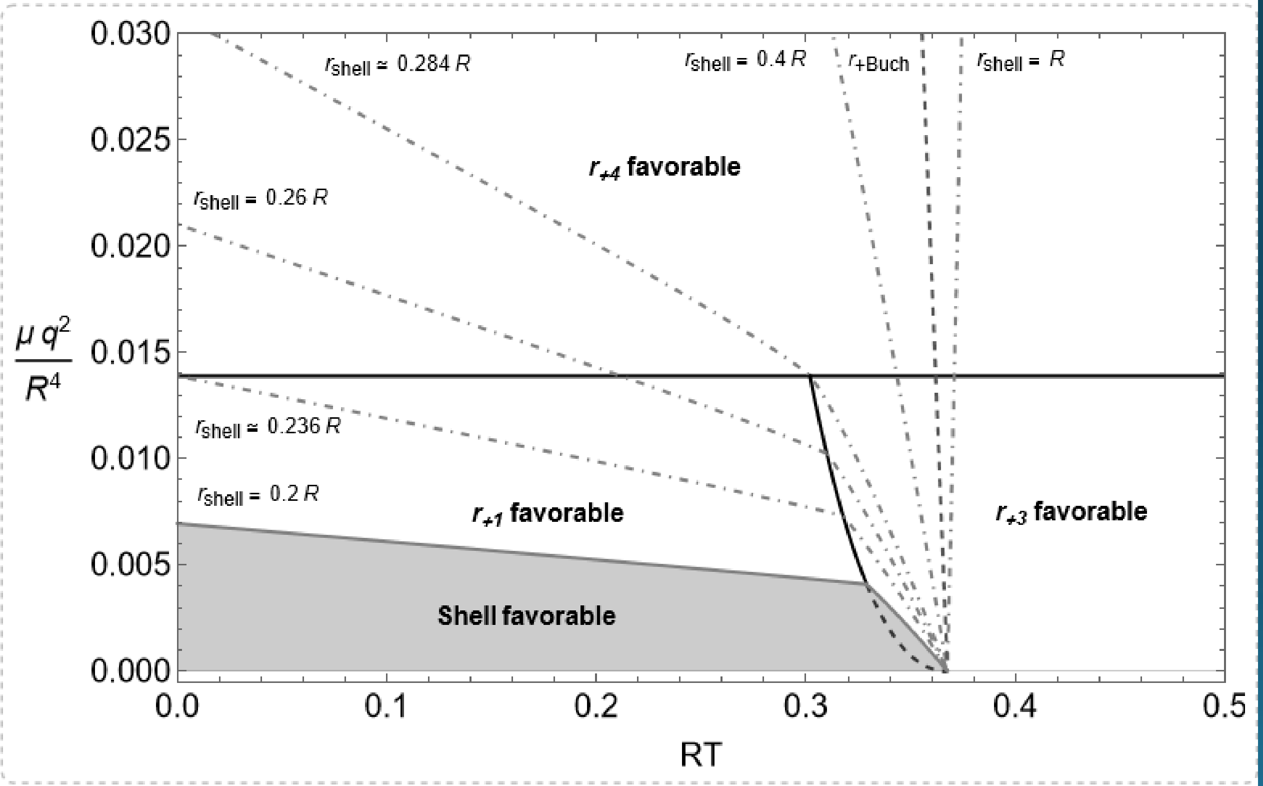
$$S = \frac{\Omega\, r_+^{d-2}[\beta,q,R]}{4}$$

If $C_{A,q} > 0$, there is stability

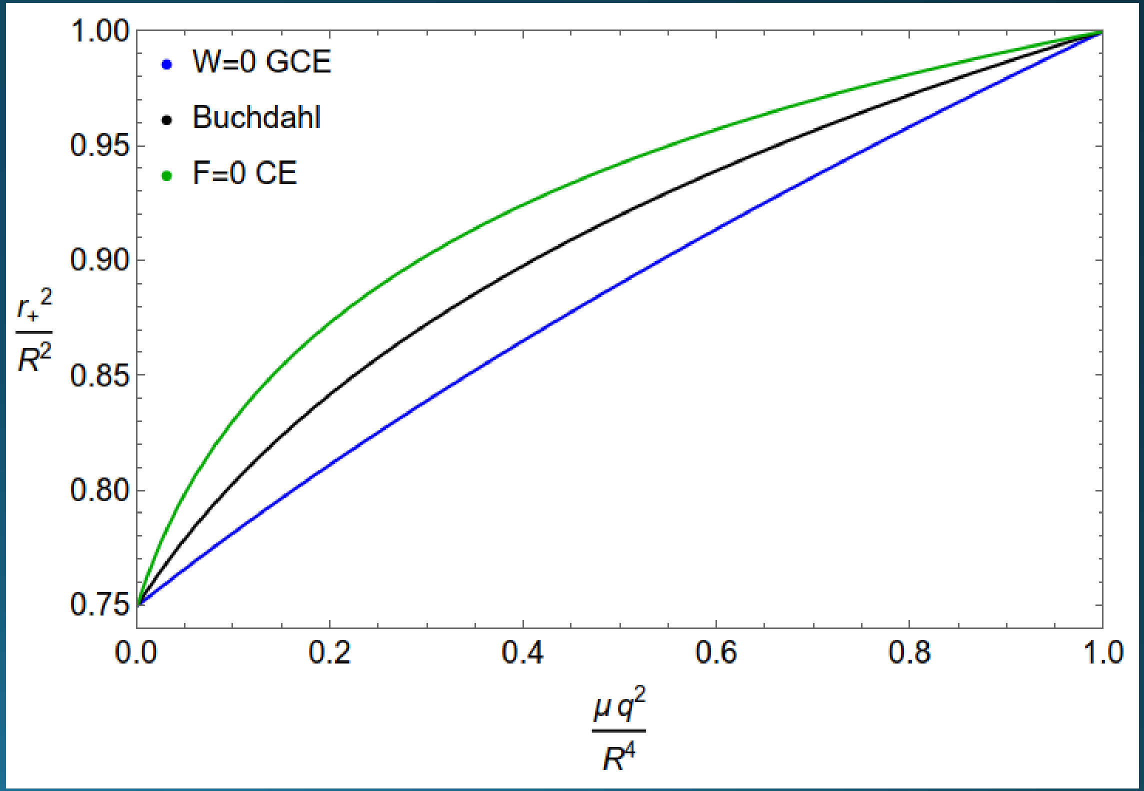
(Contrary to the grand canonical which the condition is $C_{A,\phi} > 0$)

Phase transitions

Charged shell (no grav.) vs Charged black hole



Zero free energy radius vs Buchdahl vs Grand can.



Conclusions

- Three possibilities:

For $q < q_c$, there are three solutions for the black hole, the intermediate solution is unstable, others are stable.

For $q = q_c$, there are two solutions, both stable.

For $q > q_c$ there is one solution, which is stable.

- Higher dimensions imply a lower q_c

- Thermodynamic quantities have the same expressions as in the grand canonical

but solutions are different!

- Radius of zero free energy larger than Buchdahl and larger than radius of zero grand potential.

Extra Slides

Boundary Conditions

$$ds^2 = b(y)^2 d\tau^2 + \alpha(y)^2 dy^2 + r(y)^2 d\Omega^2 \quad , \tau \in [0,2\pi[\quad , y \in]0,1]$$

At $\boldsymbol{y = 0}$ (Horizon)

$(b'\alpha^{-1})|_0 = 1$ (Regularity)

$$\left.\begin{aligned} b(0) &= 0 \\ \left(\frac{r'}{\alpha}\right)^2\Bigg|_0 &= 0 \end{aligned}\right\} \begin{gathered}\text{Killing Horizon} \\ \mathbb{R} \times \mathbb{S}^{d-2}\end{gathered}$$

$A_\tau(0) = 0$ (Zero Potential)

$r(0) = r_+$

At $\boldsymbol{y = 1}$ (Boundary of Cavity)

$\beta = 2\,\pi\, b(1)$ (Inverse Temp.)

$\dfrac{R^{d-2} A'_\tau(1)}{b(1)\,\alpha(1)} = -i\, q$ (Electric Flux)

$r(1) = R$

Critical points and Stability

We need to minimize further the action in variations of r_+ and q so that

$Z = e^{-I_E^0(\beta,q,R)}$, where $I_E^0(\beta, q, R) = I_E^*(\beta, q, R; r_+[\beta, q, R])$

Critical Points

Stability (Minima)

$$y = \frac{\mu\, q^2}{R^{2d-6}}, \qquad x = \frac{r_+}{R}$$

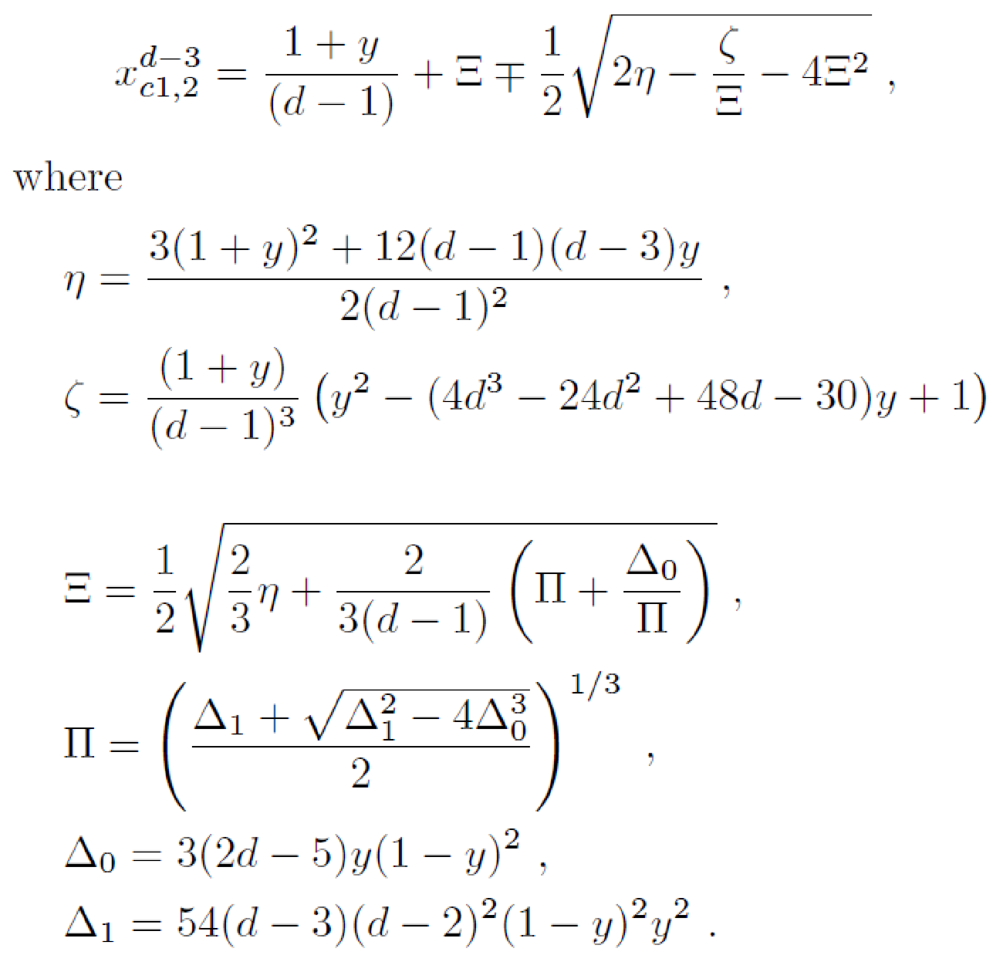
$$(x^{2d-6} - y)^2 B^2 - x^{3d-7}(1 - x^{d-3})(x^{d-3} - y) = 0$$

$$B = \frac{(d-3)\beta}{4\pi\, R}$$

$$x < x_{1c}\,, x > x_{2c}$$

Critical points of β

$$y = \frac{\mu\, q^2}{R^{2d-6}}, \qquad x = \frac{r_+}{R}$$



$$x_{c1,2}^{d-3} = \frac{1+y}{(d-1)} + \Xi \mp \frac{1}{2}\sqrt{2\eta - \frac{\zeta}{\Xi} - 4\Xi^2}\ ,$$

where

$$\eta = \frac{3(1+y)^2 + 12(d-1)(d-3)y}{2(d-1)^2}\ ,$$

$$\zeta = \frac{(1+y)}{(d-1)^3}\left(y^2 - (4d^3 - 24d^2 + 48d - 30)y + 1\right)$$

$$\Xi = \frac{1}{2}\sqrt{\frac{2}{3}\eta + \frac{2}{3(d-1)}\left(\Pi + \frac{\Delta_0}{\Pi}\right)}\ ,$$

$$\Pi = \left(\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}\right)^{1/3}\ ,$$

$$\Delta_0 = 3(2d-5)y(1-y)^2\ ,$$

$$\Delta_1 = 54(d-3)(d-2)^2(1-y)^2y^2\ .$$