# A Monte Carlo method for stationary solutions of general-relativistic Vlasov systems

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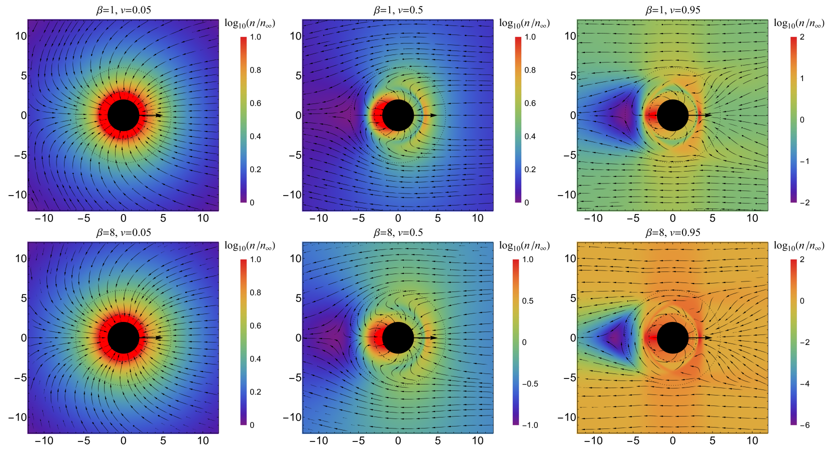
- General-relativistic kinetic theory provides an alternative to hydrodynamical (or magnetohydrodynamical) models when collisions between particles are rare and the mean free path is large.
- Kinetic models are usually obtained by solving the Vlasov equation (collisionless case).
- In many cases the solution can be obtained formally. The difficulty lies in computing observable quantities, usually in a form of integrals over momenta. This requires a good knowledge of the regions in the phase space available for the motion.
- In this talk — an attempt to compute observable quantities using Monte Carlo techniques. We approximate a continuous distribution function by a sample of particles moving along their trajectories.
- Difficulties: Selecting an appropriate sample of trajectories. Introducing appropriate averaging (coarse-graining) scheme.
- Particular examples: *Stationary accretion models in the Schwarzschild spacetime.*
- Stationary solutions — no time dependence of trajectories is needed.




Schwarzschild black hole moving through a cloud of gas (PM and Odrzywołek 2021, 2022)

Let $(M, g)$ be a spacetime manifold. The cotangent bundle of $M$:

$$T^*M = \{(x, p) \colon x \in M,\ p \in T^*_x M\}.$$

The one-particle distribution function: $\mathcal{F} \colon T^*M \supseteq U \to [0, +\infty)$. Let $S$ denote a 3-dimensional spacelike hypersurface in $M$, and let $s$ be a future-directed unit vector normal to $S$. An averaged number of particle trajectories in $U$, whose projections on $M$ intersect $S$:

$$\mathcal{N}[S] = -\int_S \mathcal{J}_\mu s^\mu \eta_S, \quad \mathcal{J}_\mu(x) = \int_{P_x^+} \mathcal{F}(x, p) p_\mu \mathrm{dvol}_x(p),$$

where

$$P_x^+ = \{p \in T^*_x M \colon g^{\mu\nu} p_\mu p_\nu < 0,\ p \text{ is future-directed}\}$$

and $\eta_S$ denotes the volume element on $S$. The volume element on $P_x^+$ is given (in local adapted coordinates) by

$$\mathrm{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)}\, dp_0 dp_1 dp_2 dp_3.$$

The energy-momentum tensor:

$$\mathcal{T}_{\mu\nu}(x) = \int_{P_x^+} p_\mu p_\nu \mathcal{F}(x,p)\mathrm{dvol}_x(p).$$

The Hamiltonian of a particle traveling along a timelike geodesic:

$$H(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu.$$

Here $(x^\mu, p_\mu)$ are understood as canonical variables. Momenta are defined by $p^\mu = dx^\mu/d\tau$. The motion of the particle is described by the Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}.$$

We require that $H = -\frac{1}{2}m^2$, where $m$ denotes the rest-mass of the particle.

The Vlasov gas of noncolliding particles is described by the probability function $\mathcal{F} = \mathcal{F}(x^\mu, p_\nu)$, which should be invariant along a geodesic:

$$\frac{d}{d\tau}\mathcal{F}(x^\mu(\tau), p_\nu(\tau)) = \frac{dx^\mu}{d\tau}\frac{\partial\mathcal{F}}{\partial x^\mu} + \frac{dp_\nu}{d\tau}\frac{\partial\mathcal{F}}{\partial p_\nu}$$
$$= \frac{\partial H}{\partial p_\mu}\frac{\partial\mathcal{F}}{\partial x^\mu} - \frac{\partial H}{\partial x^\nu}\frac{\partial\mathcal{F}}{\partial p_\nu} = \{H, \mathcal{F}\} = 0,$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. I will refer to the above relation as to the Vlasov equation. In more explicit terms, it can be written as

$$g^{\mu\nu}p_\nu\frac{\partial\mathcal{F}}{\partial x^\mu} - \frac{1}{2}p_\alpha p_\beta\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\frac{\partial\mathcal{F}}{\partial p_\mu} = 0.$$

The Vlasov equation implies that $\nabla_\mu\mathcal{J}^\mu = 0$ and $\nabla_\mu\mathcal{T}^{\mu\nu} = 0$.
An excellent modern introduction to the general-relativistic kinetic theory: Acuña-Cárdenas, Gabarrete, and Sarbach (2022).

*Monte Carlo simulation.* Replace the continuous system by a discrete one:

$$\mathcal{F}^{(N)}(x^\mu, p_\nu) = \sum_{i=1}^{N} \int \delta^{(4)}\left(x^\mu - x^\mu_{(i)}(\tau)\right) \delta^{(4)}\left(p_\nu - p^{(i)}_\nu(\tau)\right) d\tau,$$

where $\left(x^\mu_{(i)}(\tau), p^{(i)}_\nu(\tau)\right)$ correspond to trajectories of individual particles. This gives

$$\begin{aligned} \mathcal{J}^{(N)}_\mu(x) &= \int \mathcal{F}^{(N)}(x,p) p_\mu \sqrt{-\det g^{\alpha\beta}(x)} dp_0 \dots dp_3 \\ &= \sum_{i=1}^{N} \int \delta^{(4)}\left(x^\alpha - x^\alpha_{(i)}(\tau)\right) p^{(i)}_\mu(\tau) \sqrt{-\det g^{\alpha\beta}(x)} d\tau. \end{aligned}$$

We will estimate the value of $\mathcal{J}_\mu$ at a point $x$, by selecting a (small) region $\sigma$ (a cell) in a hypersurface $\Sigma$, such that $x \in \sigma$, and computing

$$\langle \mathcal{J}_\mu(x) \rangle = \left( \int_\sigma \eta_\Sigma \right)^{-1} \int_\sigma \mathcal{J}^{(N)}_\mu \eta_\Sigma.$$

*Some justification.* Suppose that $\mathcal{F}^{(N)}$ tends to a smooth distribution function $\mathcal{F}$ in the sense that

$$\int dx^0 \dots dx^3 dp_0 \dots dp_3 \frac{\mathcal{F}^{(N)}(x,p) - \mathcal{F}(x,p)}{N} \phi(x,p)$$
$$= \int \sqrt{-\det g_{\alpha\beta}(x)} dx^0 \dots dx^3 \sqrt{-\det g^{\delta\kappa}(x)} dp_0 \dots dp_3$$
$$\times \frac{\mathcal{F}^{(N)}(x,p) - \mathcal{F}(x,p)}{N} \phi(x,p) \to 0,$$

as $N \to \infty$, where $\phi(x,p)$ is a test function on $T^*\mathcal{M}$. Choosing a suitable $\phi(x,p)$, we get

$$\int_\sigma \frac{\mathcal{J}_\mu^{(N)}(x)}{N} \eta_\Sigma \to \int_\sigma \frac{\mathcal{J}_\mu(x)}{N} \eta_\Sigma = \frac{\mathcal{J}_\mu(x_0)}{N} \int_\sigma \eta_\Sigma,$$

where $x_0 \in \sigma$. The last equality follows from the mean value theorem (for each vector component separately). The formula for $\langle \mathcal{J}_\mu(x) \rangle$ follows directly.

*Examples: Stationary Bondi-type accretion in the Schwarzschild spacetime (e.g. Rioseco and Sarbach 2017).*
Asymptotic conditions:

$$\mathcal{F} = A\delta(\sqrt{-p_\mu p^\mu} - m_0) f_0,$$

where

$$f_0 = \delta(p_t + E_0) \quad \text{or} \quad f_0 = \exp(p_t/(k_B T)),$$

and where $T$ denotes the asymptotic temperature.
For a *planar* model (gas restricted to the equatorial plane) set

$$\mathcal{F} = A\delta(\sqrt{-p_\mu p^\mu} - m_0)\delta(\theta - \pi/2)\delta(p_\theta) f_0.$$

For planar models we define the particle current surface density $J_\mu$ by

$$\mathcal{J}_\mu(t, r, \theta, \varphi) = \frac{1}{r}\delta(\theta - \pi/2) J_\mu(t, r, \varphi).$$

In all cases solutions can be obtained exactly.

For a planar stationary accretion flow in the Schwarzschild spacetime we take a segment

$$S = \{(r, \theta, \varphi) \colon r_1 \leq r \leq r_2,\ \theta = \pi/2,\ \varphi = \varphi_0\}$$

a surface

$$\tilde{\Sigma} = \{(t, r, \theta, \varphi) \colon t \in \mathbb{R},\ r_1 \leq r \leq r_2, \theta = \pi/2,\ \varphi = \varphi_0\}.$$

and a cell $\tilde{\sigma} \subset \tilde{\Sigma}$,

$$\tilde{\sigma} = \{(t, r, \theta, \varphi) \colon t_1 \leq t \leq t_2,\ r_1 \leq r \leq r_2, \theta = \pi/2,\ \varphi = \varphi_0\}.$$

Let $\Phi_\tau(x_0^i)$ denote the orbit of the timelike Killing vector field $\xi^\mu = (1, 0, 0, 0)$, passing through $x_0^i$ at $\tau = 0$, i.e., $\Phi_0(x_0^i) = x_0^i$. Then $\Sigma$ can be expressed as the image

$$\tilde{\sigma} = \Phi_{[t_1, t_2]}(S).$$

The particle current surface density can be now approximated as

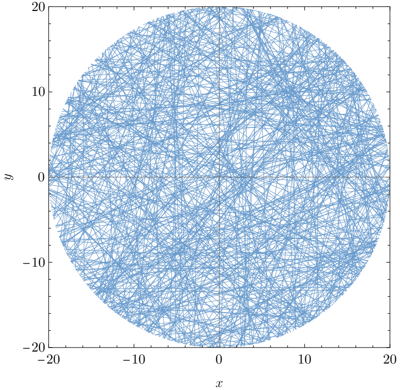
$$\langle J_\mu \rangle \quad = \quad \frac{1}{(t_2 - t_1)(r_2 - r_1)} \sum_{i=1}^{N_{\rm int}} \frac{p_\mu^{(i)} r_{(i)}}{\left| p_\varphi^{(i)} \right|},$$

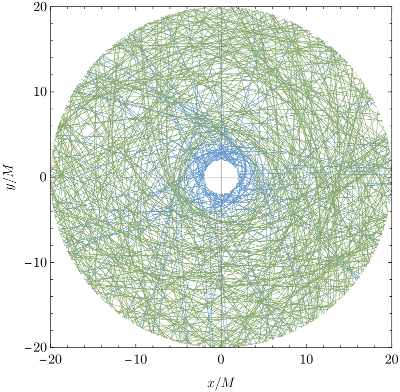
where the index $i$ numbers all *intersections* of trajectories with the segment $S$.

For stationary problems, the result should be independent of the choice of $t_1$ and $t_2$ in a sense that the number of trajectories that intersect $\Sigma$ should be proportional to the length $t_2 - t_1$, if the latter is sufficiently large. In practice, we omit the factor $t_2 - t_1$ and normalize the results by the number of trajectories taken into account.

- How to select the appropriate distribution of geodesics?
- Start with a homogeneous distribution of gas within a 2D plane in the Minkowski spacetime.
- For practical reasons, a Monte Carlo simulation has to be restricted to a compact region in space, say a disk of a radius $r_0$. The problem we are facing turns out to be a variation of a the classic Bertrand problem.
- Bertrand's paradox is usually formulated as follows. On a fixed circle, one randomly selects a chord. What is the probability that the length of this chord is larger than the side of an equilateral triangle inscribed in this circle. It is then shown that different methods of selecting the chord "at random" lead to different answers (different probabilities).

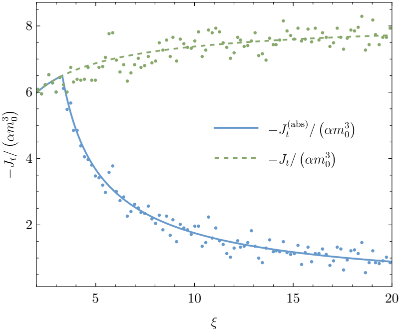
- Selecting chords belonging to straight lines uniformly distributed in a plane. In Cartesian coordinates $(x, y)$: $ux + vy + 1 = 0$. The probability distribution
$$F(u, v)dudv = (u^2 + v^2)^{-\frac{3}{2}}dudv$$
is invariant under the group of Euclidean symmetries on the plane—rotations and translations (Kendall and Moran 1963). But: $u$ and $v$ not well suited to a description of chords within a given circle.
- Parametrizing the chords by polar coordinates $(p, \theta)$ of the point on the chord with a smallest distance to the center of the circle:
$$(u^2 + v^2)^{-\frac{3}{2}}dudv = dpd\theta.$$
Given that $dudv = pdpd\theta$, we have $F(p, \theta) = p^{-1}$.
- In the context of Bertand's paradox this distribution has already been proposed by Poincaré in Calcul des probabilités, 1912.
- In our context (monoenergetic particles): $p \sim |p_\varphi|$.

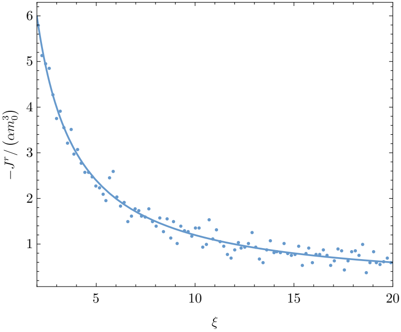





Left: Randomly distributed straight lines on a plane. Right: Randomly distributed monoenergetic geodesics in an equatorial plane of the Schwarzschild spacetime. Angular momenta are uniformly distributed. Absorbed orbits are depicted in blue. Scattered orbits are plotted in green.

*Example: Planar model, monoenergetic particles.*



*Spherically symmetric solutions.* Repeating the calculation as in the planar case for

$$\Sigma \quad = \quad \{(t, r, \theta, \varphi) \colon t_1 \le t \le t_2,\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2,\ \varphi = \varphi_0\},$$

we get the Monte Carlo estimator

$$\langle \mathcal{J}_\mu \rangle \quad = \quad \frac{1}{(t_2 - t_1)(r_2 - r_1)(\theta_2 - \theta_1)} \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} \sin \theta_{(i)}}{l_{(i)} \cos \iota_{(i)}},$$

where we have assumed that $r_2 - r_1 \ll 1$, and, consequently, $\frac{1}{2}(r_1 + r_2) \approx r_{(i)}$. Here $\iota_{(i)}$ refers to the inclination of the orbital plane, and $l_{(i)}$ is the total angular momentum defined as

$$l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}.$$

The key difference with respect to the planar case is related to the selection of geodesics. For a three dimensional distribution angular momenta should be selected with the probability measure $\propto l dl$. To see this, consider uniformly distributed straight lines in $\mathbb{R}^3$. Kendall and Moran (1963) parametrize such lines by

$$x = az + p, \tag{1a}$$

$$y = bz + q, \tag{1b}$$

where $(x, y, z)$ denote Cartesian coordinates. The appropriate probability measure, invariant with respect to Euclidean rotations and translations, is given by

$$(1 + a^2 + b^2)^{-2} da db dp dq.$$

Consider a plane perpendicular to line (1) and passing through the center of the coordinate system $O$. Denote the intersection of this plane and line (1) by $P$. Let $(\delta, \phi)$ be the polar coordinates (in this plane) of $P$. The distance $\delta = \|\overrightarrow{OP}\|$ is given by

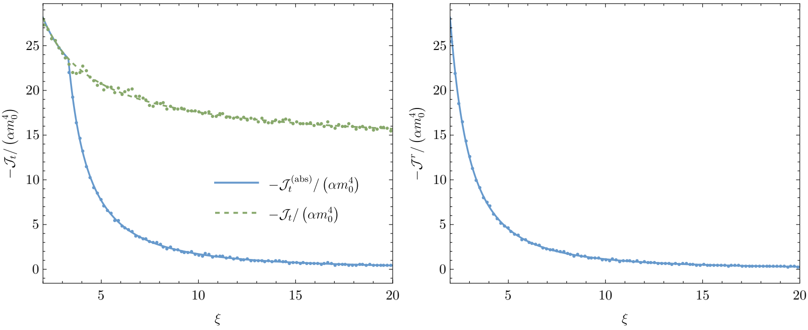
$$\delta = \sqrt{\frac{(1+b^2)p^2 - 2abpq + (1+a^2)q^2}{1+a^2+b^2}}.$$

Let $n^i$ denote a unit vector along (1). Denote the solid angle element obtained by varying $n^i$ (or the parameters $a$ and $b$) by $d\Omega$. Then

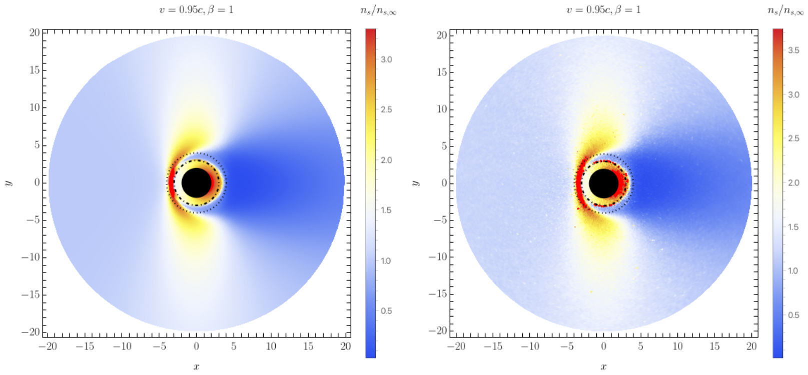
$$(1+a^2+b^2)^{-2} da\,db\,dp\,dq = \delta\, d\delta\, d\Omega.$$

Returning to the Schwarzschild (asymptotically flat case), we recall that $l \propto \delta$.

*Example: Spherically symmetric stationary accretion on the Schwarzschild black hole. Monoenergetic particles.*



*Planar accretion onto a moving black hole. A boosted Maxwell-Jüttner distribution at infinity.*




The particle surface density ratio $n_s/n_{s,\infty}$. Left: exact result. Right: Monte Carlo simulation with 168 486 945 particle trajectories.

*Summary.* A method of finding stationary solutions of the general-relativistic Vlasov equation using Monte Carlo techniques:

- Learn to compute trajecories in a given spacetime efficiently.
- Select parameters of trajectories corresponding to the required (asymptotic) conditions on the distribution function.
- Count intersections of trajectories with appropriate timelike surfaces. Count them with correct weights!

*Outlook:*

- More complicated flows on Schwarzschild spacetime (e.g., moving black holes, PM, Odrzywołek, 2021, 2022).
- Vlasov gas in the Kerr spacetime (Cieślik, PM, Odrzywołek 2022, Rioseco, Sarbach, 2018, 2023).
- Electromagnetic fields?