Greybody factors of d-dimensional Gauss-Bonnet black holes

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Perturbations on the $(d-2)$-sphere

- For a system with **spherical symmetry** (metric and all other fields);
- Metric of the type
  $ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,d\Omega^2_{d-2};$
- $a,\, b = r,\, t;\;\; i, j, k = 1, \ldots, d-2$
- General tensors of rank at least 2 on the $(d-2)$-sphere can be uniquely decomposed in their **tensorial, vectorial** and **scalar** components;
- Key point: gauge-invariant perturbation equations can be reduced to decoupled single master equations of the Schrödinger type for any kind of perturbations in this kind of background (Ishibashi, Kodama).

The Master Equations

- Each perturbation variable obeys a "master equation"

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial t^2} =: V\Phi.$$

- $d\,x/d\,r = 1/f$ ("tortoise" coordinate);
- $\Phi(x,t) = e^{i\omega t}\psi(x)$; "master" variable - multipole expansion in terms of a **multipole number** ℓ;
- V : potential;
- both Φ and V depend on the type of gravitational perturbations or field considered.
- This is also valid in the presence of **quadratic** (Gauss-Bonnet) corrections (Moura, Dotti-Gleiser)!

Higher derivative corrections

- Effective action in d dimensions:

$$\frac{1}{16\pi G_d}\int\sqrt{-g}\left[\mathcal{R}+\alpha\left(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}^2\right)\right]\mathrm{d}^d x;$$

- Asymptotically flat spherically symmetric solution (Boulware-Deser 1985):

$$\begin{aligned}
f(r) &= 1+\frac{r^2}{\alpha(d-3)(d-4)}\left(1-q(r)\right),\\
q(r) &= \sqrt{1+\frac{4\alpha(d-3)(d-4)\boldsymbol{\mu}}{(d-2)r^{d-1}}},\\
\boldsymbol{\mu} &= \frac{(d-2)R_H^{d-3}}{2}\left(1+\alpha\frac{(d-3)(d-4)}{2R_H^2}\right),\\
M &= \frac{(d-2)\Omega_{d-2}}{8\pi G_d}\boldsymbol{\mu}.
\end{aligned}$$

The corrected potentials

$$
\begin{aligned}
V_{\mathsf{T}}[f(r)] &= f(r)\left[\frac{\ell\,(\ell+d-3)}{r^2}\left(3-\frac{B(r)}{A(r)}\right)+K(r)\left(\frac{d^2K}{dr^2}(r)+\frac{df}{dr}(r)\frac{dK}{dr}(r)\right)\right], \\
V_{\mathsf{V}}[f(r)] &= f(r)\left[\frac{(d-2)c}{r^2}A(r)+K(r)\left(\frac{d^2K}{dr^2}(r)+\frac{df}{dr}(r)\frac{dK}{dr}(r)\right)\right], \\
A(r) &= \frac{1}{q(r)^2}\left(\frac{1}{2}+\frac{1}{d-3}\right)+\left(\frac{1}{2}-\frac{1}{d-3}\right), \\
K(r) &= \frac{1}{\sqrt{r^{d-2}A(r)q(r)}}, \\
B(r) &= A(r)^2\left(1+\frac{1}{d-4}\right)+\left(1-\frac{1}{d-4}\right), \\
c &= \frac{\ell\,(\ell+d-3)}{d-2}-1.
\end{aligned}
$$

(Dotti-Gleiser 2005).

The corrected potentials

$$
\begin{aligned}
V_{\mathsf{S}}[f(r)] &= \frac{f(r)U(r)}{64r^2(d-3)^2A(r)^2q(r)^8(4cq(r)+(d-1)R(q(r)^2-1))^2}, \\
R(r) &= \frac{r^2}{\alpha(d-3)(d-4)}, \\
U(r) &= 5(d-1)R(r)^2(R(r)+1) - 3(d-1)^5R(r)q(r)\left(24c(R(r)+1)+(d-1)R(r)^2\right) + \\
&2(d-1)^4q(r)^2\left(168c^2(R(r)+1)+24c(d-1)R(r)^2-(d-1)R(r)^2(7d(R(r)+1)\right. \\
&+2(d-1)^4R(r)q(r)^3\left(c(84d(R(r)+1)+44R(r)-84)-184\,2c+(d-1)(d+13)\right. \\
&+(d-1)^3\left(384c^3-48c((3d-5)d+2)R(r)^2+192c^2\left((d-15)R(r)^2+d-11\right)+\right. \\
&+(d-1)R(r)^2(d(7d(R(r)+1)+106R(r)+26)-3(55R(r)+7)))\Big)\,q(r)^4 + \\
&+(d-1)^3R(r)\left(-64c^2(d-38)+(d-1)((7d-90)d+71)R(r)^2+\right. \\
&+16c\left(13d^2(R(r)+1)-2d(81R(r)+73)+255R(r)+303\right)\Big)\,q(r)^5 + \\
&+4(d-1)^2\left(96c^3(d-7)-8c(d-1)\left(6d^2-74d+145\right)R(r)^2-\right. \\
&-8c^2(d(11d(R(r)+1)-34R(r)-58)-175R(r)+9)+(d-1)R(r)^2(-5(23R(r) \\
&+d(d(7d(R(r)+1)-89R(r)-81)+5(41R(r)+57))))q(r)^6-
\end{aligned}
$$

The corrected potentials

$$-4(d-1)^2R(r)\left(8c^2(d(72-13d)+43)+(d-1)(d(d(5d-49)+99)-63)+R(r)^2+\right.$$
$$+4c(d(d(17d(R(r)+1)-107R(r)-123)-39R(r)+121)+465R(r)+321))q(r)^7+$$
$$+(d-1)\left(128c^3(d-9)(d-5)+32c(d-1)(d(d(8d-55)+9)+246)R(r)^2+\right.$$
$$+64c^2(d-5)\left(d^2+((d-4)d+49)R(r)-3\right)-$$
$$-(d-1)R(r)^2(d(d(d(45d(R(r)+1)-452R(r)-548)+6(217R(r)+393))-$$
$$\left.-4(349R(r)+997))+565R(r)+1173)\right)q(r)^8+$$
$$+(d-1)R(r)\left(-64c^2(d-5)(d(3d-13)+36)+(d-1)(d(3d(d(9d-92)+294)-1204)+\right.$$
$$-8c(d-5)(d(d((d-79)R(r)+d-47)+191R(r)+127)+31R(r)+63))q(r)^9+$$
$$+2d-5\left(64c^3(d-5)(d-3)+8c(d-1)(d((d-43)d+141)-27)R(r)^2+\right.$$
$$+8c^2(d-5)(d((d-18)R(r)+d-2)+77R(r)-3)+(d-1)^2R(r)^2(-33(R(r)-7)+$$
$$+d(d(9d(R(r)+1)-35R(r)-59)+43R(r)+59)))q(r)^{10}-$$
$$-2d-5R(r)\left(24c^2(d-11)(d-5)(d-3)+(d-1)^2(d((7d-39)d+81)-65)R(r)^2+\right.$$
$$+12c(d-7)(d-5)(d-3)(d-1)(R(r)+1))q(r)^{11}+$$
$$+(d-5)^2(d-1)R(r)^2q(r)^{12}(16c((d-9)d+26)+(d-1)(d((d-2)R(r)+d-18)-3R(r)$$
$$+(d-5)^2(d-3)^2(d-1)^2R(r)^3q(r)^{13}.$$

Large black hole limit

$$
\begin{aligned}
\lambda &:= \frac{\alpha}{\boldsymbol{\mu}^{\frac{2}{d-3}}} \ll 1; \\
f(r) &= f_0(r)(1+\lambda\delta f(r)), \\
f_0(r) &= 1 - \frac{2\boldsymbol{\mu}}{(d-2)r^{d-3}}, \\
\delta f(r) &= \frac{2(d-4)(d-3)}{(1-2\boldsymbol{\mu}r^{3-d}(d-2)^{-1})(d-2)^2} \frac{\boldsymbol{\mu}^{2\frac{d-2}{d-3}}}{r^{2d-4}}, \\
R_H &= \left(\frac{2\boldsymbol{\mu}}{d-2}\right)^{\frac{1}{d-3}} - 2^{\frac{1}{3-d}-1}(d-4)\lambda\left((d-2)\boldsymbol{\mu}\right)^{\frac{1}{d-3}}, \\
T_{\mathcal{H}} &= \frac{d-3}{4\pi}\left(\frac{d-2}{2\boldsymbol{\mu}}\right)^{\frac{1}{d-3}}\left(1-\lambda\frac{(d-4)(d-2)}{2}\right).
\end{aligned}
$$

Greybody factor

- Field equation is written in the Schrödinger form

$$\left[-\frac{d^2}{dx^2}+V\right]\psi(x)=\omega^2\psi(x).$$

- Hawking radiation spectrum:

$$\left\langle n(\omega)\right\rangle=\frac{\gamma(\omega)}{e^{\frac{\omega}{T_{\mathcal{H}}}}\pm 1};$$

- Greybody factor: $\gamma(\omega)$;
- real frequency - **emission** rate;
- imaginary frequency - **decay** rate.

Computing the greybody factor

- Hawking radiation transmitted and reflected by the black hole potential:

$$\psi(x) \sim T(\omega)e^{i\omega x}, r \to R_H^+,$$

$$\psi(x) \sim e^{i\omega x} + R(\omega)e^{-i\omega x}, r \to +\infty;$$

- $T(\omega), R(\omega)$: **transmission** and **reflection** coefficients.
- Complex frequency: must also consider $\omega \leftrightarrow -\omega$;
- $\widetilde{R}(\omega) = R(-\omega)$, $\widetilde{T}(\omega) = T(-\omega)$;
- $\gamma(\omega) = T(\omega)\widetilde{T}(\omega)$: transmission probability of the Hawking radiation emitted from black hole;
- Asymptotically flat spacetimes: $R(\omega)\widetilde{R}(\omega) + \gamma(\omega) = 1$.

Operational problem

- Outgoing wave at infinity ($V = 0$): exponentially small and exponentially large terms.
- Solution - analytic continuation to the complex r-plane.
- Highly damped regime: $|\mathrm{Im}\,(\omega)| \gg |\mathrm{Re}\,(\omega)|$;
- **Stokes line**:

$$\mathrm{Im}\,(\omega x) = 0 \Rightarrow \mathrm{Re}\,(x) = 0$$

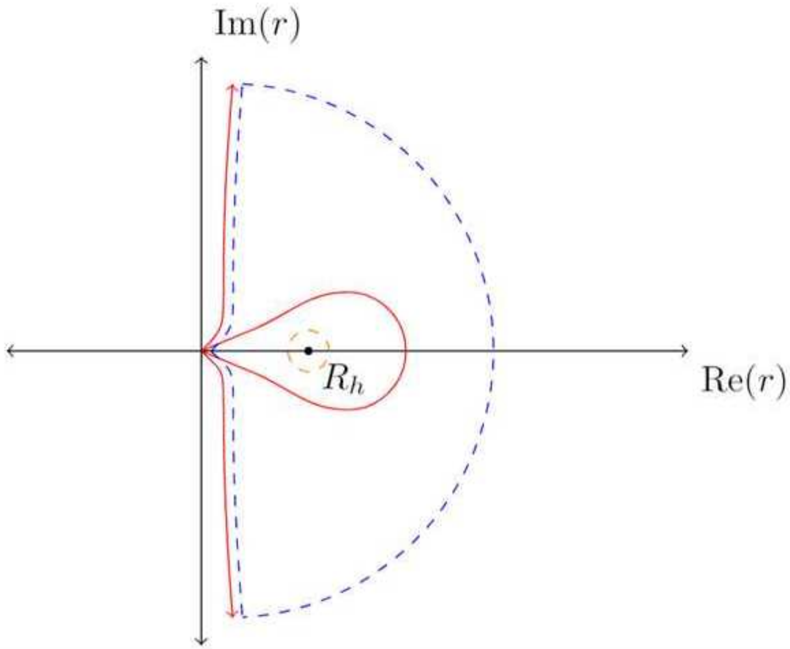
.

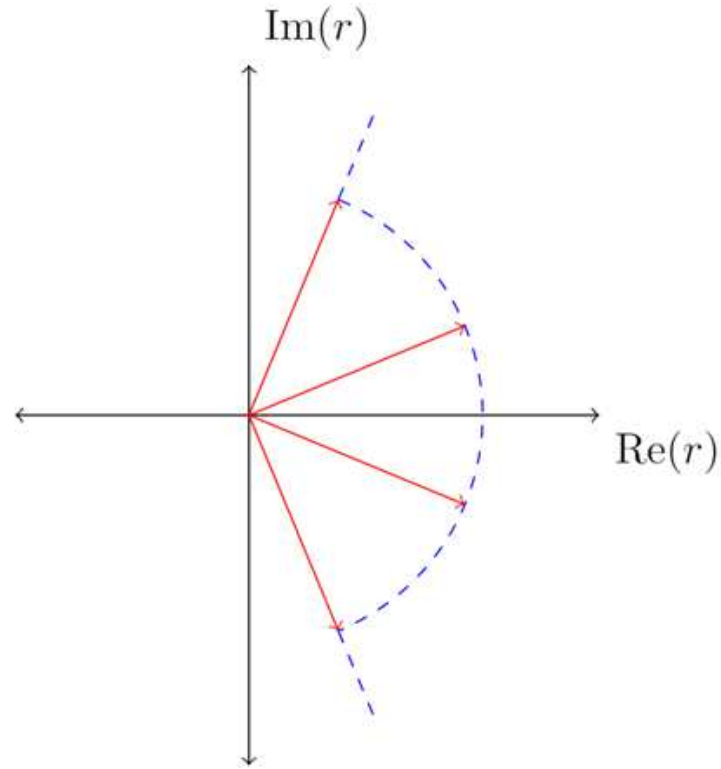
- In a contour along a Stokes line, $|e^{\pm i\omega x}| = 1$: the asymptotic behavior of $e^{\pm i\omega x}$ is always oscillatory.
- Imposing the boundary condition at infinity does not pose a problem in this case.

Monodromy method

- We pick two closed homotopic curves in the complex r-plane, enclosing the event horizon. We consider the monodromy of the perturbation associated with a full loop around these curves.
- In one monodromy, we encode the information of the boundary condition in the event horizon. In the other one, we encode the information of the boundary condition in spatial infinity.
- **Monodromy theorem**: homotopic curves share the same monodromy.
- Equating the monodromies allows us to solve for $R(\omega), \widetilde{R}(\omega)$ (Neitzke 2003, Harmark-Natário-Schiappa 2007).

The contours and Stokes lines





Schematic depiction of the small (orange) and big (blue) contours. The orange contour is to be interpreted as arbitrarily close to R_H. Some Stokes lines are depicted by red curves.

Perturbative approach

$$\psi(z) = \psi_0 + \lambda\psi_1;$$

$$V(z) = V_0(z) + \lambda V_1(z);$$

$$dz = \frac{dr}{f_0(r)}, \;\; x \mapsto z;$$

$$\frac{d^2\psi_0}{dz^2}(z) + (\omega^2 - V_0(z))\psi_0(z) = 0;$$

$$\frac{d^2\psi_1}{dz^2}(z) + (\omega^2 - V_0(z))\psi_1(z) = \xi(z);$$

$$\xi = \xi_1\frac{d^2\psi_0}{dz^2} + \xi_2\frac{d\psi_0}{dz} + \xi_3\psi_0;$$

$$\xi_1(r) = -2\delta f(r), \;\; \xi_2(r) = -\frac{d(\delta f)}{dr}(r)f(r), \;\; \xi_3(r) = V_a^1(r).$$

Near the origin

- $V^0_{\mathsf{T}} = V^0_{\mathsf{S}} \neq V^0_{\mathsf{V}}$;
- **asymptotically** we also have $V^1_{\mathsf{T}} = V^1_{\mathsf{S}} \neq V^1_{\mathsf{V}}$.

$$\frac{d^2\psi_0}{dz^2} + \left(\omega^2 - \frac{j^2-1}{4z^2}\right)\psi_0 = 0, \;\; j = 0(\mathsf{T},\mathsf{S}), \; j = 2(\mathsf{V});$$

$$\psi_0(z) = A_+\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z) + A_-\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z).$$

- Procedure: solve the differential equation for arbitrary j; at the end take the limit $j \to 0$ or $j \to 2$.
- Boundary condition gives two algebraic conditions relating $A_+, A_-, R(\omega)$.
- Equating the monodromies gives extra condition and allows to solve for $A_+, A_-, R(\omega)$.

Near the origin

- **Variation of constants**:

$$\psi_1(z) = 2\pi\sqrt{\omega z}\left(J_{-\frac{j}{2}}(\omega z)\int\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\frac{\xi(z)}{W}dz - J_{\frac{j}{2}}(\omega z)\int\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\frac{\xi(z)}{W}dz\right)$$

$$W = \frac{d}{dz}\left(\sqrt{2\pi}J_{-\frac{j}{2}}(\omega z)\right)\sqrt{2\pi}J_{\frac{j}{2}} - \frac{d}{dz}\left(\sqrt{2\pi}J_{\frac{j}{2}}(\omega z)\right)\sqrt{2\pi}J_{-\frac{j}{2}} = -4\omega\sin\left(\frac{\pi j}{2}\right).$$

- $z \sim -\frac{1}{d-2}\frac{r^{d-2}}{R_H^{d-3}}$.
- We take a $\frac{3\pi}{d-2}$ rotation in the complex r-plane $\Rightarrow 3\pi$ rotation in the complex z-plane.
- We need to compute the changes of ψ_0 and ψ_1 under this rotation.

Closing the big contour

- Known asymptotic expansions of ψ agree with the WKB approximation!

$$\psi(z) = \psi_0(z) + \lambda' \psi_1(z) \sim \sum_{k=1}^{3} \left(\Omega_k^+ e^{i\omega z} + \Omega_k^- e^{-i\omega z} \right).$$

- As we abandon the Stokes line we can no longer rely on the coefficient multiplying the $e^{i\omega z}$ term.
- Small corrections become important given that $|e^{i\omega z}| \ll 1$.
- WKB theory tells us that the dominant term (proportional to $e^{-i\omega z}$) remains unchanged.
- Using this information we can finally close the contour and compute the monodromy of ψ.

Monodromy of the big contour

- Each $\Omega_k^{\pm}$ has a different form, because of the different coefficients multiplying each ξ_k, but always involving Gamma functions (depending on j and d) in the form

$$\mathcal{H}(m,n,k) := \frac{\Gamma\left(\frac{1}{2}-\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{m}{2}+\frac{n}{2}+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma\left(-\frac{k}{2}+\frac{m}{2}-\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{k}{2}+\frac{n}{2}-\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{k}{2}+\frac{m}{2}+\frac{n}{2}+\frac{1}{2}\right)}.$$

- The complete expression for each monodromy is a sum of 96 of these terms, all with different coefficients.
- After a huge amount of work, each monodromy can be simplified using the properties $\Gamma(x+1) = x\Gamma(x),\ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$.
- Rather surprisingly, because of taking $j=2$, despite $(\xi_{\mathsf{V}})_3 \neq (\xi_{\mathsf{T}})_3,\ (\xi_{\mathsf{S}})_3$, the monodromy is equal in all cases!

Monodromy of the small contour

- Near $r = R_H$, $V(z) \approx 0 \implies$ harmonic oscillator.
- Imposing the boundary condition,

$$z = r + \frac{1}{2} \sum_{n=0}^{d-4} \frac{\exp\left(\frac{2\pi i n}{d-3}\right)}{d-3} \log\left(1 - \frac{r}{R_H} \exp\left(-\frac{2\pi i n}{d-3}\right)\right).$$

- The tortoise has a branch point in the event horizon R_H from which we can compute the monodromy of ψ around the small contour.

Equating monodromies

$$\gamma_a(\omega) = 1 - R_a(\omega)\widetilde{R}_a(\omega) = (\gamma_0)_a(\omega)(1 + \lambda(\delta\gamma)_a(\omega)),$$

$$(\gamma_0)_a(\omega) = \frac{e^{\frac{\omega}{T_{\mathcal{H}}}} - 1}{3 + e^{\frac{\omega}{T_{\mathcal{H}}}}},$$

$$(\delta\gamma)_a(\omega) = -\frac{4}{3 + e^{\frac{\omega}{T_{\mathcal{H}}}}} \left[\frac{3}{16}(d-4)(d-2)\frac{\omega}{T_{\mathcal{H}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}} \left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}} \varrho_a \right],$$

$$\varrho_a = \mathrm{e}^{-\frac{2\pi i}{d-2}} \frac{\pi^{\frac{3}{2}}(d-2)^{-\left(\frac{1}{d-2}+1\right)}(d-4)((d-5)d+2)\Gamma\left(\frac{1}{2(d-2)}\right)}{(d-1)\Gamma\left(\frac{1}{2}+\frac{1}{2(d-2)}\right)^3} \sin\left(\frac{\pi}{2(d-2)}\right),$$

$a =$ tensorial, vectorial, scalar.

Because of the $\mathrm{e}^{-\frac{2\pi i}{d-2}}$ term the λ correction is **complex**.

Conclusions

- λ corrections depend strongly on the dimension d.
- In Einstein gravity, all types of gravitational perturbations give rise to the same greybody factor in the highly damped limit.
- The d-dimensional Gauss-Bonnet correction preserves this property up to first order in λ (perturbative regime).
- This result agrees with the isospectrality of quasinormal modes of all types of gravitational perturbations, for the same solution and same order in λ in the same limit (Moura-Rodrigues 2023).
- These corrections match the analogous result for tensorial gravitational perturbations of the Callan-Myers-Perry d-dimensional black hole in string theory.