Universidad de Valladolid

Propagation and emission of gravitational waves in $f(\mathcal{R})$ gravity within the Palatini formalism

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Introduction

Figure 1: VIRGO interferometer.

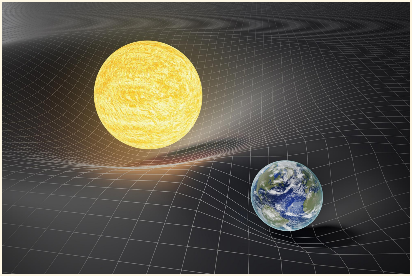

Figure 2: Artistic representation of GR.

Our aim

Can we tell apart Palatini $f(\mathcal{R})$ gravity from GR through GW detections?

Palatini $f(\mathcal{R})$ gravity

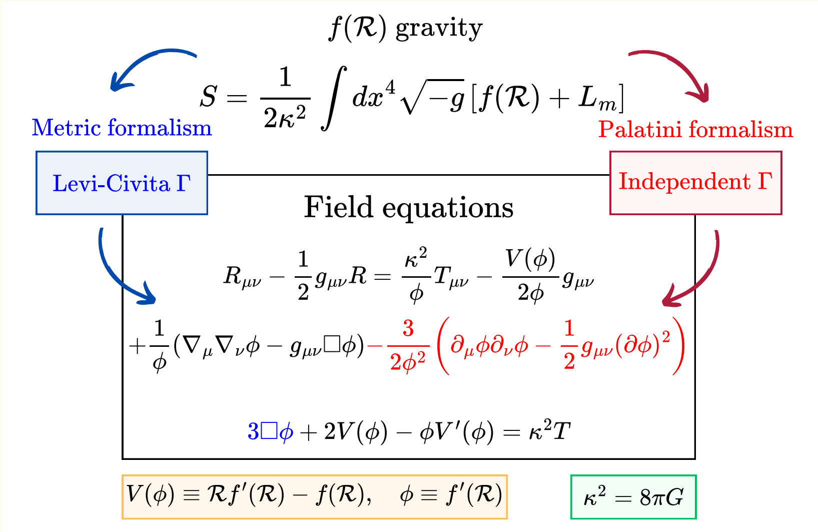

Figure 3: Main differences between metric $f(\mathcal{R})$ and Palatini $f(\mathcal{R})$ gravity.

Linearisation of the field equations

Perturbation theory

We assume a small perturbation and detach it from the background.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \quad \phi = \bar{\phi} + \delta\phi \ , \tag{1}$$

The background quantities satisfy the unperturbed field equations. Therefore, the perturbed field equation for ϕ reads:

$$\left(V'(\bar{\phi}) - \bar{\phi} V''(\bar{\phi}) \right) \delta\phi = \kappa^2 \delta T. \tag{2}$$

Linearisation of the field equations

The perturbed Einstein equations in the $f(\mathcal{R})$ Palatini formalism read:

$$\frac{1}{2}\Bigg(\nabla^\lambda\nabla_\mu h_{\lambda\nu} + \nabla^\lambda\nabla_\nu h_{\lambda\mu} - \Box h_{\mu\nu} - \nabla_\nu\nabla_\mu h + \bar{g}_{\mu\nu} - \bar{R}h_{\mu\nu}$$

$$\left(\Box h - \nabla_\alpha\nabla_\beta h^{\alpha\beta} + h^{\alpha\beta}\bar{R}_{\alpha\beta}\right)\Bigg) = \kappa^2\frac{\delta T_{\mu\nu}}{\bar{\phi}} - \kappa^2\frac{\bar{T}_{\mu\nu}}{\bar{\phi}}\frac{\delta\phi}{\bar{\phi}} + \frac{V'(\bar{\phi})}{2}g_{\mu\nu}\frac{\delta\phi}{\bar{\phi}}$$

$$+\frac{V(\bar{\phi})}{2\bar{\phi}}\left(h_{\mu\nu} + \bar{g}_{\mu\nu}\frac{\delta\phi}{\bar{\phi}}\right) - \frac{3}{2\bar{\phi}^2}\Bigg[\partial_\mu\bar{\phi}\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\bar{\phi}$$

$$-\frac{1}{2}h_{\mu\nu}(\partial\bar{\phi})^2 - \bar{g}_{\mu\nu}\partial^\alpha\bar{\phi}\partial_\alpha\delta\phi\Bigg] + \frac{3}{\bar{\phi}^2}\frac{\delta\phi}{\bar{\phi}}\left[\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - \frac{1}{2}\bar{g}_{\nu\mu}(\partial\bar{\phi})^2\right]$$

$$+\frac{1}{\bar{\phi}}\Bigg[\nabla_\mu\nabla_\nu\delta\phi - \frac{\bar{g}^{\lambda\alpha}}{2}(\nabla_\nu h_{\alpha\mu} + \nabla_\mu h_{\alpha\nu} - \nabla_\alpha h_{\mu\nu})\partial_\lambda\bar{\phi} - \bar{g}_{\mu\nu}\Box\delta\phi - h_{\mu\nu}\Box\bar{\phi}$$

$$+\bar{g}_{\mu\nu}\frac{\bar{g}^{\lambda\alpha}}{2}(\nabla_\beta h_{\lambda\alpha} + \nabla_\alpha h_{\lambda\beta} - \nabla_\lambda h_{\alpha\beta})\partial_\beta\bar{\phi} - \nabla_\mu\nabla_\nu\bar{\phi}\frac{\delta\phi}{\bar{\phi}} - \bar{g}_{\mu\nu}\Box\bar{\phi}\frac{\delta\phi}{\bar{\phi}}\Bigg]. \quad (3)$$

Propagation of GWs in vacuum

Vacuum

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \bar{R} = V'(\bar{\phi}) = 0, \quad \bar{T}_{\mu\nu} = 0\,, \quad \delta T_{\mu\nu} = 0, \quad \bar{\phi} = \phi_0\,. \qquad (4)$$

Defining the new tensor $\theta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, and imposing the Lorenz gauge ($\partial^\alpha \theta_{\alpha\beta} = 0$), the perturbed field equations read:

$$\Box\theta_{\mu\nu} = -2\nabla_\mu\nabla_\nu\frac{\delta\phi}{\phi_0} + 2\eta_{\mu\nu}\Box\frac{\delta\phi}{\phi_0}\,, \quad -\phi_0 V''(\phi_0)\delta\phi = 0\,. \qquad (5)$$

Alert!

Note that $2V(\phi) - \phi V'(\phi) = \kappa^2 T$ allows us to find a relation $\phi(T)$. As $T = 0$, ϕ must be constant and, as $\bar{\phi} = \phi_0$, $\delta\phi$ has to be constant as well.

$$\Box\theta_{\mu\nu} = 0\,. \qquad (6)$$

Emission of GWs in the weak-field limit

Weak-field limit

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \bar{R} = V'(\bar{\phi}) = 0, \quad \bar{T}_{\mu\nu} = 0\,, \quad \delta T_{\mu\nu} \longrightarrow T_{\mu\nu}, \quad \bar{\phi} = \phi_0\,. \tag{7}$$

Defining again the tensor $\theta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, and imposing the Lorenz gauge ($\partial^\alpha \theta_{\alpha\beta} = 0$), the perturbed Einstein equations read:

$$\Box\theta_{\mu\nu} = -\frac{2\kappa^2}{\phi_0}T_{\mu\nu} - 2\nabla_\mu\nabla_\nu\frac{\delta\phi}{\phi_0} + 2\eta_{\mu\nu}\Box\frac{\delta\phi}{\phi_0}\,, \tag{8}$$

while the auxiliary field perturbed equation reads:

$$-\phi_0 V''(\phi_0)\delta\phi = \kappa^2 T\,. \tag{9}$$

Emission of GWs in the weak-field limit

Combining both equations we obtain the equation for the gravitational perturbation in terms of T:

$$\Box\theta_{\mu\nu} = -2\kappa^2 T^{\text{eff}}_{\mu\nu} \, , \tag{10}$$

where, for convenience, we defined an effective energy-momentum tensor:

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{\phi_0} T_{\mu\nu} - \frac{1}{\kappa^2 \phi_0^2 V''(\phi_0)} \left(\nabla_\mu \nabla_\nu T - \eta_{\mu\nu} \Box T\right) \, . \tag{11}$$

And we can use the retarded Green function method to obtain a solution:

$$\theta_{\mu\nu} = \frac{\kappa^2}{2\pi} \int d^4x' \, \frac{T^{\text{eff}}_{\mu\nu}(x')}{|\vec{x} - \vec{x}'|} \delta\left(x^0 - |\vec{x} - \vec{x}'| - x^{0\prime}\right) \, , \tag{12}$$

Emission of GWs in the weak-field limit

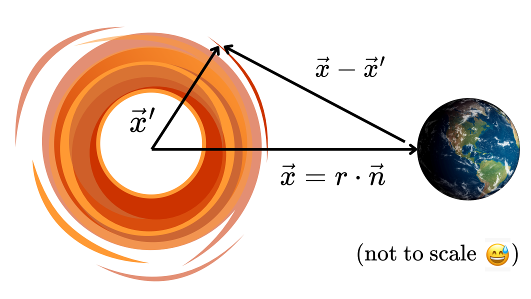

Figure 4: Diagram showing the relation (13).

Far source

Note that, for a sufficiently far source we can write:

$$|\vec{x} - \vec{x}'| \simeq r - \vec{x}' \cdot \vec{n} \,. \tag{13}$$

Emission of GWs in the weak-field limit

Morover, we need to know the part gravitational perturbation that actually contributes to detectable gravitational waves, so we need to apply the Transverse-traceless gauge.

Transverse-traceless gauge (TT gauge)

$$\theta_{ij}^{TT} = \Lambda_{ijkl}\theta^{kl} \,, \quad \Lambda_{ijkl}(\vec{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \,, \quad P_{ij} = \delta_{ij} - n_i n_j \,. \tag{14}$$

The gravitational perturbation in the TT-gauge, for a sufficiently far source reads:

$$\theta_{ij}^{TT} \simeq \frac{4G}{r}\Lambda_{ijkl} \int d^3x' \, T_{\text{eff}}^{kl}(t - r + \vec{x}' \cdot \vec{n}, x') \,. \tag{15}$$

Emission of GWs in the weak-field limit

If the velocities inside the source are small compared to c, we can expand the effective energy-momentum tensor as:

$$T_{kl}^{\text{eff}}(t - \frac{r}{c} + \frac{\vec{x}' \cdot \vec{n}}{c}, x') = T_{kl}^{\text{eff}}(t - \frac{r}{c}, x')$$
$$+ \frac{\partial T_{kl}^{\text{eff}}}{\partial t} \frac{(\vec{x}' \cdot \vec{n})}{c} + \frac{\partial^2 T_{kl}^{\text{eff}}}{\partial t^2} \frac{(\vec{x}' \cdot \vec{n})(\vec{x}' \cdot \vec{n})}{2c^2} + \mathcal{O}\left(\frac{1}{c^3}\right) . \quad (16)$$

And consequently, we can expand the gravitational perturbation:

$$\theta_{ij}^{TT} = \frac{4G}{r} \Lambda_{ijkl} \left(S^{kl} + \frac{1}{c} n_m \dot{S}^{klm} + \frac{1}{2c^2} n_m n_p \ddot{S}^{klmp} + \ldots \right) , \quad (17)$$

where we have defined the tensors $S^{kl..}$ as follows:

$$S^{kl} = \int d^3x' \, T_{\text{eff}}^{kl} , \quad S^{klm} = \int d^3x' \, T_{\text{eff}}^{kl} x^{m\prime} , \ \ldots \quad (18)$$

Emission of GWs in the weak-field limit

If we focus on the first order (or quadrupolar) term, using integration by parts, we obtain a GR-like term plus a Palatini $f(\mathcal{R})$ contribution:

$$S^{kl} = \frac{1}{\phi_0}\frac{\partial^2}{\partial t^2}\int d^3x' \left(T^{00} - \frac{1}{\kappa^2\phi_0 V_0''}\partial_j\partial^j T\right) x^{k\prime}x^{l\prime} = S^{kl}_{GR} + S^{kl}_{PA} \,. \quad (19)$$

Integrating by parts we see that:

$$\int d^3x' \left(\partial_j\partial^j T\right) x^{k\prime}x^{l\prime} = 2\delta^{kl}\int d^3x' T \,. \quad (20)$$

And since $\Lambda_{ijkl}\delta^{kl} = 0$, we get:

$$\left[\theta^{TT}_{ij}\right]_{\text{quadrupolar}} = \frac{4G}{r}\Lambda_{ijkl}\left(S^{kl}_{GR} + S^{kl}_{PA}\right) = \frac{4G}{r}\Lambda_{ijkl}S^{kl}_{GR} \,. \quad (21)$$

Emission of GWs in the weak-field limit

For the second order (or octupolar) term, the procedure is more complex, but straightforward. Integrating by parts,we obtain a GR-like term plus a Palatini $f(\mathcal{R})$ contribution:

$$\begin{aligned}\dot{S}^{klm} &= \frac{1}{\phi_0}\int d^3x'\left[\frac{1}{2}x'^m\dddot{T}^{00}+\ddot{T}^{m0}-\frac{1}{\kappa^2\phi_0 V''(\phi_0)}\left(\frac{1}{2}x'^m\partial_j\partial^j\dddot{T}+\partial^m\dddot{T}\right)\right]x^{l\prime}x^{k\prime} \\ &= \dot{S}^{klm}_{GR}+\dot{S}^{klm}_{PA}\ . \end{aligned} \quad (22)$$

Integrating by parts we find out that:

$$\int d^3x'\left(\frac{1}{2}x'^m\partial_j\partial^j\dddot{T}+\partial^m\dddot{T}\right)x^{l\prime}x^{k\prime} = \delta^{kl}\int d^3x'\dddot{T}x'^m \quad (23)$$

And, again, as $\Lambda_{ijkl}\delta^{kl}=0$, we get:

$$\left[\theta^{TT}_{ij}\right]_{\text{octupolar}} = \frac{4G}{r}\Lambda_{ijkl}n_m\left(\dot{S}^{klm}_{GR}+\dot{S}^{klm}_{PA}\right) = \frac{4G}{r}\Lambda_{ijkl}n_m\dot{S}^{klm}_{GR}\ . \quad (24)$$

Main conclusions

- We obtained a general equation for a $f(\mathcal{R})$ Palatini gravitational perturbation in terms of the auxiliary field ϕ and its potential $V(\phi)$.
- We showed that, in Palatini $f(\mathcal{R})$ gravity, the propagation of GWs in vacuum coincides with the GR prediction.
- We found that, in the weak-field limit, the emission of GWs within the Palatini $f(\mathcal{R})$ formalism coincides with the GR prediction except for an effective gravitational constant, at least to quadrupolar and octupolar order.

Main References

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Palatini f(R) gravity

First things first:

Palatini formalism

- The affine connection Γ is, a priori, independent to the metric $g_{\mu\nu}$.
- The curvature tensors are defined in terms of this independent connection (for example $\mathcal{R}_{\mu\nu}(\Gamma)$).

$f(\mathcal{R})$ gravity

- The Einstein-Hilbert action is generalized through a function $f(\mathcal{R})$ of the scalar curvature $\mathcal{R}$.

$$S = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}\,[f(\mathcal{R}) + L_m] \ . \quad (25)$$

The curvature tensor is defined as:

$$\mathcal{R}_{\mu\nu}(\Gamma) = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} \ . \quad (26)$$

Palatini f(R) gravity

Varying the action with respect to the metric we obtain:

$$f_{\mathcal{R}} \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f = \kappa^2 T_{\mu\nu} \,, \tag{27}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}$ and $f_{\mathcal{R}} = f'(\mathcal{R})$. While varying the action respect to the independent connection:

$$\nabla^*_\lambda \left(\sqrt{-g} f_{\mathcal{R}} g^{\mu\nu} \right) = 0 \,. \tag{28}$$

Conformal transformation

$$q_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{and} \quad \Omega^2 = f_{\mathcal{R}} \quad \Rightarrow \quad \nabla^*_\lambda \left(\sqrt{-q} q^{\mu\nu} \right) = 0 \tag{29}$$

We recover the metricity condition from GR for the metric $q_{\mu\nu}$.

Palatini f(R) gravity

$q_{\mu\nu}$

- Γ can be written as the Levi-Civita connection of the metric q.
- $\mathcal{R}$ depends of q.

$g_{\mu\nu}$

- We can construct the curvature tensors and scalars $R(g)$, but $R(g) \neq \mathcal{R}(q)$.

From the metric equation we obtain:

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = \frac{\kappa^2}{f_{\mathcal{R}}}T_{\mu\nu} - g_{\mu\nu}\frac{\mathcal{R}f_{\mathcal{R}} - f}{2f_{\mathcal{R}}}$$
$$-\frac{3}{2f_{\mathcal{R}}^2}\left[\nabla_\mu f_{\mathcal{R}}\nabla_\nu f_{\mathcal{R}} - \frac{1}{2}g_{\nu\mu}\nabla_\lambda f_{\mathcal{R}}\nabla^\lambda f_{\mathcal{R}}\right] + \frac{1}{f_{\mathcal{R}}}\left[\nabla_\mu\nabla_\nu f_{\mathcal{R}} - g_{\mu\nu}\Box f_{\mathcal{R}}\right] . \tag{30}$$

And taking the trace of the metric equation:

$$f_{\mathcal{R}}\mathcal{R} - 2f = \kappa^2 T . \tag{31}$$

Scalar-tensor equivalence

Scalar-tensor theory equivalence

We define the following auxiliary field:

$$\phi = f'(\mathcal{R}), \qquad V(\phi) = \mathcal{R}\phi - f(\mathcal{R}) \,. \tag{32}$$

Then, we obtain a GR-like equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa^2}{\phi}T_{\mu\nu} - \frac{V(\phi)}{2\phi}g_{\mu\nu} - \frac{3}{2\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) \,, \tag{33}$$

as well as an equation governing the auxiliary field:

$$2V(\phi) - \phi V'(\phi) = \kappa^2 T \,. \tag{34}$$