# Integrability in perturbed BHs: background hidden structures

Spanish and Portuguese Relativity Meeting (EREP 2024)


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In collaboration with J.L. Jaramillo and C.F. Sopuerta

- J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, 2407.14196 (2024), J. L. Jaramillo, B. Krishnan, and C. F. Sopuerta, Int. J. Mod. Phys. D 32, 2342005 (2023)
- M. Lenzi and C. F. Sopuerta, Phys. Rev. D 104, 084053 (2021), Phys. Rev. D 104, 124068 (2021), Phys. Rev. D 107, 044010 (2023), Phys. Rev. D 107, 084039 (2023), Phys. Rev. D 109, 084030 (2024)
- E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. 39, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X 11, 031003 (2021)

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# BH perturbation theory

# BH perturbation theory

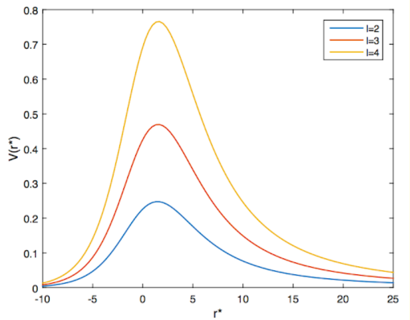
Perturbed Einstein equations at linear order $\delta G_{\mu\nu} = 0$

$$\Downarrow$$

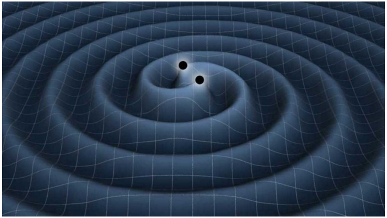
## The master equations

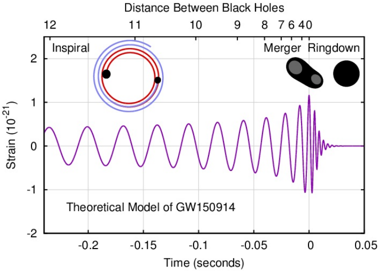
$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V^{\ell}_{\mathrm{even/odd}}\right)\Psi^{\ell m}_{\mathrm{even/odd}} = 0$$

- Master functions $\Psi = \Psi(h, \partial h)$
- Effective potential $V^{\ell}_{\mathrm{even/odd}}$
- Known ones: $(V_{\mathrm{RW}}, \Psi_{\mathrm{RW}})$, $(V_{\mathrm{RW}}, \Psi_{\mathrm{CPM}})$ (odd parity), $(V_{\mathrm{Z}}, \Psi_{\mathrm{ZM}})$ (even parity)



# BH binary systems






1. GW signal can be written in terms of master functions

$$h_{+/\times} \propto \Psi \Rightarrow h = h_+ - i h_\times \propto \Psi$$

2. Energy and angular momentum emission (luminosity) at infinity

$$\frac{dE}{dt} \propto |\dot{\Psi}|^2, \quad \frac{dJ}{dt} \propto \Psi\dot{\Psi}$$

3. Extreme Mass Ratio Inspirals (EMRIs) detectable by LISA

K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005)

# The standard branch

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-{}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}\right){}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=0$$

- Standard branch potentials

$${}_{\mathrm{S}}V_\ell^{\mathrm{odd/even}}=\begin{cases} V_\ell^{\mathrm{RW}} & \text{odd parity} \\ V_\ell^{\mathrm{Z}} & \text{even parity}\end{cases}$$

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- Most general master function

$$ {}_{\mathrm{S}}\Psi^{\mathrm{odd/even}}=\begin{cases} \mathcal{C}_1\Psi^{\mathrm{CPM}}+\mathcal{C}_2\Psi^{\mathrm{RW}} & \text{odd parity} \\ \mathcal{C}_1\Psi^{\mathrm{ZM}}+\mathcal{C}_2\boxed{\Psi^{\mathrm{NE}}} & \text{even parity} \end{cases}$$

$$\Psi^{\mathrm{NE}}(t,r)=\frac{1}{\lambda(r)}t^a\left(r\tilde{K}_{:a}-\tilde{h}_{ab}r^b\right)$$

# The Darboux branch

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - {}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}\right) {}_{\mathrm{D}}\Psi^{\mathrm{odd/even}} = 0$$

- Family of potentials ${}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}}$ satisfying

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{V^{\mathrm{RW/Z}}_{\ell,x}}{\delta V}\right)_{,x} - \delta V = 0\,,$$

with $\delta V = {}_{\mathrm{D}}V_\ell^{\mathrm{odd/even}} - V_\ell^{\mathrm{RW/Z}}$.

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023), Phys. Rev. D **107**, 084039 (2023)

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - {}_{\rm D}V_\ell^{\rm odd/even}\right){}_{\rm D}\Psi^{\rm odd/even} = 0$$

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with $\delta V = {}_{\rm D}V_\ell^{\rm odd/even} - V_\ell^{\rm RW/Z}$.

- Most general (potential dependent) master function

$${}_{\rm D}\Psi^{\rm odd/even} = \begin{cases} \mathcal{C}\left(\Sigma^{\rm odd}\,\Psi^{\rm CPM} + \Phi^{\rm odd}\right) & \text{odd parity} \\ \mathcal{C}\left(\Sigma^{\rm even}\,\Psi^{\rm ZM} + \Phi^{\rm even}\right) & \text{even parity} \end{cases}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023), Phys. Rev. D **107**, 084039 (2023)

# Darboux covariance

# Darboux transformation

- Darboux transformation between $(v, \Phi)$ and $(V, \Psi)$

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = 0 \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0$$

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$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

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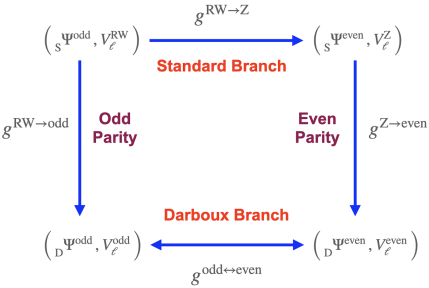
- Darboux covariance of perturbations of spherically-symmetric BHs




M. Lenzi and C. F. Sopuerta,
Phys. Rev. D **104**, 124068
(2021)

- Darboux transformation between $(v, \Phi)$ and $(V, \Psi)$

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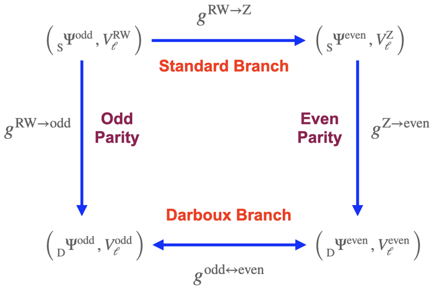


**Isospectral symmetry**

$$\Psi = e^{ikt}\psi$$

$$\phi_{,xx} - v\phi = -k^2\phi$$
$$\psi_{,xx} - V\psi = -k^2\psi$$

# Darboux transformation

- Darboux transformation between $(v, \Phi)$ and $(V, \Psi)$

$$\left(-\partial_t^2 + \partial_x^2 - v\right)\Phi = \boxed{\sigma} \longrightarrow \begin{cases} \Psi = \Phi_{,x} + W\,\Phi \\ V = v + 2\,W_{,x} \\ W_{,x} - W^2 + v = \mathcal{C} \end{cases} \longrightarrow \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = \boxed{S}$$

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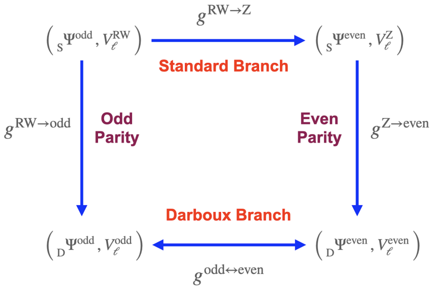


Darboux covariance with perturbative sources

$$\boxed{S = \sigma_{,x} + W\sigma}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **109**, 084030 (2024)

# KdV structures in Cauchy slices

- Korteweg-de Vries deformations: isospectral symmetries of the master equation

- A triangle of integrable structures: KdV-Virasoro-Schwarzian derivative

- Conformal transformations of the master equation: Schwarzian derivative modification in the potential

J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, 2407.14196 (2024)
M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021), Phys. Rev. D **107**, 044010 (2023)

# Korteweg-de Vries isospectral deformations

$$V_{,\sigma} - 6VV_{,x} + V_{,xxx} = 0$$

- Darboux transformation + inverse scattering solves the KdV equation

C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095–1097 (1967)

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- KdV deformations of the frequency domain master equation

$$\begin{cases} V(x) \to V(\sigma, x) \\ \psi(x) \to \psi(\sigma, x) \\ k \to k(\sigma) \end{cases}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$\partial_\sigma V = \{V, \mathcal{H}\} \quad \longrightarrow \quad I_n[V] = \int_{-\infty}^{\infty} dx\, P_n(V, V_{,x}, V_{,xx}, \ldots)$$

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$\boxed{I_n[V] = I_n[V_{\mathrm{RW}}]}$$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

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- KdV equation as an integrable Hamiltonian system with infinite conserved quantities

$$I_n[V] = I_n[V_{\mathrm{RW}}]$$

- BH greybody factors from KdV integrals $\boxed{T = T\left[\{I_n\}\right]}$

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **107**, 044010 (2023), Phys. Rev. D **107**, 084039 (2023)

# The KdV-Virasoro-Schwarzian derivative triangle

- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\mathrm{GFZ}} = \{V, \mathcal{H}_1\}_{\mathrm{M}} \ ,$$

- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

$$\partial_\sigma V = \{V, \mathcal{H}_2\}_{\text{GFZ}} = \{V, \mathcal{H}_1\}_{\text{M}} \ ,$$

- Magri brackets are the classical realization of the Virasoro algebra

$$V(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+1}} \longrightarrow \pi i \{L_n, L_m\}_{\text{M}} = (n-m)L_{n+m} - \frac{n(n^2-1)}{2}\delta_{n+m,0}$$

J.-L. Gervais, Phys. Lett. B **160**, 277–278 (1985), J.-L. Gervais, Physics Letters B **160**, 279–282 (1985)

- Bi-Hamiltonian structure of KdV: Gardner-Zakharov-Faddeev brackets and Magri brackets

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- Solitonic potential as a CFT energy-momentum tensor:
  - Infinitesimal conformal transformation of V: $w(z) = z + \epsilon(z)$

$$\delta_\epsilon V(w) = \{V(w), F_\epsilon\}_{\mathrm{M}} \ , \quad F_\epsilon = -\frac{1}{2} \int dz \, \epsilon(z) V(z)$$

  - Finite conformal transformation of V: Schwarzian derivative

$$V(w) = \left(\frac{dw}{dz}\right)^{-2} \left[ V(z) + \frac{1}{2} \mathcal{S}(w(z)) \right] \ , \quad \mathcal{S}\left(w(z)\right) \equiv \frac{w_{zzz}}{w_z} - \frac{3}{2} \left( \frac{w_{zz}}{w_z} \right)^2$$

$$\psi_{,xx} - V\psi = -k^2\psi$$

- Perform the following general transformation

$$\begin{cases} x & \mapsto & x = x(\tilde{x})\,, \\ \psi & \mapsto & \psi(x) = \omega(\tilde{x})\tilde{\psi}(\tilde{x}) \end{cases} \longrightarrow a(\tilde{x})\tilde{\psi}_{,\tilde{x}\tilde{x}} + b(\tilde{x})\tilde{\psi}_{,\tilde{x}} + c(\tilde{x})\tilde{\psi} = -k^2\omega\tilde{\psi}$$

- Cancel first order derivative terms to preserve the operator structure, i.e. $b(\tilde{x}) = 0$ to obtain

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \left(k^2 x_{\tilde{x}}^2 - \tilde{V}\right)\tilde{\psi} = 0\,, \quad \tilde{V}(\tilde{x}) = \left(\frac{d\tilde{x}}{dx}\right)^{-2}\left[V(x) + \frac{1}{2}\mathcal{S}(\tilde{x}(x))\right]$$

$$\psi_{,xx} - V\psi = -k^2\psi$$

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The Schwarzian derivative tracks the KdV (hidden) integrable structure

# KdV structures in hyperboloidal slices

- Review of hyperboloidal slicing

- Covariance of the hyperboloidal slicing under general scale tranformations

- Conformal transformations of the master equation

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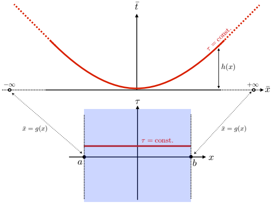
J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, 2407.14196 (2024)
E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)

# Hyperboloidal slicing

$$\left(-\partial_t^2 + \partial_x^2 - V_\ell\right)\phi = 0$$

- Perform the following transformation

$$(t,x) \to (\tau,\xi) \;:\; \left\{ \begin{array}{lcl} t & = & \tau - h(\xi) \\ x & = & g(\xi) \end{array} \right.$$




- With $\psi = \partial_\tau \phi$, the master equation becomes

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = i\mathbb{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \mathbb{L} = \frac{1}{i}\begin{pmatrix} 0 & 1 \\ \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix}, \qquad \begin{array}{ll} \mathcal{L}_1 & \text{bulk} \\ \mathcal{L}_2 & \text{boundary} \end{array}$$

E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)

- The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are related to "bulk" and "boundary" features of the system respectively

E. Gasperin and J. L. Jaramillo, Class. Quant. Grav. **39**, 115010 (2022), J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, Phys. Rev. X **11**, 031003 (2021)

- The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are related to "bulk" and "boundary" features of the system respectively

- The hyperboloidal formulation is covariant under the scale transformation $\phi(\tau, \xi) = \Omega(\xi)\tilde{\phi}(\tau, \xi)$

- Fix $\Omega$ by cancelling the term containing $\partial_\xi \tilde{\phi}$

$$\mathcal{L}_1[V] \to \mathcal{L}_1[\tilde{V}] \qquad \tilde{V}_\ell = g'^2 V_\ell - \frac{1}{2}\mathcal{S}(g)$$

The KdV/Virasoro/Schwarzian (hidden) integrable structure is now embedded in the hyperbolidal setting

J. L. Jaramillo, M. Lenzi, and C. F. Sopuerta, 2407.14196 (2024)

# Conclusions

- Interplay with asymptotic dynamics
- Implementation of covariant and conformal transformations of hyperboloidal approach
- Symmetry consistency with physical properties (e.g. tidal Love numbers)
- Hidden symmetries as a tool for algebraic estimation of QNMs
- Inverse scattering (BH spectroscopy)