Algebraic classification of 2+1 spacetimes

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Novel algebraic classification

Conformal invariant tensor in 3D — Cotton tensor: [Cotton 1899]

$$C_{abc} \equiv 2\Big(\nabla_{[a}R_{b]c} - \tfrac{1}{4}\nabla_{[a}R\, g_{b]c}\Big)$$

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Newman–Penrose type scalars

$$\Psi_0 \equiv C_{abc}\, k^a\, m^b\, k^c \qquad \Psi_2 \equiv C_{abc}\, k^a\, m^b\, l^c \qquad \Psi_3 \equiv C_{abc}\, l^a\, k^b\, l^c$$
$$\Psi_1 \equiv C_{abc}\, k^a\, l^b\, k^c \qquad \Psi_4 \equiv C_{abc}\, l^a\, m^b\, l^c$$

Projections of the Cotton tensor on the null triad

$$\mathbf{k} \cdot \mathbf{l} = -1 \qquad \mathbf{m} \cdot \mathbf{m} = 1$$

Novel algebraic classification

The algebraic classification is then given by the gradual vanishing of these scalars

algebraic type	the conditions	
I	$\Psi_0 = 0\,,$	$\Psi_1 \neq 0$
II	$\Psi_0 = \Psi_1 = 0\,,$	$\Psi_2 \neq 0$
III	$\Psi_0 = \Psi_1 = \Psi_2 = 0\,,$	$\Psi_3 \neq 0$
N	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,,$	$\Psi_4 \neq 0$
D	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4\,,$	$\Psi_2 \neq 0$
O	all $\Psi_{\mathrm{A}} = 0$	

Bel–Debever criteria

The algebraic classification can be related to an important property of the Cotton tensor

The Bel–Debever criteria

$$
\begin{aligned}
k_{[d}\, C_{a]bc}\, k^b\, k^c = 0 \quad &\Leftrightarrow \quad \Psi_0 = 0 \\
C_{abc}\, k^b\, k^c = 0 \quad &\Leftrightarrow \quad \Psi_0 = \Psi_1 = 0 \\
k_{[d}\, C_{a]bc}\, k^b = 0 \quad &\Leftrightarrow \quad \Psi_0 = \Psi_1 = \Psi_2 = 0 \\
C_{abc}\, k^b = 0 \quad &\Leftrightarrow \quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0
\end{aligned}
$$

The null vector $\mathbf{k}$ is then the principal null direction of the Cotton tensor

Lorentz transformations of the Cotton scalars

The choice of the null triad satisfying $\mathbf{k} \cdot \mathbf{l} = -1$ and $\mathbf{m} \cdot \mathbf{m} = 1$ is not unique

The freedom is given by the local Lorentz transformation

3 subgroups of the Lorentz transformation

- Boost:

$$\mathbf{k}' = B\,\mathbf{k} \qquad \mathbf{l}' = B^{-1}\,\mathbf{l} \qquad \mathbf{m}' = \mathbf{m}$$

- Null rotation with fixed $\mathbf{k}$:

$$\mathbf{k}' = \mathbf{k} \qquad \mathbf{l}' = \mathbf{l} + \sqrt{2}L\,\mathbf{m} + L^2\mathbf{k} \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}L\,\mathbf{k}$$

- Null rotation with fixed $\mathbf{l}$:

$$\mathbf{k}' = \mathbf{k} + \sqrt{2}K\,\mathbf{m} + K^2\mathbf{l} \qquad \mathbf{l}' = \mathbf{l} \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}K\,\mathbf{l}$$

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Transformation of the Newman–Penrose scalars

- Boost: [Milson et al. 2005]

$$\Psi'_{\mathrm{A}} = B^{2-\mathrm{A}}\,\Psi_{\mathrm{A}}$$

- Null rotation with fixed $\mathbf{l}$:

$$\Psi'_0 = \Psi_0 + 2\sqrt{2}\,K\,\Psi_1 + 6K^2\,\Psi_2 - 2\sqrt{2}\,K^3\,\Psi_3 - K^4\,\Psi_4$$

$$\Psi'_1 = \Psi_1 + 3\sqrt{2}\,K\,\Psi_2 - 3K^2\,\Psi_3 - \sqrt{2}\,K^3\,\Psi_4$$

$$\Psi'_2 = \Psi_2 - \sqrt{2}\,K\,\Psi_3 - K^2\,\Psi_4$$

$$\Psi'_3 = \Psi_3 + \sqrt{2}\,K\,\Psi_4$$

$$\Psi'_4 = \Psi_4$$

Cotton aligned null directions (CANDs)

We can always set $\Psi_0 = 0$ by solving the equation

$$\Psi_4\, K^4 - 2\sqrt{2}\,\Psi_3\, K^3 - 6\,\Psi_2\, K^2 + 2\sqrt{2}\,\Psi_1\, K - \Psi_0 = 0$$

These directions are called the Cotton aligned null directions

Multiplicity of CANDs

algebraic type	CANDs	multiplicity	canonical Cotton scalars	
I	$\nwarrow\nwarrow\nearrow\nearrow$	$1+1+1+1$	$\Psi_0 = 0\,,$	$\Psi_1 \neq 0$
II	$\nwarrow\nwarrow\Nearrow$	$1+1+2$	$\Psi_0 = \Psi_1 = 0\,,$	$\Psi_2 \neq 0$
D	$\Nwarrow\Nearrow$	$2+2$	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4\,,$	$\Psi_2 \neq 0$
III	$\nwarrow\ \nearrow\!\!\!\nearrow\!\!\!\nearrow$	$1+3$	$\Psi_0 = \Psi_1 = \Psi_2 = 0\,,$	$\Psi_3 \neq 0$
N	$\nearrow\!\!\!\nearrow\!\!\!\nearrow\!\!\!\nearrow$	4	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,,$	$\Psi_4 \neq 0$
O		N/A	all $\Psi_A = 0$	

Algebraic classification using invariants

A basis independent classification can be constructed using polynomial invariants [d'Inverno et al. 1971]

Quadratic and cubic invariants of the Cotton tensor

$$C_{abc}\, C^{abc} = 4\,(\Psi_0\Psi_4 - 2\,\Psi_1\Psi_3 - 3\,\Psi_2^2)$$

$$C_{abc}\, C^{abd}\, Y^c{}_d = 6\,(\Psi_0\Psi_3^2 - \Psi_1^2\Psi_4 + 2\,\Psi_0\Psi_2\Psi_4 + 2\,\Psi_1\Psi_2\Psi_3 + 2\,\Psi_2^3)$$

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Quadratic and cubic invariants of the Cotton tensor

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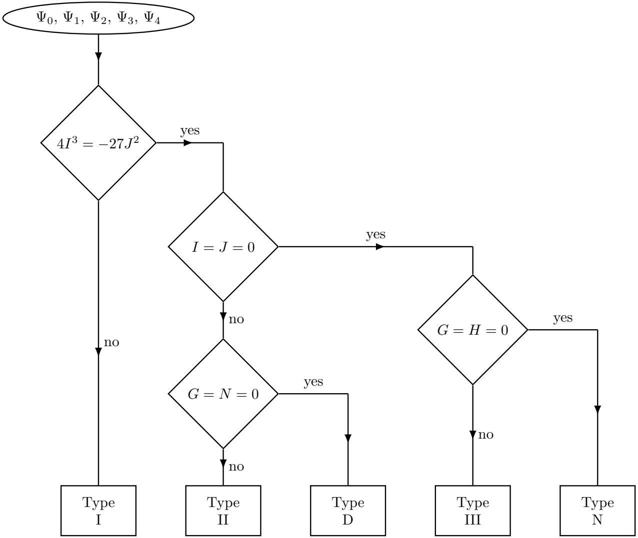
$$C_{abc}\,C^{abd}\,Y^c{}_d = 6\,(\Psi_0\Psi_3^2 - \Psi_1^2\Psi_4 + 2\,\Psi_0\Psi_2\Psi_4 + 2\,\Psi_1\Psi_2\Psi_3 + 2\,\Psi_2^3)$$

We then define important quantities $I \equiv \frac{1}{4}C_{abc}\,C^{abc}$ and $J \equiv \frac{1}{6}C_{abc}\,C^{abd}\,Y^c{}_d$ together with

the remaining covariants

$$G \equiv \Psi_1\Psi_4^2 - 3\,\Psi_2\Psi_3\Psi_4 - \Psi_3^3$$
$$H \equiv 2\,\Psi_2\Psi_4 + \Psi_3^2$$
$$N \equiv 3\,H^2 + \Psi_4^2\,I$$

Algebraic classification using invariants



Relation to the previous method of classification

Key geometric object is the Cotton–York tensor
[Barrow et al. 1986, García-Díaz et al. 2004]

$$Y_{ab} = \tfrac{1}{2}\, g_{ak}\, \omega^{kmn}\, C_{mnb}$$

Idea is to solve the eigenvalue problem

$$Y_a{}^b\, \nu_b = \lambda\, \nu_a \quad \Leftrightarrow \quad \mathsf{det}(Y_a{}^b - \lambda\, \delta_a{}^b) = 0$$

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However, we can express the Cotton–York tensor in terms of the Cotton scalars and find their relation to the eigenvalues

$$Y_a{}^b = \begin{pmatrix} \Psi_2 + \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{2}(\Psi_0 + \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) \\ \frac{1}{2}(\Psi_0 + \Psi_4) & \Psi_2 - \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) \\ \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix}$$

Relation to the previous method of classification

This will give us the values of the Cotton scalars in some basis

Using the invariants we can show the equivalence of these methods

algebraic type	Jordan normal form of ${Y_a}^b$	special Cotton scalars	invariants
I	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}$	$\Psi_1 = 0 = \Psi_3$ $\Psi_0 = \frac{1}{2}(\lambda_1 - \lambda_2) = -\Psi_4$ $\Psi_2 = \frac{1}{2}(\lambda_1 + \lambda_2)$	$I = \lambda_1\lambda_2 - (\lambda_1+\lambda_2)^2$ $J = \lambda_1\lambda_2\,(\lambda_1+\lambda_2)$
II	$\begin{pmatrix} \lambda_1 - 1 & -1 & 0 \\ 1 & \lambda_1 + 1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	$\Psi_0 = 0\,,\ \Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$ $\Psi_4 = 2$	$I = -3\,\lambda_1^2$ $J = 2\,\lambda_1^3$ $G = 0\,,\ N = 36\,\lambda_1^2$
D	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$	$I = -3\,\lambda_1^2$ $J = 2\,\lambda_1^3$ $G = 0 = N$
III	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_2$ $\Psi_3 = \sqrt{2}$	$I = 0 = J$ $G = -2\sqrt{2}\,,\ H = 2$
N	$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_2$ $\Psi_1 = 0 = \Psi_3$ $\Psi_4 = 2$	$I = 0 = J$ $G = 0 = H$

Example(s)

Robinson–Trautman spacetimes with an EM field

$$\mathrm{d}s^2 = \frac{r^2}{P^2}\left(\mathrm{d}x + e\,P^2\mathrm{d}u\right)^2 - 2\,\mathrm{d}u\,\mathrm{d}r - 2H\,\mathrm{d}u^2$$

$$2H = -m + \kappa_0\,Q^2 \ln\left|\frac{Q}{r}\right| - 2\,(\ln Q)_{,u}\,r - \Lambda\,r^2$$

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Robinson–Trautman spacetimes with an EM field

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$$2H = -m + \kappa_0\,Q^2\ln\left|\frac{Q}{r}\right| - 2\,(\ln Q)_{,u}\,r - \Lambda\,r^2$$

With respect to the null basis

$$\mathbf{k} = \partial_r \quad \mathbf{l} = \partial_u - H\,\partial_r - eP^2\,\partial_x \quad \mathbf{m} = \frac{P}{r}\,\partial_x$$

The Newman–Penrose scalars are

$$\Psi_0 = 0 = \Psi_4 \qquad \Psi_2 = 0$$

$$\Psi_1 = -\frac{\kappa_0 Q^2}{2r^3} \qquad \Psi_3 = \frac{1}{2}\left(m - \kappa_0 Q^2\log\left|\frac{Q}{r}\right| + \Lambda r^2\right)\Psi_1$$

Summary

- We proposed an effective method of algebraic classification in 3D based on the Newman–Penrose scalars
- We derived a simple algorithm for the classification based on the polynomial invariants
- We showed that the classification is fully equivalent with the older method

Thank you for your attention!