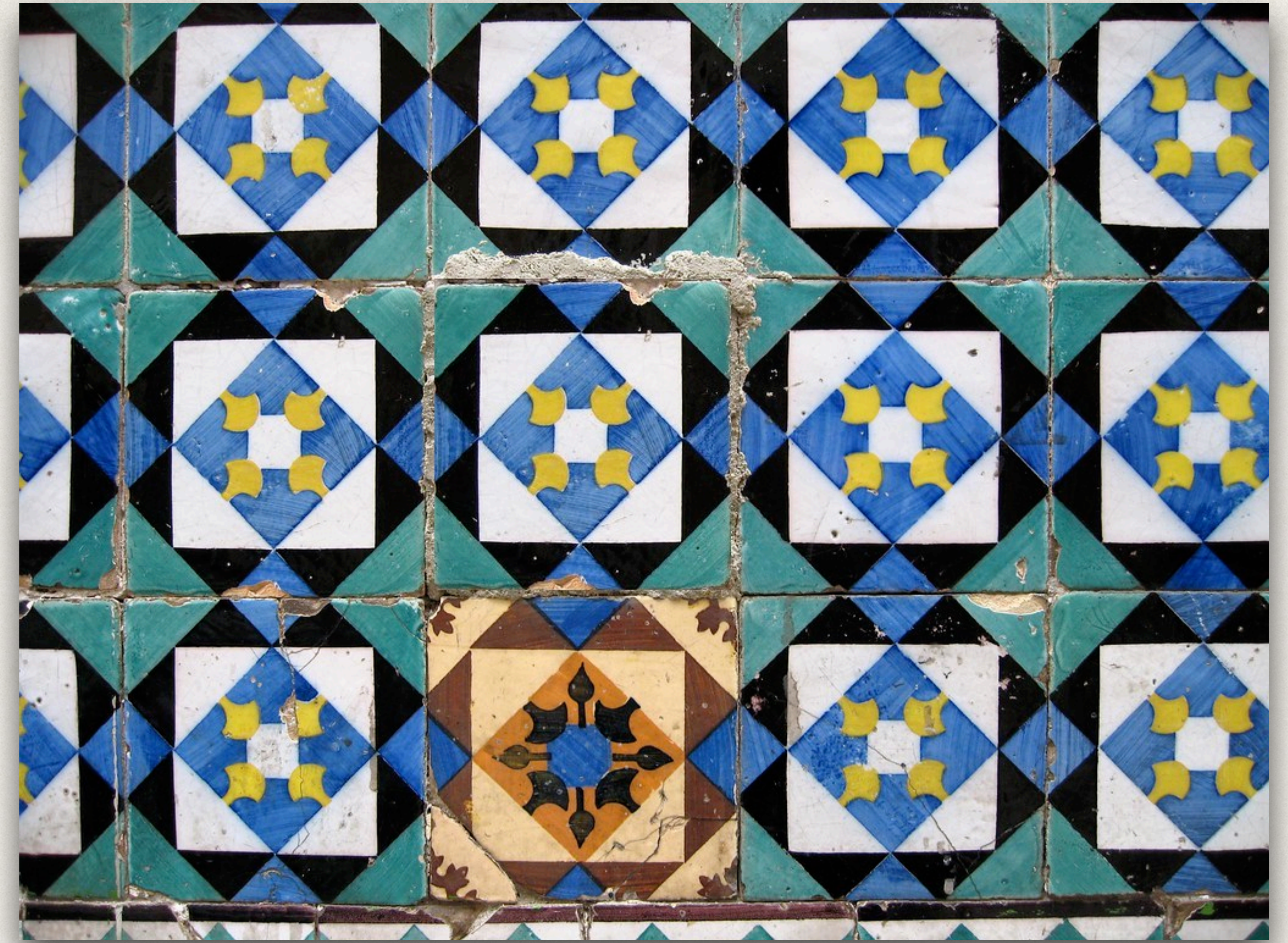


# Heat Kernel and Revelations

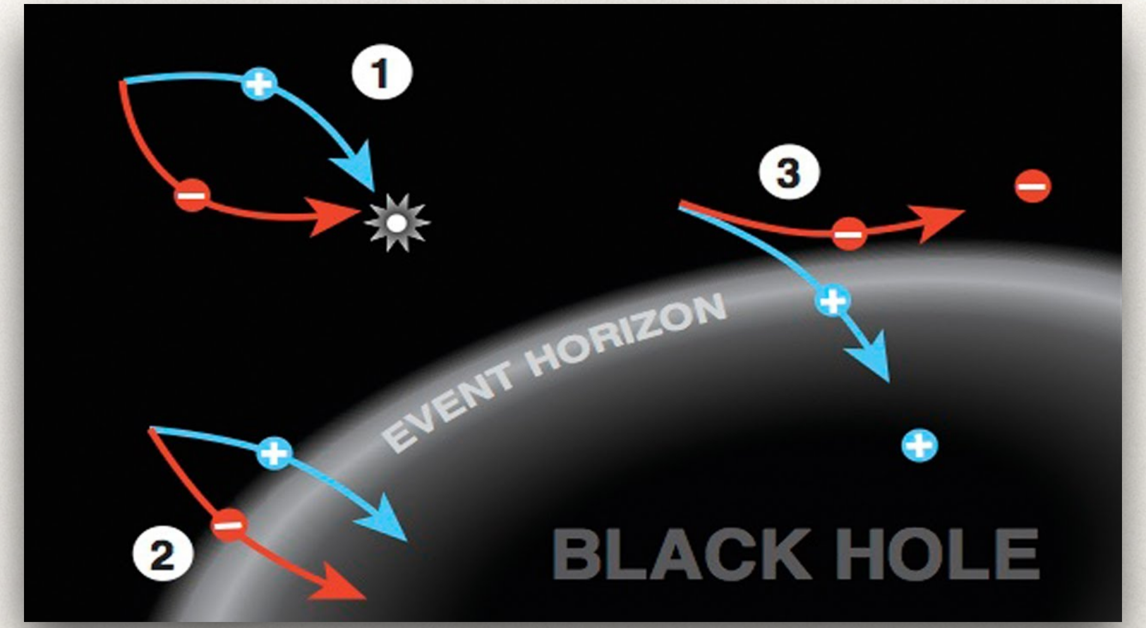
*Vincenzo Vitagliano (University of Genova & INFN), 23 Jul 2024*

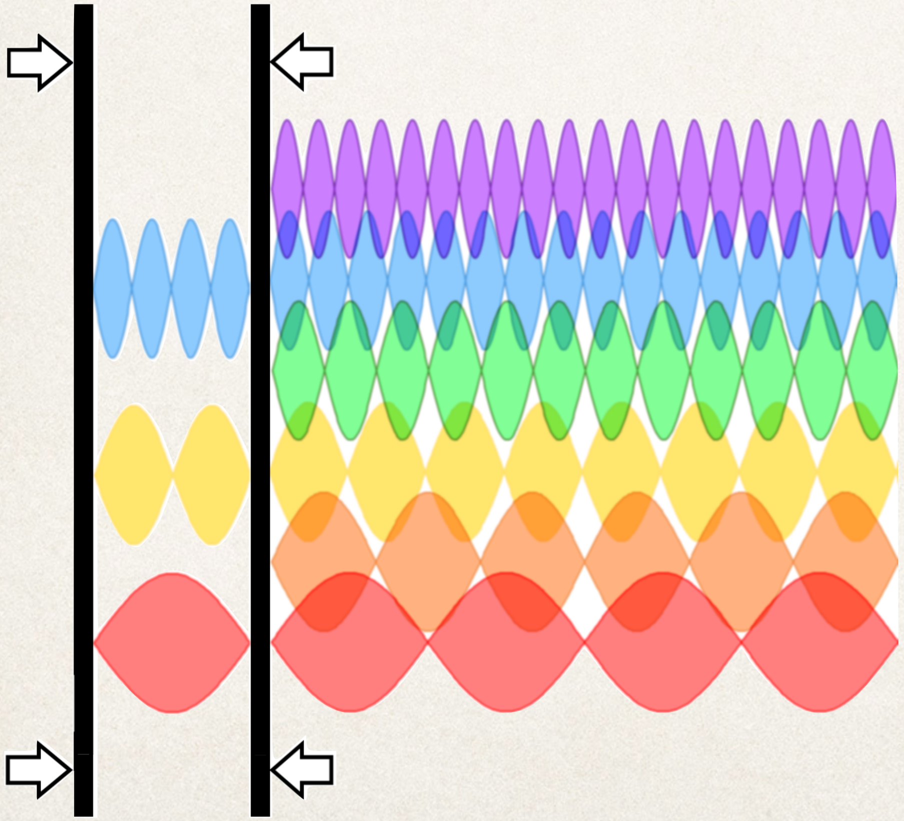
Università di Genova

INFN Istituto Nazionale di Fisica Nucleare

# Crumbs of semiclassical phenomena…

- Particle production in the early universe
- Hawking radiation



- Schwinger effect
- Casimir effect
- …

$$\mathscr{W} = \frac{1}{2} \ln \det \mathscr{O}$$

(with $\mathscr{O} = -\Delta + \mathrm{End}$)

$$\mathscr{W} = -\frac{1}{2}\mathrm{Tr}\int \frac{dt}{t} K(t; x, y; \mathcal{O})$$

$$\mathrm{Tr}\, K(t; x, y; \mathcal{O}) \underset{t\to 0^+}{\simeq} \sum_{k\geq 0} t^{\frac{k-n}{2}} a_k(\mathcal{O})$$

# Try this at home…

$$K[t; x, y; \mathcal{O} = -\partial^2 + m^2] = \frac{1}{(4\pi t)^{n/2}} \exp\left[-\frac{|x-y|^2}{4t} - tm^2\right]$$

$$K[t; x, y; \mathscr{O} = -\Box + \xi R] = \frac{\Delta_{VM}^{1/2}}{(4\pi t)^{n/2}} \exp\left[-\frac{\sigma(x, y)}{2t}\right] F[t; x, y]$$

where $F[t; x, y] = \sum_{k \geq 0} t^k f_k(x, y)$

## Examples of partially resummed HK expansions…

derivatives contributions up to
second and third order in the curvatures

**Barvinsky&Vilkovisky, NPB (1990)**

**Codello&Zanusso, JMathPhys (2013)**

**Barvinsky&Vilkovisky, NPB (1990)**

resummations
in abelian bundles, QED , symmetric spaces

**Avramidi&Fucci, CommMathPhys (2009)**

**Gusynin&Shovkovy, JMathPhys (1999)**

**Avramidi, JMathPhys (1996)**

$$K[t; x, y; \mathcal{O} = -\Box + \xi R] = \frac{\Delta_{VM}^{1/2}}{(4\pi t)^{n/2}} \exp\left[-\frac{\sigma(x,y)}{2t}\right] F[t; x, y]$$

where $F[t; x, y] = \exp\left[-tR(y)\left(\xi - 1/6\right)\right]\left(1 + \sum_{j\geq 1} t^j \tilde{f}_j(x,y)\right)$

**Parker & Toms, PRD (1985)**

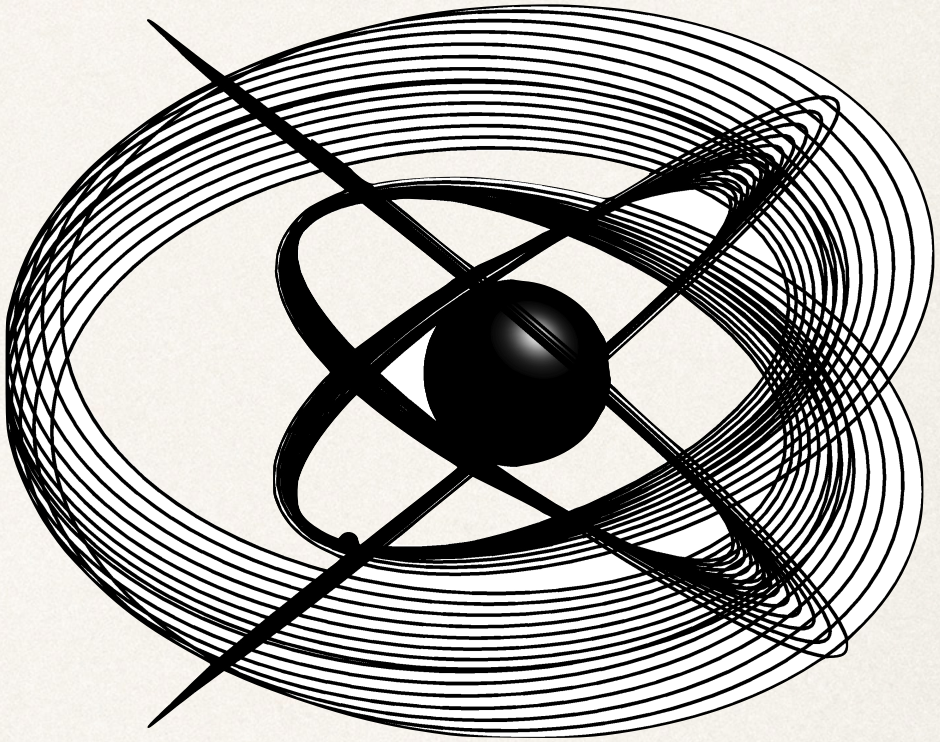
**Jack & Parker, PRD (1985)**

# Take home

Symmetry breaking mechanisms can be modified in curved spaces by effective masses of purely geometrical origin

## How does the interplay between strong interactions and geometry work?

BHs outskirts:curvature effects
comparable to $\Lambda_{QCD}$
(cf. Hawking-Moss picture )



$T_{BH} \sim 1/m_{BH}$

**Photons, neutrinos and gravitons…** → **…electrons…muons…** → **…pions and heavier hadrons**

# The *gourmet* recipe

NJL + Large $N$ approximation

+ Hubbard-Stratonovich transf.

$$\Gamma = -\int d^4x\sqrt{g}\left(\frac{\sigma^2}{2\lambda}\right) + \mathrm{Tr}\ln(i\gamma^\mu\nabla_\mu - \sigma)$$

with $\sigma[r] \equiv -\frac{\lambda}{N}\bar{\psi}\psi$

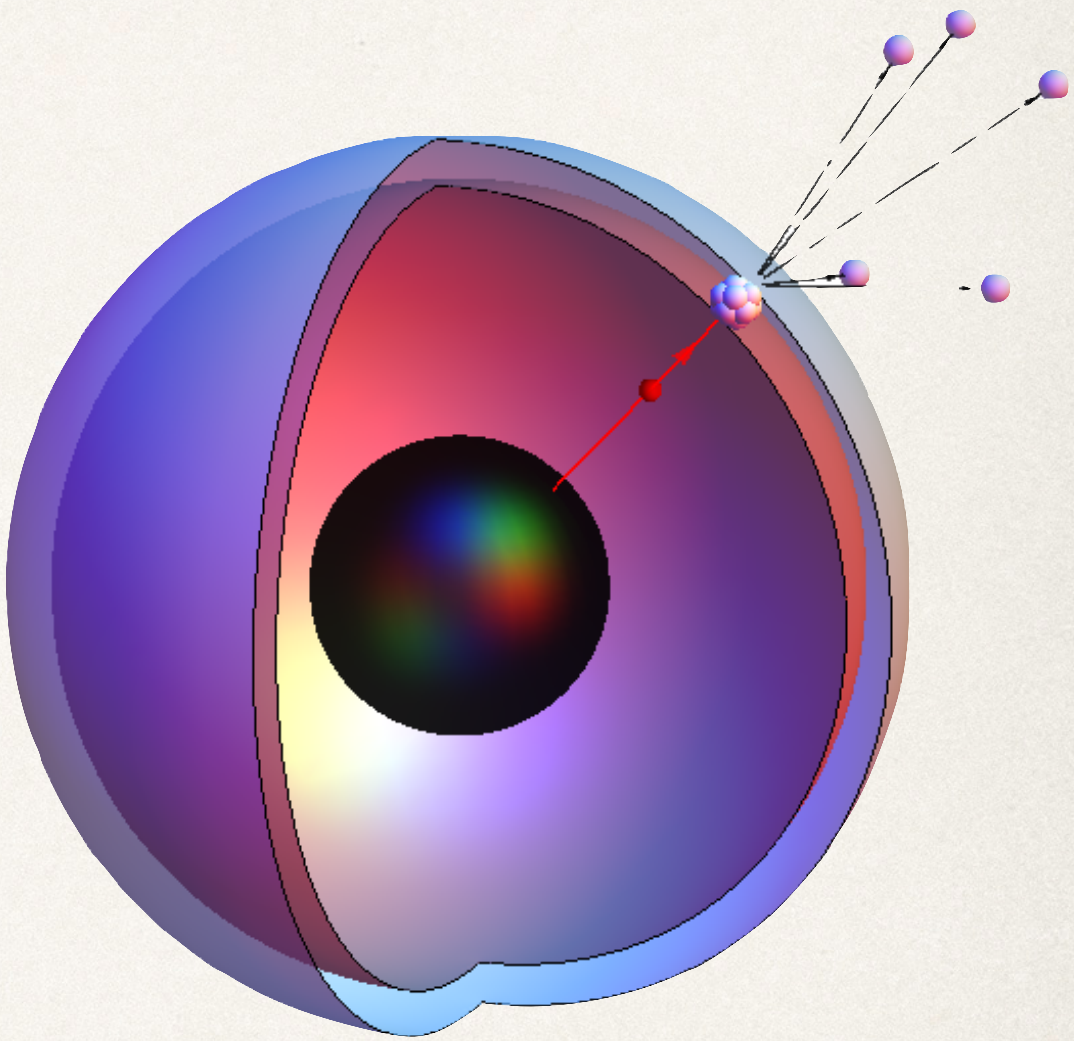
Flachi, PRL (2013)

Flachi & Fukushima, PRL (2014)

Flachi, Fukushima & me, PRL (2015)

# The *chiral gap effect*

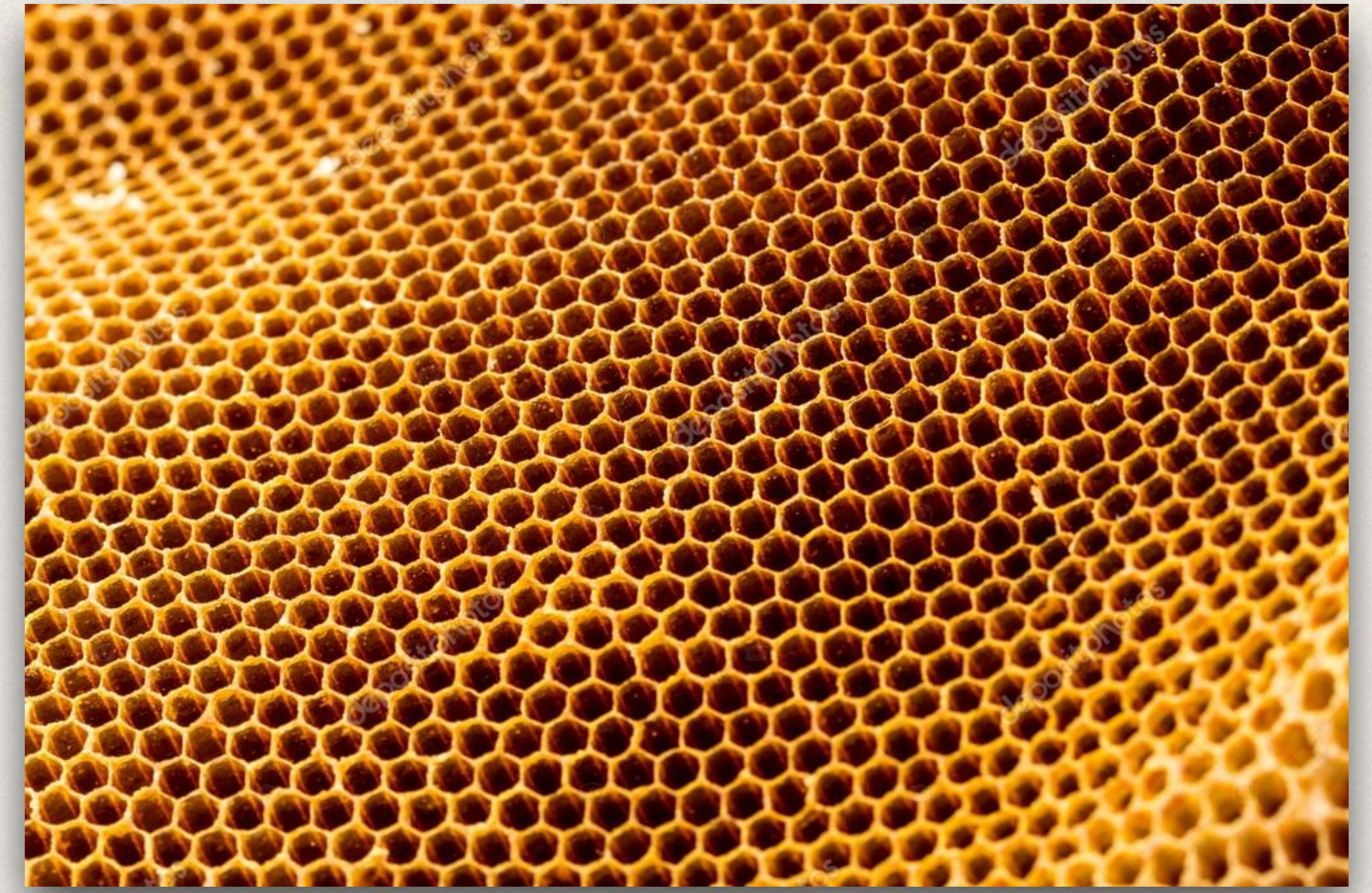
QCD phase diagram is in principle much more complicated…

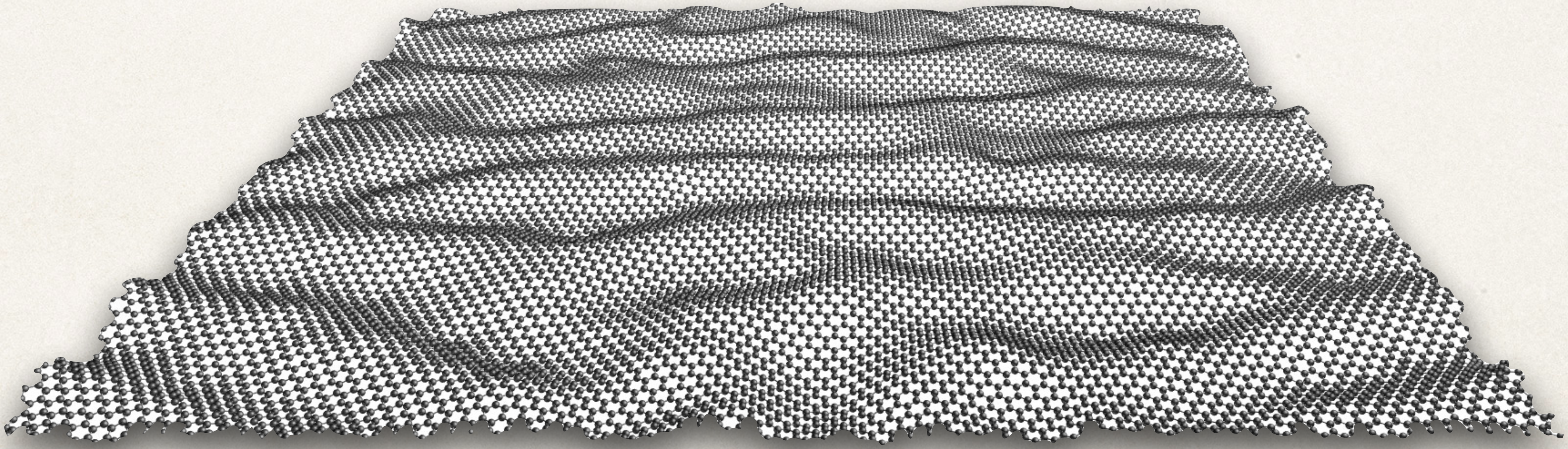


**Flachi, PRL (2013)**

**Flachi & Fukushima, PRL (2014)**

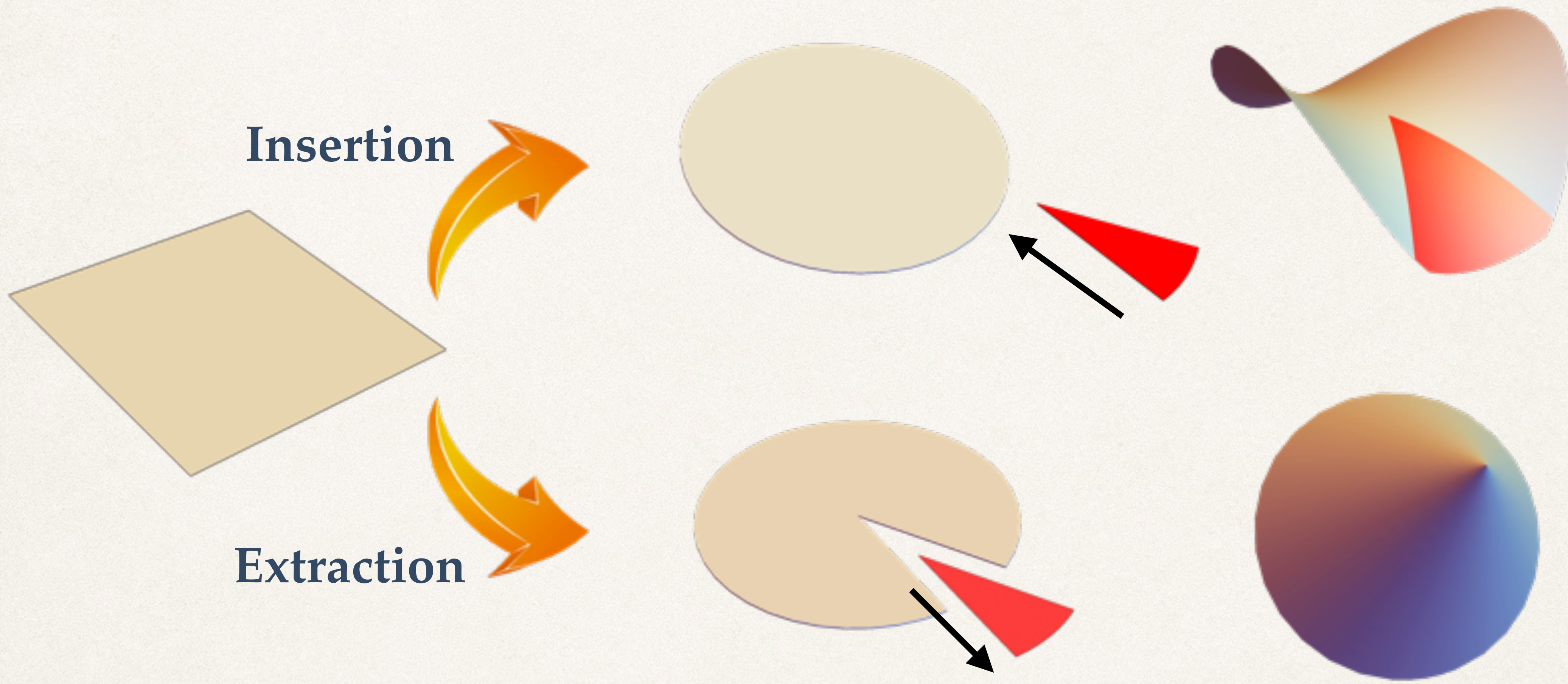
**Flachi, Fukushima & me, PRL (2015)**





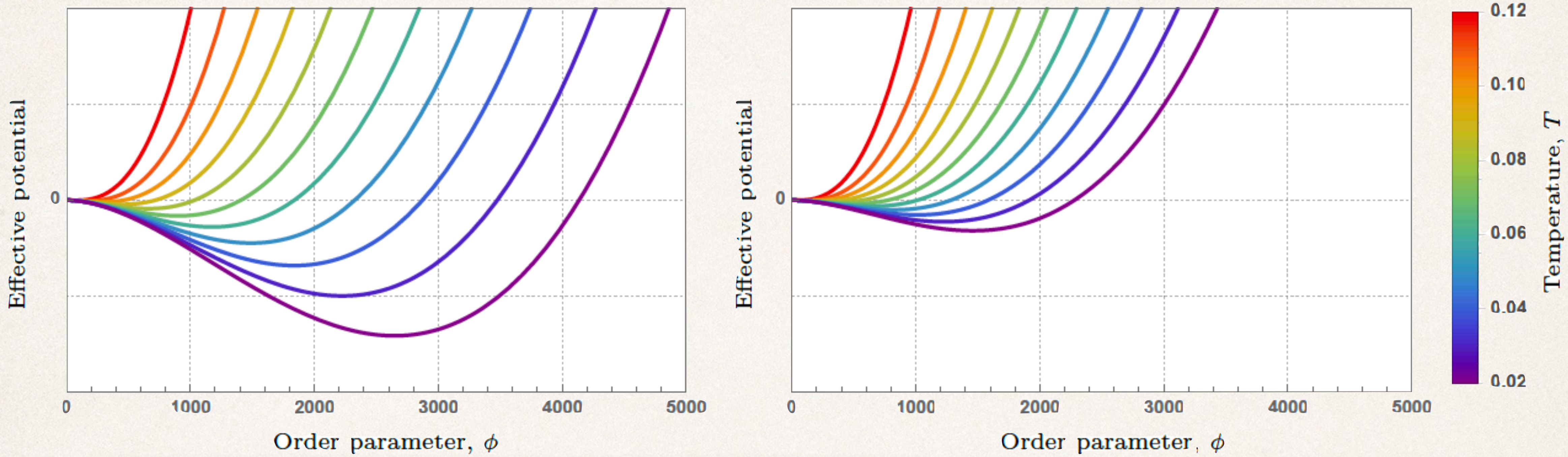
Fasolino et al, Nature Materials (2011)

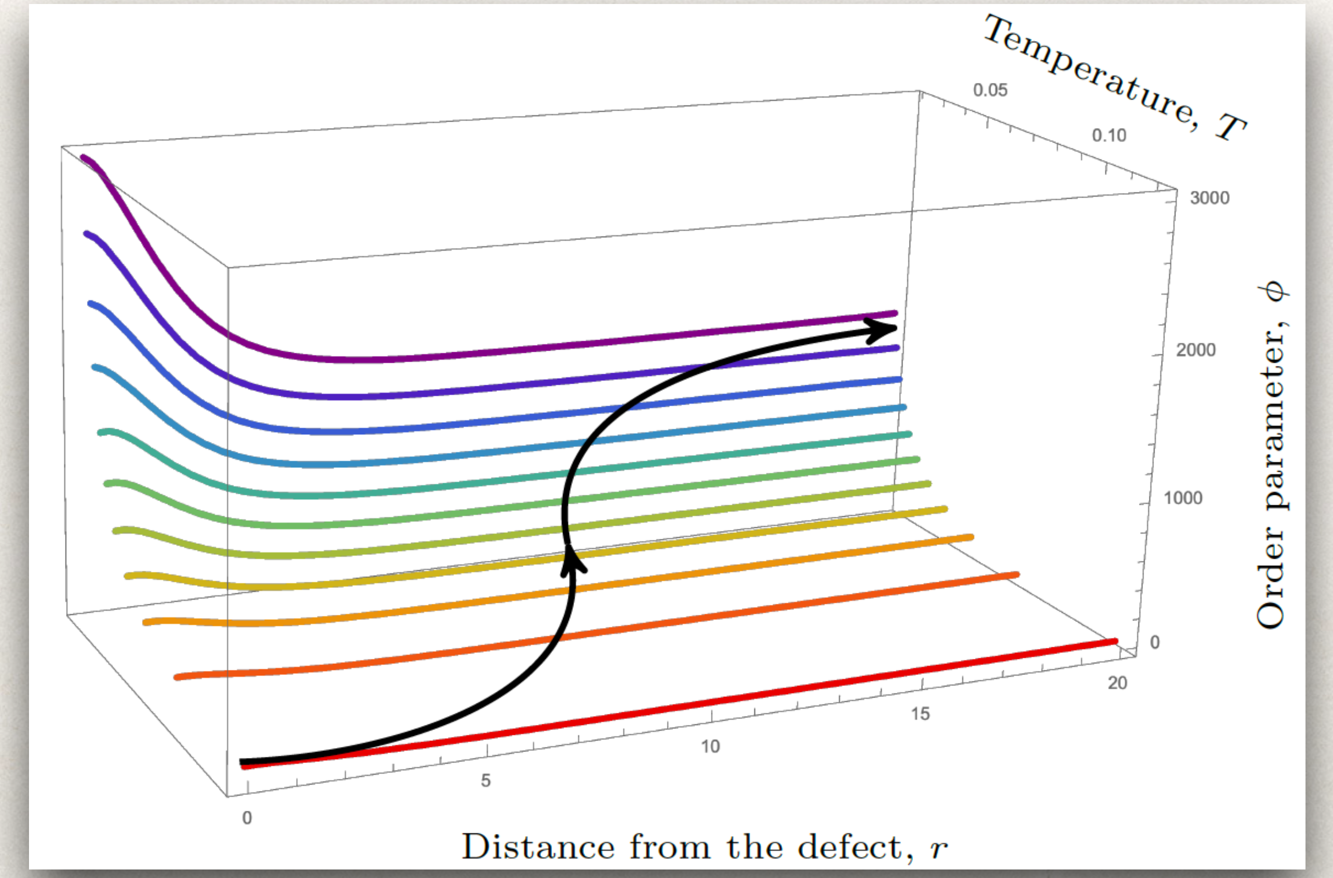
# 切り紙




Castro, Flachi, Ribeiro & me, PRL (2018)
Flachi & me, PRD (2019)

# Results




**Castro, Flachi, Ribeiro & me, PRL (2018)**

**Flachi & me, PRD (2019)**

…back for a moment to resummed HK's…

$$\mathcal{L}^{(1)}_{\text{scalar}} = \int_0^\infty \frac{d\tau}{\tau} \det{}^{1/2}\left(\frac{e\tau F^\mu_{\ \nu}}{\sinh(e\tau F^\mu_{\ \nu})}\right)\bar{g}(x,\tau)$$

$$\mathcal{L}^{(1)}_{\text{spinor}} = \int_0^\infty \frac{d\tau}{\tau} \det{}^{1/2}\left(\frac{e\tau F^\mu_{\ \nu}}{\sinh(e\tau F^\mu_{\ \nu})}\right)\text{tr}[e^{\imath e\tau F_{\mu\nu}\sigma^{\mu\nu}/2}]\bar{h}(x,\tau)$$

**Brown & Duff, PRD (1975)**

**Gusynin & Shovkovy, JMathPhys (1999)**

**Navarro-Salas & Pla, PRD (2021)**

# Derivative resummations: a proof

$$K[\tau; x, x; \mathcal{O}] = \frac{e^{-\tau V + \nabla^\alpha V \left[\gamma^{-3}\left(\gamma\tau - 2\tanh(\gamma\tau/2)\right)\right]_{\alpha\beta} \nabla^\beta V}}{(4\pi\tau)^{d/2} \det^{1/2}\left((\gamma\tau)^{-1}\sinh(\gamma\tau)\right)} \Omega(x, x; \tau)$$

$[\gamma^2_{\mu\nu} = 2\,\nabla_{\mu\nu} V]$

¡¡¡works for Yukawa, SQED, QED, “axial” QED!!!

**Franchino-Viñas, Garcia-Perez, Mazzitelli, Wainstein & me, PLB (2024)**

**Franchino-Viñas, Garcia-Perez, Mazzitelli, Pla & me, in preparation**

# Some food for thought…

## A scalar Schwinger effect

$$\Gamma_E = -\int d^d x \int_0^\infty \frac{d\tau}{\tau} K[\tau; x, x]$$

$$\Gamma_M = -\int d^d x \int_0^\infty \frac{d\tau}{\tau} \det{}^{-1/2}[(\tilde{\gamma}_\epsilon \tau)^{-1} \sinh[\tilde{\gamma}_\epsilon \tau]] \cdot \text{"reg"}$$

# Avenues of Quantum Field Theory in Curved Spacetime IV

January 22-24, 2025

University of Tours

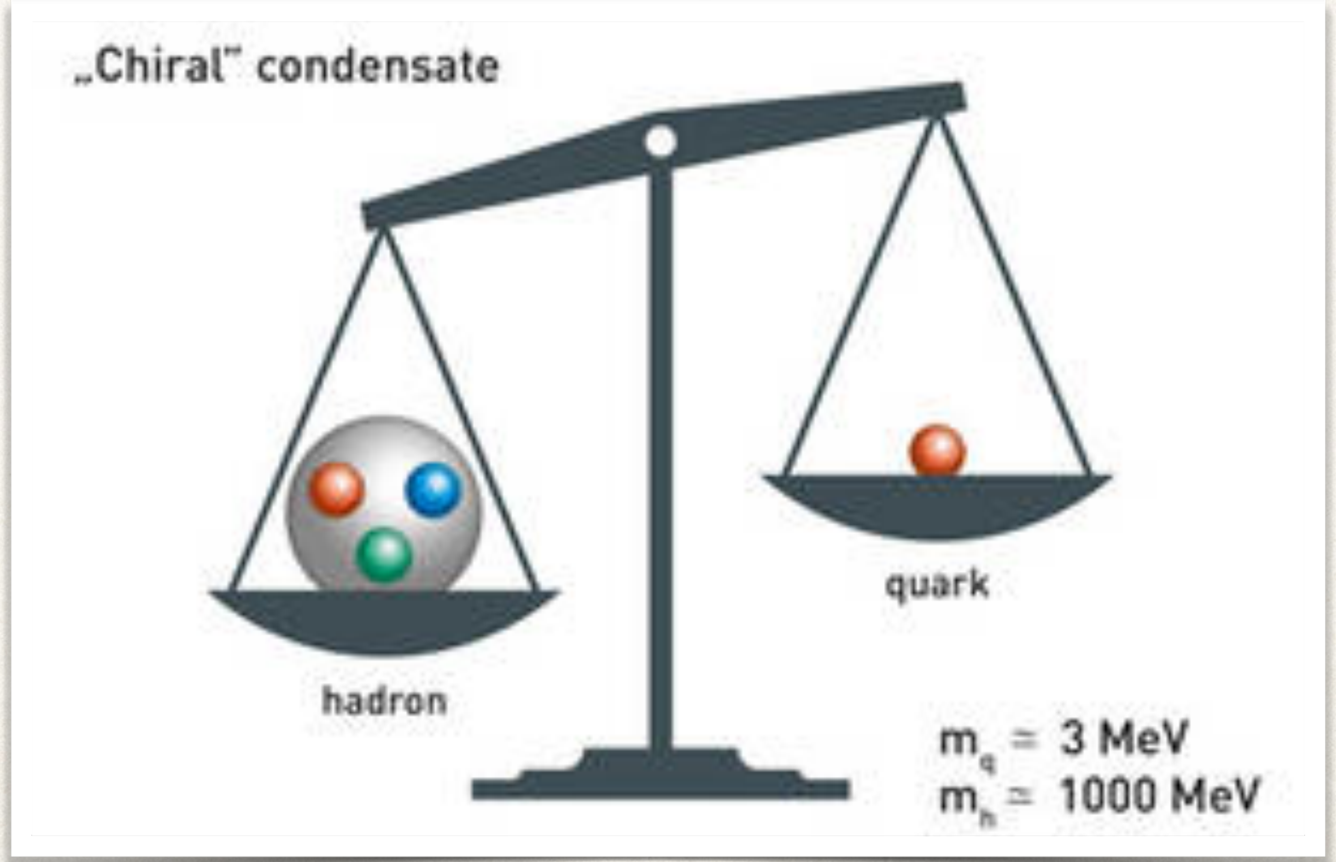
***Organizers***: M Chernodub (Tours U), O Corradini (UMoRe), A Flachi (Keio U), J V Rocha (ISCTE-IUL), D Trancanelli (UMoRe), V Vitagliano (UniGe)

# Effective field theory models

Massive fermions: spontaneously broken symmetry

$$S_{\rm NJL} = \int d^4x \sqrt{g} \left\{ \bar{\psi} i \gamma^\mu \nabla_\mu \psi + \frac{\lambda}{2N} (\bar{\psi}\psi)^2 \right\}$$

Generation dynamical effective mass $M_{\rm eff} \sim \langle \bar{\psi}\psi \rangle$



# The *chiral gap effect*

In the chiral limit $M_{\rm eff} \sim \langle\bar\psi\psi\rangle$; more in general $M_{\rm eff}$ solves the gap equation, *viz.* minimizes the grand potential $\Omega[M_{\rm eff}] \sim$ tree level+quantum

$$\Omega_{\rm loop}[\sigma] = \frac{N}{2}\ln\det\left[\Box + \sigma^2 + \frac{R}{4} + f\sigma'\right]$$

Taking $g_{tt} = 1$ and using the heat kernel expansion

$$\mathrm{Tr}_{\rm space}\exp^{-t\cdot(-\partial_t^2-\Delta+\sigma^2+R/4+f\sigma')} = \frac{1}{(4\pi t)^2}\exp^{-t\,(\sigma^2+R/12+f\sigma')}\sum_k \mathrm{Tr}\,a_k t^k$$

**Flachi, PRL (2013)**

**Flachi & Fukushima, PRL (2014)**

**Flachi, Fukushima & me, PRL (2015)**

# Boundary conditions calculation

$$\imath\bar{\psi}\gamma^\mu\nabla_\mu\psi + \frac{\lambda}{2\mathcal{N}}(\bar{\psi}\psi)^2 = \imath\psi'^\dagger A^\dagger\gamma^0\gamma^\mu\nabla_\mu(A\psi') + \frac{\lambda}{2\mathcal{N}}(\psi'^\dagger A^\dagger\gamma^0 A\psi')^2 =$$

$$= \imath\psi'^\dagger A^\dagger\gamma^0\gamma^\mu(\nabla_\mu A)\psi' + \imath\psi'^\dagger\gamma^0\gamma^\mu\nabla_\mu\psi' + \frac{\lambda}{2\mathcal{N}}(\overline{\psi'}\psi')^2 =$$

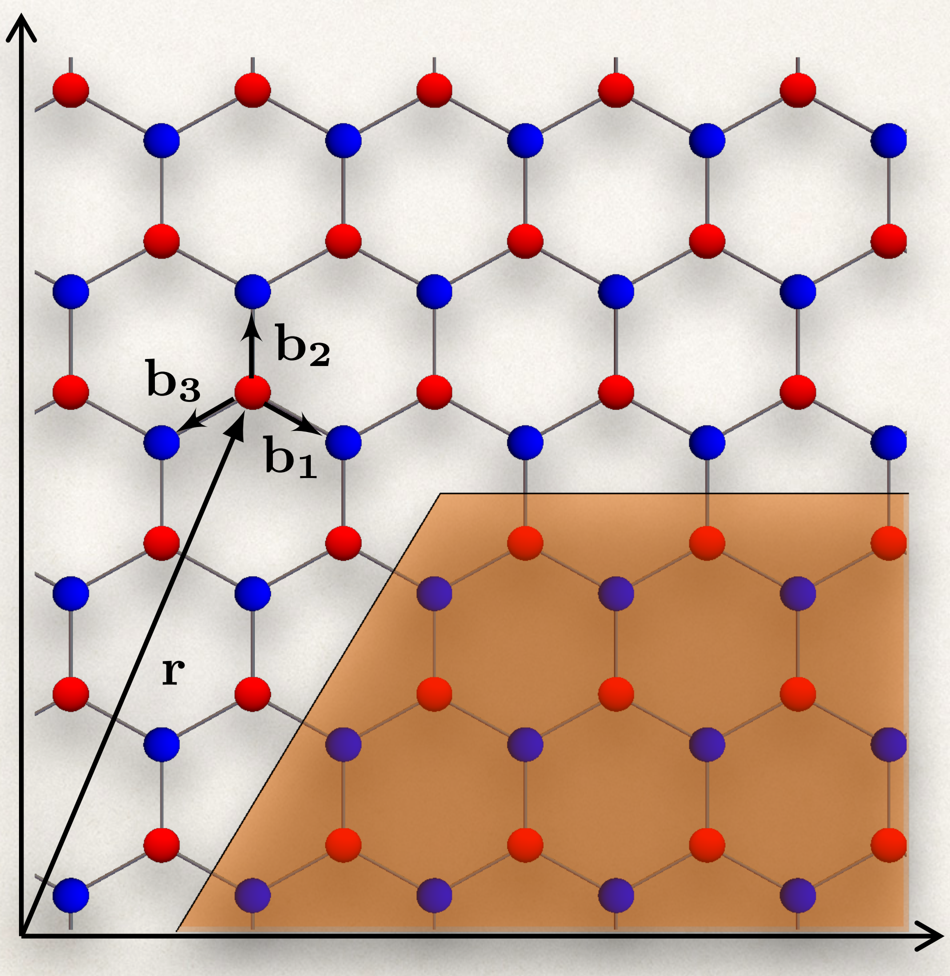
$$= \imath\psi'^\dagger\gamma^0\gamma^\mu A^\dagger(\nabla_\mu A)\psi' + \imath\overline{\psi'}\gamma^\mu\nabla_\mu\psi' + \frac{\lambda}{2\mathcal{N}}(\overline{\psi'}\psi')^2 =$$

$$= \imath\overline{\psi'}\gamma^\mu\left(-\imath\delta^\phi_\mu\frac{N_d}{4}R\right)\psi' + \imath\overline{\psi'}\gamma^\mu\nabla_\mu\psi' + \frac{\lambda}{2\mathcal{N}}(\overline{\psi'}\psi')^2$$

$$\imath\overline{\psi'}\gamma^\mu\left(\nabla_\mu - \imath\mathcal{B}_\mu\right)\psi' + \frac{\lambda}{2\mathcal{N}}(\overline{\psi'}\psi')^2 \equiv \imath\overline{\psi'}\gamma^\mu\mathcal{D}_\mu\psi' + \frac{\lambda}{2\mathcal{N}}(\overline{\psi'}\psi')^2$$

# Hubbard model

$$\mathsf{H} = -t \sum_{\mathbf{r},\ i,\ \sigma=\pm} u^\dagger_\sigma(\mathbf{r}) v_\sigma(\mathbf{r}+\mathbf{b}_i) + \text{H.C.} + \frac{U}{4} \sum_{\mathbf{r},\sigma,\sigma',i} \left(n_\sigma(\mathbf{r}) n_{\sigma'}(\mathbf{r}) + n_\sigma(\mathbf{r}+\mathbf{b}_i) n_{\sigma'}(\mathbf{r}+\mathbf{b}_i)\right)$$



# HS transformation

$$H = -t\sum_{\langle i,j\rangle\sigma}(c_{i\sigma}^{\dagger}c_{j\sigma} + \text{H.c.}) + U\sum_{j} n_{i\uparrow}n_{i\downarrow}$$

$$n_{i\uparrow}n_{i\downarrow} = \frac{\rho_i^2}{4} - (S_i^z)^2$$

$$\rho_i = n_{i\uparrow} + n_{i\downarrow}$$

$$S_i^z = \tfrac{1}{2}\sum_{\sigma} c_{i\sigma}^{\dagger}\sigma_z c_{i\sigma}$$

$$e^{U\sum_i n_{i\uparrow}n_{i\downarrow}} = \int \prod_i \frac{d\phi_i\, d\Delta_i\, d^2\mathbf{n}_i}{4\pi^2 U} \exp\sum_i\left(\frac{\phi_i^2}{U} + i\phi_i\rho_i + \frac{\Delta_i^2}{U} - 2\Delta_i \mathbf{n}_i\cdot\mathbf{S}_i\right)$$

$$Z = \int \prod_i \frac{dc_i^{\dagger}\, dc_i\, d\phi_i\, d\Delta_i\, d^2\mathbf{n}_i}{4\pi^2 U} \exp\left(-\int_0^{\beta} L(\tau)\right)$$

$$L(\tau) = \sum_{i\sigma} c_{i\sigma}^{\dagger}\partial_{\tau} c_{i\sigma} - t\sum_{\langle i,j\rangle\sigma}(c_{i\sigma}^{\dagger}c_{j\sigma} + \text{H.c.}) + \sum_i\left(\frac{\phi_i^2}{U} + (i\phi_i - \mu)\rho_i + \frac{\Delta_i^2}{U} - 2\Delta_i \mathbf{n}_i\cdot\mathbf{S}_i\right)$$

# Hubbard model

Bosonization

$$\mathscr{L} = \bar{\psi}_\sigma \imath \partial\!\!\!/ \psi_\sigma + \left(\sigma \bar{\psi}_\sigma \phi \psi_\sigma\right) + \frac{\phi^2}{2\lambda} \;; \qquad \sigma = \pm$$

$$\psi_\sigma^T = \left(\psi_\sigma^{A1}, \psi_\sigma^{B1}, \psi_\sigma^{A2}, \psi_\sigma^{B2}\right)$$

$$\psi_\sigma^{IJ} = \int d^2p\, e^{-\imath \mathbf{p}\cdot\mathbf{r}} z_\sigma^{IJ}(\mathbf{p})$$

**e.g. Weng et al, PLB[R] (1990); Schultz, PRL (1990)**

# The metric

A conical metric…

$$ds^2 = d\tau^2 + dr^2 + \alpha^2 r^2 d\varphi^2$$

…and its regularisation

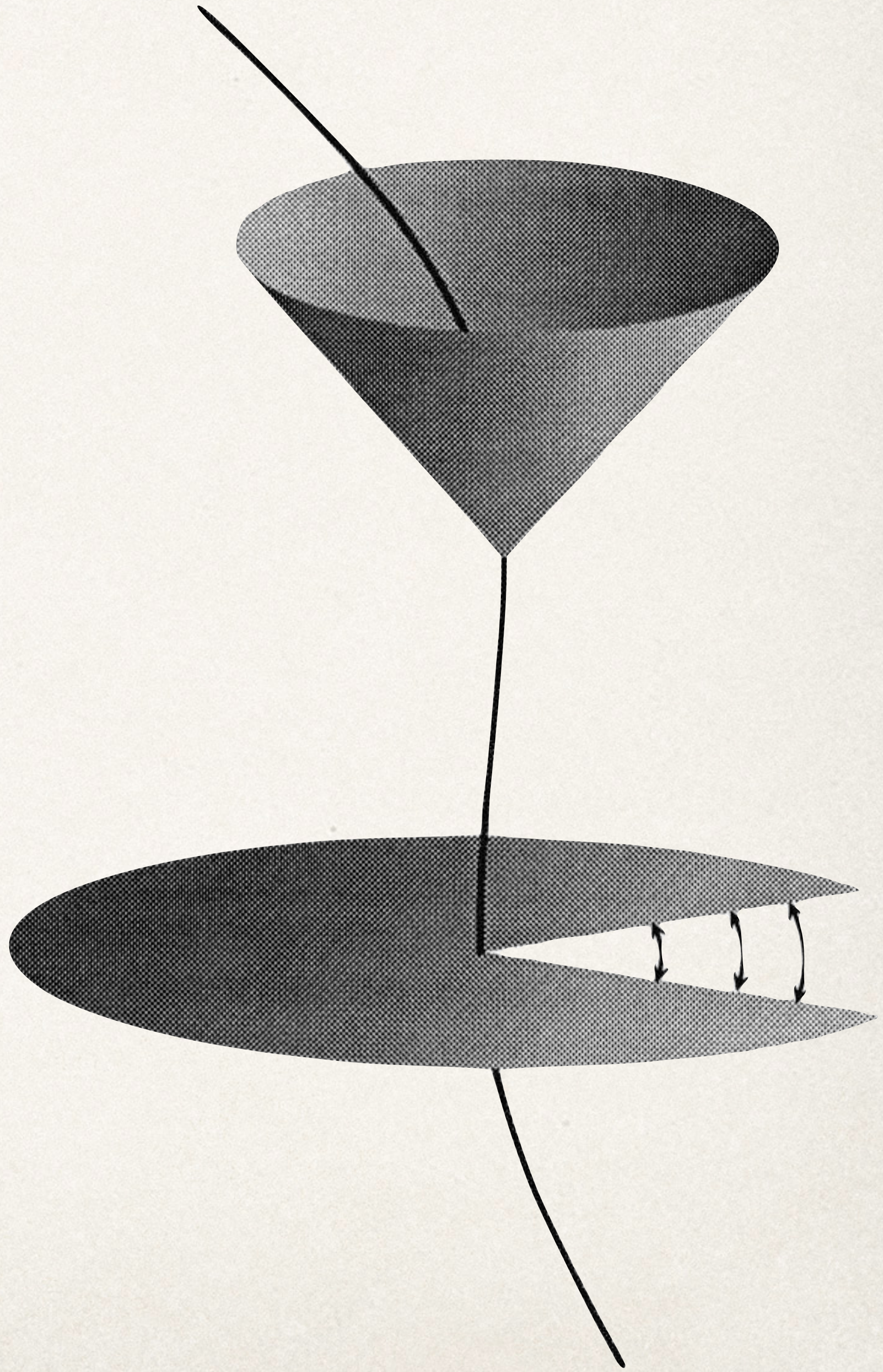
$$d\tilde{s}^2 = d\tau^2 + f_\epsilon(r) dr^2 + \alpha^2 r^2 d\varphi^2$$

With

1) $\lim_{\epsilon \to 0} f_\epsilon(r) = 1;$

2) $f_\epsilon(r) \approx 1$ for $r \gg \epsilon;$

3) $f_\epsilon(r) = \text{const}$ for $r = 0$



# Boundary conditions

Circulating the defect…

$$\psi(r,\varphi+2\pi) \;=\; -\exp\left(i(6-n_s)\pi\gamma_5/2\right)\psi(r,\varphi)$$

Redefinition of the field

$$\psi'(r,\varphi) = \exp\left(-i\varphi(6-n_s)\gamma_5/4\right)\psi(r,\varphi)$$

$$\Longrightarrow \qquad \mathscr{A}_\mu = -\delta_\mu^\varphi(6-n_s)\gamma_5/4$$

**Sitenko&Vlasii, NPB (2007)**

# The effective action

$$\tilde{\Gamma}\left[\phi\right] = -\int d^3x \sqrt{\tilde{g}}\frac{\phi^2}{2\lambda} + \mathrm{Tr}\log\left(\imath\gamma^\mu \tilde{D}_\mu \pm \phi\right)$$

$$\tilde{\Gamma}\left[\phi\right] = -\int d^3x \sqrt{\tilde{g}}\frac{\phi^2}{2\lambda} + \frac{1}{2}\sum_{p=\pm} \log\det\left(\tilde{\Box} + \frac{\tilde{R}}{4} + \phi^2 \pm \sqrt{\tilde{g}^{rr}}\phi'\right)$$

**Castro, Flachi, Ribeiro & me, PRL (2018)**

**Flachi & me, PRD (2019)**