Perturbations of bimetric gravity on most general spherically symmetric spacetimes

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Spanish and Portuguese Relativity Meeting
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Outline

- Introduction to the bimetric theory of gravity
- 2+2 decomposition of spherically symmetric backgrounds
- Perturbative analysis
  - Harmonic decomposition of the perturbations
  - Perturbative equations of motion
- Application: static nonbidiagonal backgrounds
- Final remarks

Introduction to the bimetric theory of gravity

- **Bimetric gravity** is a consistent theory with non-linear interactions between massive and massless spin-2 fields that admits self-accelerated cosmological solutions.

- In addition to the usual metric, $\tilde{g}_{ab}$, we have an extra one, $\tilde{f}_{ab}$. These two metrics are coupled to each other.

- **Action**:

$$S_{\text{Bi}} = \overbrace{\underset{\substack{\text{Planck Mass}\\ \text{for } \tilde{g}_{\mu\nu}}}{\frac{M_g^2}{2}} \int \mathrm{d}^4x \sqrt{-\tilde{g}}\, \underset{\substack{\text{Ricci scalar}\\ \text{for } \tilde{g}_{\mu\nu}}}{R^{(\tilde{g})}}}^{\text{EH term for } \tilde{g}_{ab}} + \overbrace{\underset{\substack{\text{“Planck Mass”}\\ \text{for } \tilde{f}_{\mu\nu}}}{\frac{M_f^2}{2}} \int \mathrm{d}^4x \sqrt{-\tilde{f}}\, \underset{\substack{\text{Ricci scalar}\\ \text{for } \tilde{f}_{\mu\nu}}}{R^{(\tilde{f})}}}^{\text{EH term for } \tilde{f}_{ab}} - \overbrace{\underset{\substack{\text{mass}\\ \text{parameter}}}{m^2} M_g^2 \int \mathrm{d}^4x \sqrt{-\tilde{g}} \sum_{n=0}^{4} \underset{\substack{\text{dimensionless}\\ \text{couplings constants}}}{\beta_n} \underset{\substack{\text{symmetric}\\ \text{polynomials}}}{e_n\left(\sqrt{\tilde{g}^{ac}\tilde{f}_{cb}}\right)}}^{\text{Interaction term}}$$

[Hassan, Rosen (2012)]

- Matter fields are assumed to couple only to $\tilde{g}_{ab}$. Total action:

$$S[\tilde{g}_{ab}, \tilde{f}_{ab}, \psi] = S_{\text{Bi}}[\tilde{g}_{ab}, \tilde{f}_{ab}] + S_{\text{m}}[\tilde{g}_{ab}, \psi]$$

Introduction to the bimetric theory of gravity

- The bimetric action is invariant under the simultaneous replacements

$$\tilde{g} \leftrightarrow \tilde{f}, \quad \beta_n \leftrightarrow \beta_{4-n}, \quad M_g \leftrightarrow M_f, \quad m^2 \leftrightarrow m^2 M_g^2 / M_f^2$$

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- Field equations:

$$G_{\mu\nu}^{(\tilde{g})} + m^2 V_{\mu\nu}^{(\tilde{g})}(\tilde{g}, \tilde{f}, \beta_n) = \frac{1}{M_g^2} \mathscr{T}_{\mu\nu}$$

$$G_{\mu\nu}^{(\tilde{f})} + \frac{m^2}{\alpha^2} V_{\mu\nu}^{(\tilde{f})}(\tilde{g}, \tilde{f}, \beta_n) = 0$$

where

- $\alpha \equiv M_f / M_g$ is the ratio between the 'Planck masses',
- $V_{\mu\nu}^{(\tilde{g})}(\tilde{g}, \tilde{f}, \beta_n)$ and $V_{\mu\nu}^{(\tilde{f})}(\tilde{g}, \tilde{f}, \beta_n)$ describe the interaction between both metrics, and are given in terms of

$$\mathbb{S}^{\mu}{}_{\nu} := \sqrt{\tilde{g}^{\mu\rho} \tilde{f}_{\rho\nu}}\,,$$

- $\mathscr{T}_{\mu\nu}$ is the matter stress-energy tensor.

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$$G^{(\tilde{g})}_{\mu\nu} = 8\pi t^{(\tilde{g})}_{\mu\nu} \qquad t^{(\tilde{g})}_{\mu\nu} := \frac{1}{8\pi M_g^2}\mathscr{T}_{\mu\nu} - \frac{m^2}{8\pi} V^{(\tilde{g})}_{\mu\nu}$$

$$G^{(\tilde{f})}_{\mu\nu} = 8\pi t^{(\tilde{f})}_{\mu\nu} \qquad t^{(\tilde{f})}_{\mu\nu} := -\frac{m^2}{8\pi\alpha^2} V^{(\tilde{f})}_{\mu\nu}$$

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Spherically symmetric background

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- The background metric tensors can then be written in block-diagonal form,

$$g_{\mu\nu}(x^\lambda)\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{AB}(x^D)\mathrm{d}x^A \mathrm{d}x^B + r_g^2(x^D)\gamma_{ab}(x^d)\mathrm{d}x^a \mathrm{d}x^b$$

$$f_{\mu\nu}(x^\lambda)\mathrm{d}x^\mu \mathrm{d}x^\nu = f_{AB}(x^D)\mathrm{d}x^A \mathrm{d}x^B + r_f^2(x^D)\gamma_{ab}(x^d)\mathrm{d}x^a \mathrm{d}x^b$$

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  - **We work covariantly on $\mathscr{M}^2$**
  - $\mathbb{S}^\mu{}_\nu$ is also diagonal by blocks with $\mathbb{S}^a{}_b = \dfrac{r_f}{r_g}\delta^a{}_b$

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Perturbative analysis

- Perturbative ansatz:

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exact solutions (background)

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Well known (GR)

Non trivial computation,
requires $\Delta[\mathbb{S}^{\mu}{}_{\nu}]$
$(\mathbb{S}^{\mu}{}_{\nu} = \sqrt{\tilde{g}^{\mu\rho}\tilde{f}_{\rho\nu}}\ )$

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$(\mathbb{S}^{\mu}{}_{\nu} = \sqrt{\tilde{g}^{\mu\rho}\tilde{f}_{\rho\nu}})$

- On a spherically symmetric background the problem can be simplified by decomposing $\Delta[\mathbb{S}^{\mu}{}_{\nu}]$ in a basis of tensor spherical harmonics

$$\tilde{\mathbb{S}}^{\mu}{}_{\alpha}\tilde{\mathbb{S}}^{\alpha}{}_{\nu} = \tilde{g}^{\mu\alpha}\tilde{f}_{\alpha\nu} \longrightarrow \mathbb{S}^{\mu}{}_{\alpha}\Delta[\mathbb{S}^{\alpha}{}_{\nu}] + \Delta[\mathbb{S}^{\mu}{}_{\alpha}]\mathbb{S}^{\alpha}{}_{\nu} = g^{\mu\alpha}h^{(f)}_{\alpha\nu} - g^{\mu\alpha}h^{(g)}_{\alpha\beta}g^{\beta\sigma}f_{\sigma\nu}$$

Harmonic decomposition of the perturbations

Tensor spherical harmonics

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polar (Z)

axial (X)

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- Basis for **scalars** on the sphere: $\{Y_l^m\}$
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  $Z_l^m{}_a := \partial_a Z_l^m \qquad X_l^m{}_a := \epsilon_a{}^b Z_l^m{}_b$

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- Basis for **rank-two tensors** on the sphere: $\{Z_l^m{}_{ab}, X_l^m{}_{ab}, \gamma_{ab} Z_l^m, \epsilon_{ab} Z_l^m\}$

  $$Z_l^m{}_{ab} := Z_l^m{}_{:ab} + \frac{l(l+1)}{2}\gamma_{ab} Z_l^m \qquad X_l^m{}_{ab} := \frac{1}{2}(X_l^m{}_{a:b} + X_l^m{}_{b:a})$$

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As in GR, **different polarities decouple at the linear level**,
so long as the background is spherically symmetric.

Harmonic decomposition of the perturbations

- Decomposition of the perturbations into tensor spherical harmonics:

$$h^{(i)}_{AB}(x^D, x^d) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H^{(i)m}_{l\,AB} Z^m_l$$

$$h^{(i)}_{Ab}(x^D, x^d) := \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left[H^{(i)m}_{l\,A} Z^m_{l\ \ b} + h^{(i)m}_{l\,A} X^m_{l\ \ b} \right]$$

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$l = 0$ and $l = 1$

are special cases

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- And similarly for $\Delta[t^{(i)}_{\mu\nu}]$, $\Delta[\mathscr{T}_{\mu\nu}]$ and $\Delta[\mathbb{S}^{\mu}{}_{\nu}]$

Perturbative equations of motion

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- **Example:** axial sector, scalar equation ($l \geq 2$)

$$2 \overset{(g)}{\nabla}{}^{A} h^{(g)}_{A} - \overset{(g)}{\nabla}{}^{A} \overset{(g)}{\nabla}{}_{A} h^{(g)} + 2 \overset{(g)}{\nabla}{}_{A} \left(h^{(g)} \frac{\overset{(g)}{\nabla}{}^{A} r_g}{r_g} \right) = 16\pi \left(t^{(g)} - \frac{Q_g}{2} h^{(g)} \right)$$

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$$t^{(g)} = \frac{1}{4}(Q_g - Q_m)h^{(g)} + \frac{m^2}{32\pi\,\omega}\left[\beta_1 + \beta_2\mathbb{S}^{A}{}_{A} + \beta_3\det\left(\mathbb{S}^{A}{}_{B}\right)\right]\left(h^{(f)} - \omega^2 h^{(g)}\right) + \frac{1}{8\pi M_g^2}\psi$$

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- Number of physical propagating degrees of freedom:

	Axial	Polar
$l = 0$	0	1
$l = 1$	1	2
$l \geq 2$	3	4

Particular case: static nonbidiagonal background

static
$+$
nonbidiagonal
$+$
$\mathscr{T}_{\mu\nu} = 0$

→

$$G^{(g)}_{\mu\nu} + m^2(\beta_0 + 2\omega\beta_1 + \omega^2\beta_2)g_{\mu\nu} = 0$$

$$G^{(f)}_{\mu\nu} + \frac{m^2}{\alpha^2}(\beta_4 + 2\omega^{-1}\beta_3 + \omega^{-2}\beta_2)f_{\mu\nu} = 0$$

[Volkov (2015)]

g and f are decoupled
at background level

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$$G^{(g)}_{\mu\nu} + m^2 \overbrace{\left(\beta_0 + 2\omega\beta_1 + \omega^2\beta_2\right)}^{= \Lambda_g} g_{\mu\nu} = 0$$

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- Most general static nonbidiagonal ansatz:

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Sigma_g(r)\mathrm{d}t^2 + \frac{1}{\Sigma_g(r)}\mathrm{d}r^2 + r^2\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b, \qquad \Sigma_g(r) := 1 - \frac{2\mu_g}{r} - \frac{m^2\Lambda_g}{3}r^2$$

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two Schwarzschild-(anti)de Sitter metrics

- Most general static nonbidiagonal ansatz:

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$$\longrightarrow \qquad \begin{aligned} & G^{(g)}_{\mu\nu} + m^2 \overbrace{(\beta_0 + 2\omega\beta_1 + \omega^2\beta_2)}^{=\Lambda_g} g_{\mu\nu} = 0 \\ & G^{(f)}_{\mu\nu} + \frac{m^2}{\alpha^2} \underbrace{(\beta_4 + 2\omega^{-1}\beta_3 + \omega^{-2}\beta_2)}_{=\Lambda_f} f_{\mu\nu} = 0 \end{aligned}$$

two Schwarzschild-(anti)de Sitter metrics

- Most general static nonbidiagonal ansatz:

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Sigma_g(r)\mathrm{d}t^2 + \frac{1}{\Sigma_g(r)}\mathrm{d}r^2 + r^2\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b, \qquad \Sigma_g(r) := 1 - \frac{2\mu_g}{r} - \frac{m^2\Lambda_g}{3}r^2$$

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- From (T, r_f, θ, ϕ) to (t, r, θ, ϕ):

$$T(t,r) = ct + \int \sqrt{\left(\frac{1}{\Sigma_g} - \frac{1}{\Sigma_f}\right)\left(\frac{c^2}{\Sigma_g} - \frac{\omega^2}{\Sigma_f}\right)}\,\mathrm{d}r, \qquad r_f(r) = \omega r \;\; \text{with } \omega \text{ given by } \;\; \beta_1 + 2\beta_2\omega + \beta_3\omega^2 = 0$$

Particular case: static nonbidiagonal background

static
+
nonbidiagonal
+
$\mathscr{T}_{\mu\nu} = 0$

$$\longrightarrow$$

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Which is the physical interpretation of c?

Effect of c

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If only the mode $m = 0$ is exited, axial symmetry is preserved. In this case,

$$\text{two Schwarzschild BH} \xrightarrow{\text{perturbed}} \text{two slowly rotating Kerr BH with angular momentum}$$

$$J_g = -\frac{c_1}{4}\sqrt{\frac{3}{\pi}}, \qquad J_f = -\frac{c_2\,\omega}{4c}\sqrt{\frac{3}{\pi}}$$

Final remarks

- We have presented a formalism to study linear perturbations of bimetric gravity on any spherically symmetric background, including dynamical spacetimes.

- The setup is based on the Gerlach-Sengupta formalism for GR. Each of the two background metrics is written as a warped product between a two-dimensional Lorentzian metric and the round metric of the two-sphere.
  - A covariant notation on the Lorentzian manifold is used so that all expressions are valid for any coordinates.

- As an application we have considered the case of the most general nonbidiagonal static backgrounds (two Schwarzschild-(anti)de Sitter metrics).
  - We have solved analitically the static perturbations with $l = 1$ in the axial sector.

- **Future work:** to solve the perturbative equations in the dynamical case for the axial sector and $l = 1$ where there is one propagating degree of freedom (contrary to GR).

Harmonic decomposition of the perturbations

- Decomposition of $\Delta[\mathbb{S}^{\mu}{}_{\nu}]$:

$$\Delta[\mathbb{S}^{A}{}_{B}](x^D, x^d) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} S_l^{mA}{}_{B} Z_l^m$$

$$\Delta[\mathbb{S}^{A}{}_{b}](x^D, x^d) := \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left[S_l^{mA} Z_l^m{}_{b} + s_l^{mA} X_l^m{}_{b} \right]$$

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In general, neither $\mathbb{S}^{\mu}{}_{\nu}$ nor $\Delta[\mathbb{S}^{\mu}{}_{\nu}]$ are symmetric

- The scalar components can be solved explicitly in terms of those of g and f:

$$\tilde{S} = \frac{1}{2\omega}\left(K^{(f)} - \omega^2 K^{(g)}\right), \qquad S = \frac{1}{2\omega}\left(G^{(f)} - \omega^2 G^{(g)}\right), \qquad \check{S} = 0, \qquad s = \frac{1}{2\omega}\left(h^{(f)} - \omega^2 h^{(g)}\right)$$

Physical propagating degrees of freedom

- In vacuum GR:

10 d.o.f − 8 first-class constraints = 2 propagating d.o.f

(metric: rank-two symmetric tensor field) (4 generators of diffeomorphisms and 4 vanishing of the conjugate momenta of lapse and shift)

- Bimetric gravity: when the two metrics are coupled, a set of four first-class constraints of the system is removed, due to the now common diffeomorphism invariance

2×10 d.o.f − $4 + 2 \times 4$ first-class constraints − 2 second-class constraints = 7 propagating d.o.f

(particular form of the interaction term)

↑ 2 symmetric rank-two tensors (14 polar, 6 axial)

↑ 3 four vectors (9 polar, 3 axial)

↑ 1 scalar ghost (1 polar)

↑ 4 polar, 3 axial