A class of tilted cosmological solutions to the near equilibrium Einstein-Boltzmann system

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Objects

- To find integrable self-consistent systems for the Einstein-Boltzmann system

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$$

$$p^\mu \left(\frac{\partial f}{\partial x^\mu} - \Gamma^\alpha_{\mu\nu} p^\nu \frac{\partial f}{\partial p^\alpha} \right) = Q(f, f) \,,$$

for $g_{\mu\nu}(x^\sigma)$ and $f = f(x^\mu, p^\alpha, \mathcal{I})$, with $T_{\mu\nu}$ defined from f.

- To determine thermodynamical coefficients, like viscosity and heat conductivity, for small deviations from equilibrium.

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Boltzmann equation

- Boltzmann equation:

$$p^i X_i(f) - \gamma^a_{jk} p^j p^k \frac{\partial f}{\partial p^a} = Q(f, f)$$

Distribution function $f = f(t, x^\alpha, p^a, \mathcal{I})$, $\quad p^i p_i = m^2 c^2$
$\alpha, \beta ... = 1, 2, 3$ and $a, b ... = 1, 2, 3,$

- In Lorentz tetrad: $ds^2 = \eta_{ij} \omega^i \omega^j = g_{\mu\nu} dx^\mu dx^\nu$

where: $\quad \omega^i = \omega^i{}_\mu dx^\mu$, or $X_i = X_i{}^\mu \frac{\partial}{\partial x^\mu}$,

such that

$$\omega^0 = \sqrt{g_{00}} \left(cdt + \frac{g_{0\alpha}}{g_{00}} dx^\alpha \right), \quad \omega^a_0 = 0 \quad \text{and} \quad X_0 = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{c \partial t}$$

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Boltzmann equation

- Collision term (Pennisi & Ruggeri), BGK-like (Bhatnagar, Gross & Krook):

$$Q(f) = \frac{u_i p^i}{c^2 \tau} \left(f_{EP} - f - \frac{\gamma^\star p^i q_i}{pmc^2} \frac{A(\gamma)}{B(\gamma)} f_{EP} \right) ,$$

- where the equilibrium distribution function is given by

$$f_{EP}(t, x^\alpha, p^a, \mathcal{I}) = \frac{n}{4\pi m^2 ckTK_2(\gamma) A(\gamma)} \exp\left(-\frac{\gamma^\star}{\gamma} \frac{u_i p^i}{kT} \right) .$$

$\gamma^* \equiv \gamma \left(1 + \mathcal{I}/(mc^2)\right)$, $\gamma \equiv mc^2/(kT)$, $\mathcal{I}$ internal energy.

- $K_n(\gamma)$ modified Bessel functions of 2:nd kind, $Ki_n(\gamma)$ Bickley-Naylor functions

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Integrals of K_n, Ki_n

- $A(\gamma) = \frac{\gamma}{K_2(\gamma)} \int_0^\infty \frac{K_2(\gamma^\star)}{\gamma^\star} \Phi(\mathcal{I}) d\mathcal{I}$,
  $B(\gamma) = \frac{\gamma}{K_2(\gamma)} \int_0^\infty K_3(\gamma^\star) \Phi(\mathcal{I}) d\mathcal{I}$,
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  $D(\gamma) = \frac{1}{K_2(\gamma)} \int_0^\infty \gamma^{\star 2} K_1(\gamma^\star) \Phi(\mathcal{I}) d\mathcal{I}$,
  $E(\gamma) = \frac{1}{K_2(\gamma)} \int_0^\infty \gamma^{\star 2} \mathrm{Ki}_1(\gamma^\star) \Phi(\mathcal{I}) d\mathcal{I}$
- Here $\Phi(\mathcal{I})$ is the density of states for the internal degrees of freedom. For polytropes, $\Phi(\mathcal{I}) \propto \mathcal{I}^\alpha$, the total degrees of freedom are given by $D = 2\alpha + 5$.
- For diatomic molecule with 2 rotational degrees of freedom excited $\alpha = 0$.

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Moments

- Current density:

$$V^i = mc \int_0^\infty \int_{R^3} f p^i \Phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} \,.$$

- In Eckart frame: $V^i = nmu^i$
- Energy-momentum tensor:

$$T^{ij} = \frac{1}{mc} \int_0^\infty \int_{R^3} \left(mc^2 + \mathcal{I}\right) f p^i p^j \Phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} \,,$$

- General form:

$$T^{ij} = \frac{\mu}{c^2} u^i u^j + (p + \Pi) h^{ij} + \frac{2}{c^2} q^{(i} u^{j)} + \pi^{ij}$$

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Conservation laws

- Kinematic quantities of particle velocity u^i:

$$u^i{}_{;j} = \frac{1}{c^2} a^i u_j - \frac{1}{3} h^i{}_j \theta + \sigma^i{}_j + \omega^i{}_j$$

a_i=acceleration, θ=expansion, σ_{ij}=shear, ω_{ij}=vorticity,
$h_{ij} = u_i u_j / c^2 - \eta_{ij}$

- Particle conservation $\nabla_i V^i = 0$: $cX_0(n) + n\theta = 0$
- Energy-momentum conservation, $\nabla_j T^{ij} = 0$, give for T:

$$c\frac{X_0(T)}{T} = -\frac{k}{pc_v}\left[\theta(p+\Pi) + q^i{}_{;i} - \frac{1}{c^2} a_i q^i - \sigma_{ij}\pi^{ij}\right]$$

where the specific heat

$$c_v = \frac{\partial}{\partial T}\left(\frac{\mu}{n}\right) = k\left[-1 + 5\frac{B}{A} - \frac{B^2}{A^2} + \gamma\frac{C}{A}\right].$$

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Chapman-Enskog method

- Expand distribution function as

$$f = f_{EP}(1 + \epsilon\phi_P)$$

where f_{EP} is the equilibrium distribution and ϵ is a small parameter of the same order as the relaxation time τ.

- Then ϕ_P is given by

$$\phi_P = -\frac{c\tau}{p_0}\left(p^i\frac{X_i(n)}{n} + p^i\frac{X_i(T)}{T}\left(1 - \frac{B}{A} + \frac{p_0\gamma^\star}{mc}\right)\right)$$
$$+\frac{c\tau}{p_0}\frac{\gamma^\star}{mc^2}\left(\frac{1}{3}\theta p_a p^a + \frac{1}{c}p_a a^a p_0 + \sigma_{ab}p^a p^b\right) - p^a q_a\frac{\gamma^\star}{pmc^2}\frac{A}{B}.$$

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Thermodynamic coefficients

- From the Eckart condition $V^i = V^i_{EP}$

$$q_a = \kappa \left(X_a(T) - \frac{1}{c^2} T a_a \right)$$

- where

$$\kappa = \frac{\tau k p}{3m} \gamma^3 \frac{B}{A} \left(\frac{\Gamma B}{\gamma A^2} - \frac{3}{\gamma^2} \right) , \quad \text{with}$$

$$\Gamma = \frac{1}{K_2(\gamma)} \int K_2(\gamma^*) \left(\frac{1}{\gamma^*} - \frac{K_1(\gamma^*)}{K_2(\gamma^*)} + \frac{Ki_1(\gamma^*)}{K_2(\gamma^*)} \right) \Phi(\mathcal{I}) d\mathcal{I} \, .$$

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Thermodynamic coefficients

From the energy-momentum tensor:

- Pressure and energy density

$$p = nkT\,, \quad \text{and} \quad \mu = p\left[\frac{B}{A} - 1\right]\,,$$

- Bulk viscous pressure

$$\begin{aligned} \Pi &= -\zeta\theta\,, \quad \text{where} \\ \zeta &= p\gamma\tau\left(\frac{B}{3\gamma A} - \frac{C-D+E}{9A} - \frac{1}{\gamma}\left(\frac{5BA - B^2 + \gamma CA}{5BA - B^2 + \gamma CA - A^2}\right)\right) \end{aligned}$$

- Anisotropic viscous pressure

$$\begin{aligned} \pi_{ab} &= 2\eta\sigma_{ab}\,, \quad \text{where} \\ \eta &= \frac{\tau p\gamma}{15}\left(\frac{3B}{\gamma A} - \frac{C-D+E}{A}\right). \end{aligned}$$

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Tilted Bianchi VIII cosmology with heatflow

- Bianchi VIII structure algebra

$$d\sigma^1 = -\sigma^2 \wedge \sigma^3 \,, \quad d\sigma^2 = \sigma^3 \wedge \sigma^1; \;, \quad d\sigma^3 = \sigma^1 \wedge \sigma^2 \,.$$

with

$$\begin{aligned} \sigma^1 &= dx + (1+x^2)dy + (x - y - x^2 y)dz \,, \\ \sigma^2 &= 2xdy + (1 - 2xy)dz \,, \\ \sigma^3 &= dx + (-1+x^2)dy + (x + y - x^2 y)dz \,, \end{aligned}$$

- Form basis

$$\omega^0 = cdt + a(t)\sigma^1 \,, \; \omega^1 = b(t)\sigma^1 \,, \; \omega^2 = g(t)\sigma^2 \,, \; \omega^3 = g(t)\omega^3 \,,$$

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- The kinematic quantities of the fluid velocity $u = cX_0 = \frac{\partial}{\partial t}$ are

$$\theta = c\left(\frac{\dot{b}}{b} + 2\frac{\dot{g}}{g}\right) \ , \quad \mathcal{A} \equiv -a_1/c^2 = -\frac{\dot{a}}{b} \, , \quad \Omega \equiv \omega_{23}/c = \frac{a}{2g^2} \ ,$$

$$\Sigma \equiv -\sigma_{11}/c = 2\sigma_{22}/c = 2\sigma_{33}/c = -\frac{2}{3}\left(\frac{\dot{g}}{g} - \frac{\dot{b}}{b}\right)$$

- and the expansion and twist along the symmetry axis are given by

$$\phi = -2\gamma^1{}_{22} = -2\gamma^1{}_{33} = -\frac{a\dot{g}}{bg} \, , \ \ \xi = \gamma^1{}_{23} = -\gamma^1{}_{32} = -\frac{b}{2g^2}$$

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1:st order system

- The equations can be rewritten in terms of the scalars [1], [2]

$$S = \{\mathcal{E}, \mathcal{H}, \mu, p, \Pi, \pi_{11}, q_1, \mathcal{A}, \theta, \Sigma, \Omega, \phi, \xi\}$$

- The system the reduces to a first order coupled system of ODE's in time for the 5 kinematic quantities $\theta, \Sigma, \Omega, \mathcal{A}, \xi$ and the temperature T and the particle density n.
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$$\dot{\xi} = \frac{\xi}{3}(6\Sigma - \theta/c) , \quad \dot{\Omega} = \left(\Sigma - \frac{2}{3c}\theta\right)\Omega + \mathcal{A}\xi ,$$

$$\dot{\theta} = \dots , \quad \dot{\Sigma} = \dots , \quad \dot{\mathcal{A}} = \dots ,$$

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$$c\dot{n} = -n\theta .$$

where $\dot{S} \equiv \frac{\partial S}{c\partial t}$

[1] See 1+1+2 covariant split of spacetime, Clarkson.

[2] $\mathcal{E}$ and $\mathcal{H}$ represent the electric and magnetic parts of the Weyl tensor.

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- Viscosity and heat conductivity coefficients within Eckart theory (1:st order perturbation from equilibrium) determined for general metric.
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Thermodynamics coefficients for $\alpha = 0$ ($D = 5$)

- The density and pressure are given by

$$\mu = nkT\left(1+\gamma\frac{K_2(\gamma)}{K_1(\gamma)}\right) \quad \text{and} \quad p = nkT\,,$$

- and the thermodynamical coefficients ζ, η and κ by

$$\begin{aligned}
\zeta =& nkT\tau\left[\frac{(1-\gamma^2)}{9} - \frac{1}{1+\frac{3\gamma K_2}{K_1}+\gamma^2\left(1-\frac{K_2^2}{K_1^2}\right)}\right.\\
&\left.+\frac{\gamma K_2}{9K_1}\left(1+\frac{\gamma^2}{3}\right)+\frac{\gamma^4}{27}\left(\frac{Ki_1}{K_1}-1\right)\right]\,,\\
\eta =& \frac{\gamma nkT\tau}{15}\left[\frac{10}{\gamma}-\gamma-\frac{\gamma^3}{3}\left(1-\frac{Ki_1}{K_1}\right)+\left(1+\frac{\gamma^2}{3}\right)\frac{K_2}{K_1}\right]\,,\\
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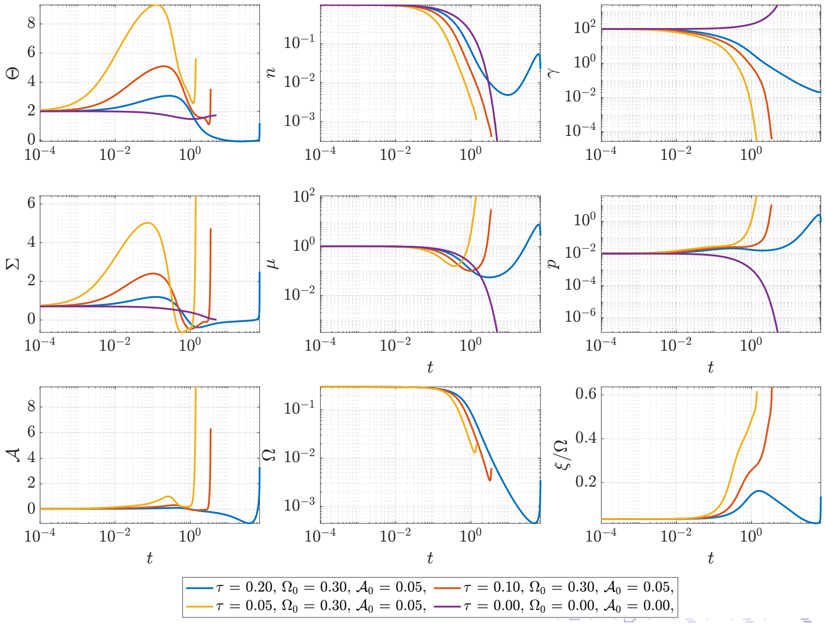
Numerics: Varying relaxation time τ



Numerics: Effects of cosmological constant Λ