# Generalized Siklos space-times

Ángel Jesús Murcia Gil, INFN Padova (Italy)

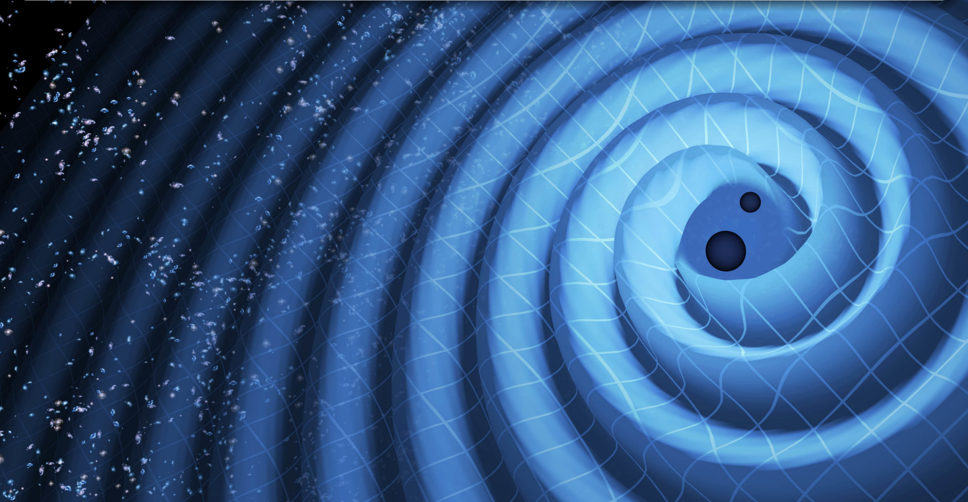
INFN PADOVA
Istituto Nazionale di Fisica Nucleare
Sezione di Padova

Spanish and Portuguese Relativity Meeting 2024, Coimbra (Portugal)

Based on arXiv:2407:01359 [gr-qc] with B. Araneda.

# Gravitational waves. A new portal to gravity

Ever since their first observation, **gravitational waves** (GWs) have acquired **central role** in the empirical exploration of **gravity**.



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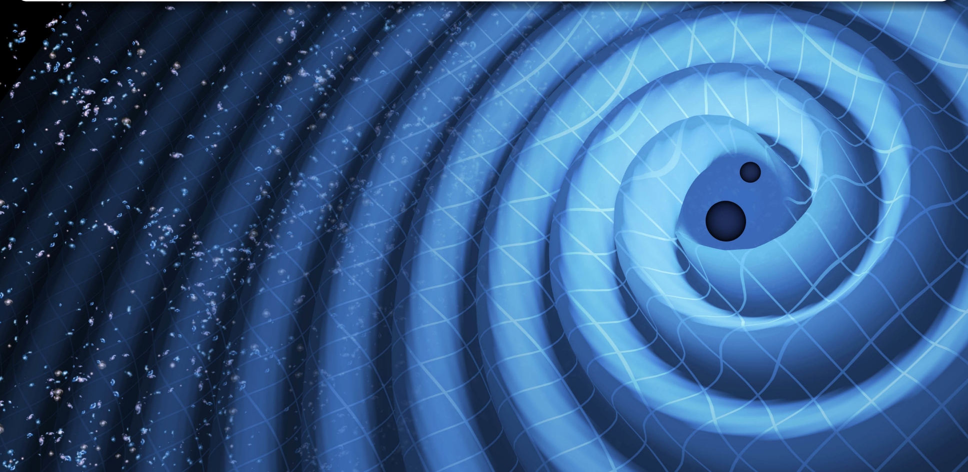
Do there exist **exact solutions** of **GR** which admit a **gravitational-wave interpretation**?

# PP-waves

**There are exact** solutions of GR with a **GW interpretation**. Prominent example: **pp-waves** [Brinkmann '25; Hély '59; Peres '59].



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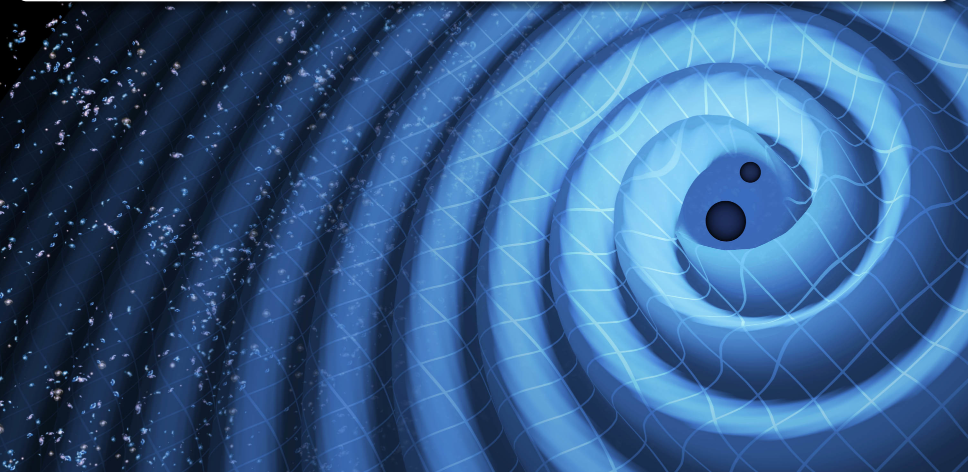

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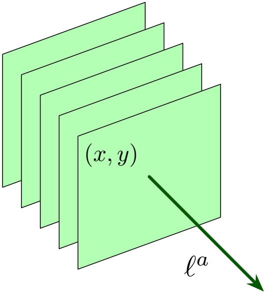


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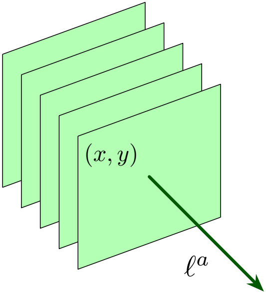


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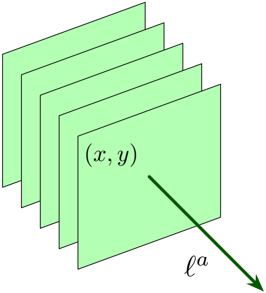


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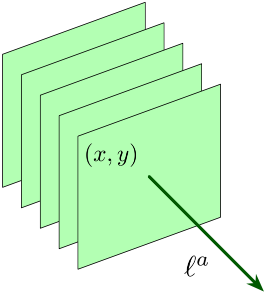


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**Physical** interpretation: **plane-fronted waves** with **parallel rays** propagating on **Minkowski** [*e.g.* Ehlers, Kundt '62; Roche, Aazami, Cederbaum '22].

# PP-waves and parallel spinors

If our space-time $(M, g)$ is spin, **pp-waves** are **equivalent** to existence of **parallel spinor**! [*e.g.* Tod '83; Bryant '00; ÁM, Shahbazi '21; Araneda '22]

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**Parallel spinor**: **Geometric technique** to solve **Einstein equations** (2nd order) through **1st order equation**!

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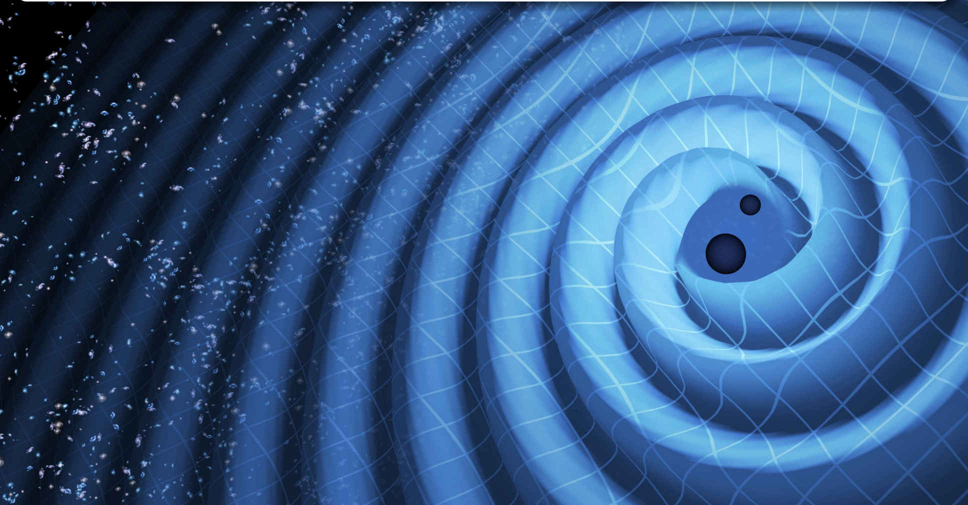

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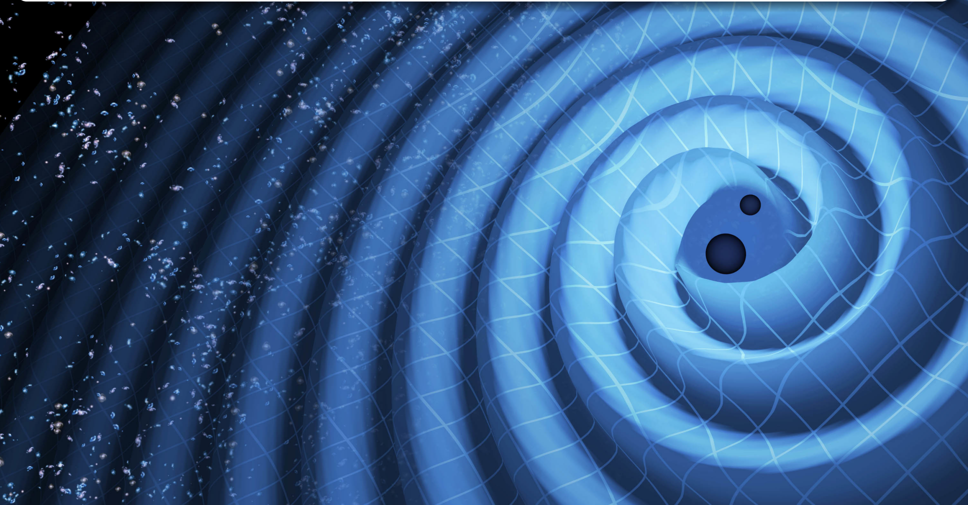

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Inspired by **supersymmetry** conditions in **supergravity**, Siklos analyzed space-times $(M, g)$ endowed with the following type of spinors [Siklos '85]:

$$\nabla_m \psi = \frac{b}{\sqrt{2}} \gamma_m \psi \,, \quad b \in \mathbb{C} \,.$$

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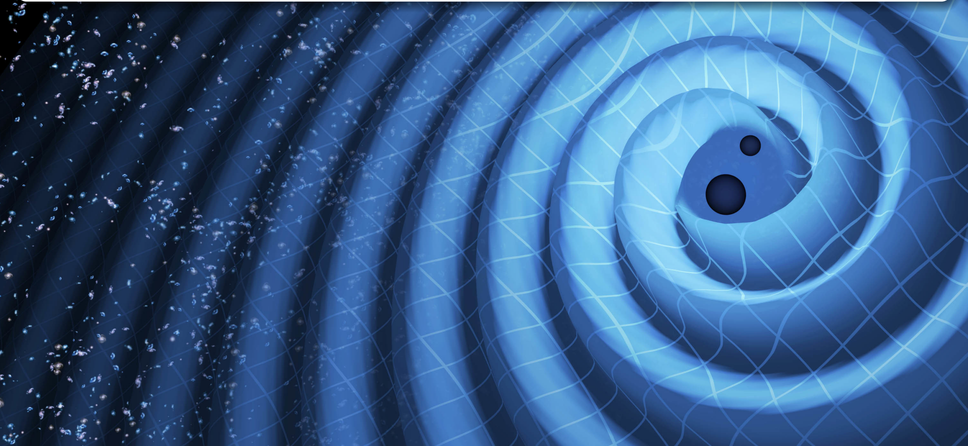
If $(M, g)$ not AdS, $b \in \mathbb{R}$ and the Killing spinor $\psi = (o_A, \tilde{o}_{A'})^T$ is **Majorana** [Araneda, ÁM '24]. Now $\ell^a = o^A \tilde{o}^{A'}$ is **no longer parallel**. It is **Killing**:

$$\nabla_{(a} \ell_{b)} = 0 \,.$$

# Properties of Siklos space-times

Siklos space-times are locally isometric to [Siklos '85]:

$$\mathrm{d}s^2_{\mathrm{Siklos}} = \frac{1}{b^2 x^2}\left[H(v,x,y)\mathrm{d}v^2 + 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2\right] .$$



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Physical interpretation: **Siklos space-times** correspond to **exact GWs** propagating on top of **AdS** [Podolský '97], with propagation vector $\ell_a$.

# Generalized Siklos space-times

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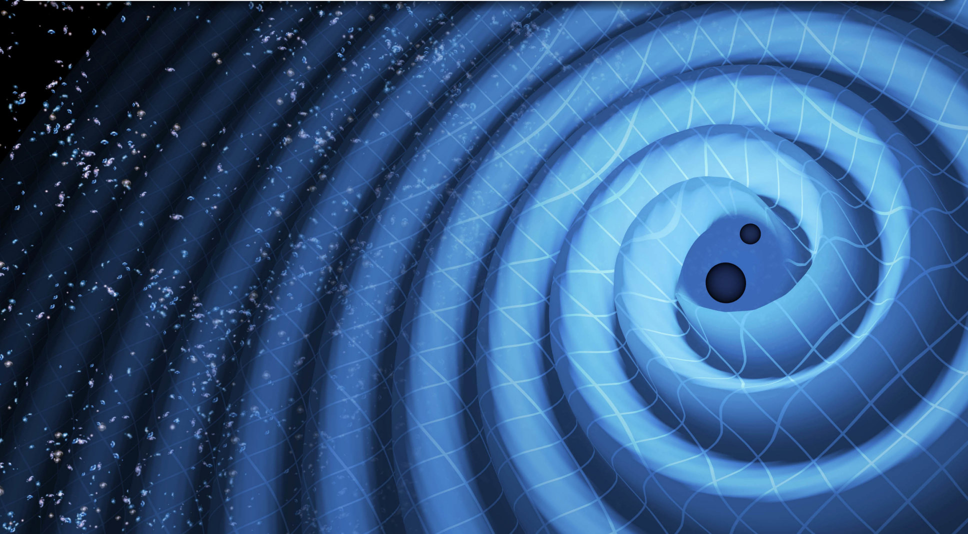

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A given **space-time** is said to be **generalized Siklos** if it possesses a spinor $\psi$ satisfying **[Araneda, ÁM '24]**:

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2. May be useful to construct **solutions** to **Einstein equations** with certain **matter content**.

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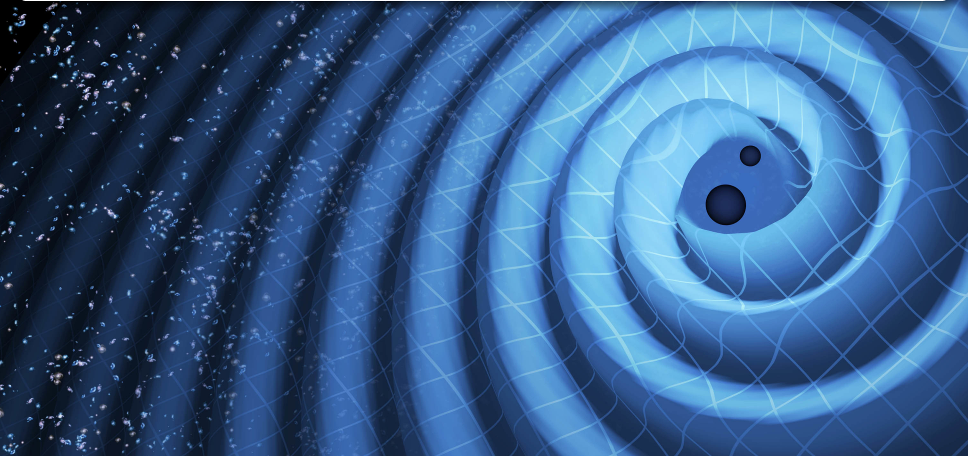
If $(M, g)$ is a **generalized Siklos space-time**:

1. If $(M, g)$ not conformally flat, Killing function $\lambda$ is **real** and $\psi = (o_A, \tilde{o}_{A'})$ may be taken to be **Majorana**.



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3. $(M, g)$ **conformal** to **pp-wave**:

$$\mathrm{d}s^2_{\mathrm{gen-Siklos}} = \frac{1}{\Omega^2(x, v)} \left[2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 + H\mathrm{d}v^2\right] ,$$

$$\Omega(x, v) = \int \lambda(x, v)\, \mathrm{d}x + f(v) , \quad f \in C^\infty(\mathbb{R}) .$$

# Generalized Siklos space-times

4. **Weyl** tensor is **type N**, so **gravitational dofs** are interpreted as **waves**: $u$ is **affine parameter** along the rays given by $\ell^a = (\partial_u)^a$ and **wave surfaces** are spanned by coordinates $(x, y)$.

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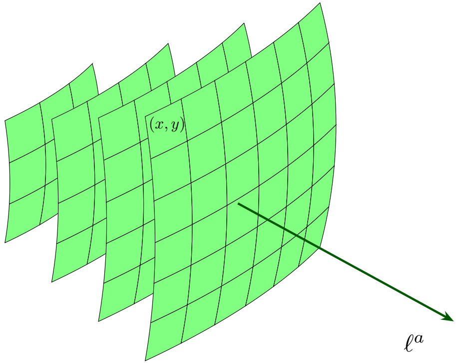

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2. **Physically relevant case**: $\Omega_{xx} = 0$, which is **equivalent** to standard **Siklos space-times**.

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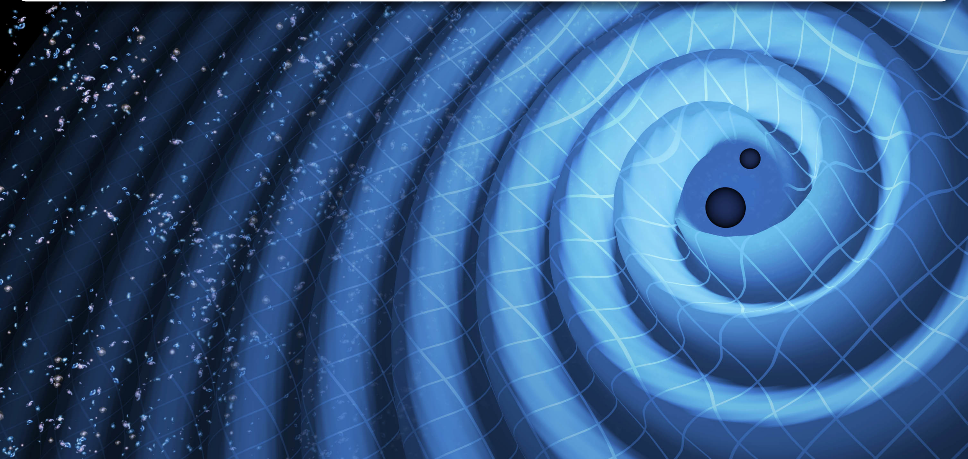
Space-times with **spinors satisfying** a given **differential condition**:



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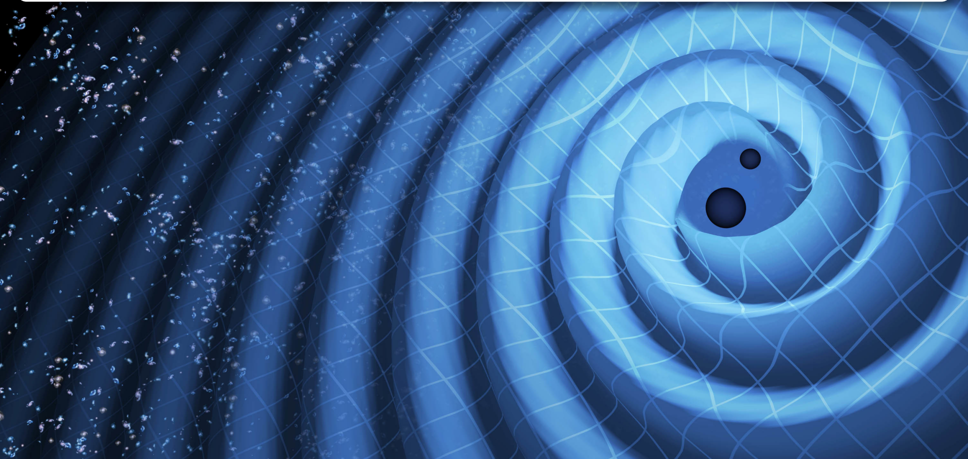

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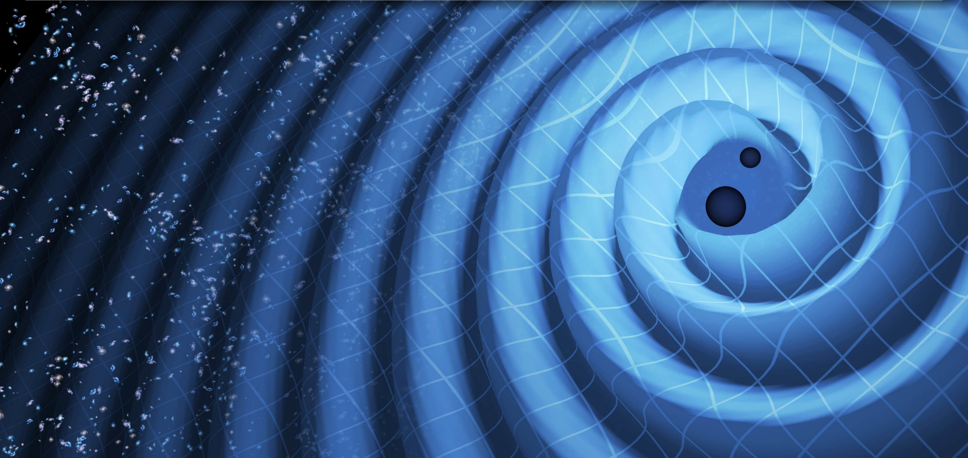

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1. Admit a **spinor** $\psi$ such that $\nabla_m \psi = \frac{\lambda}{\sqrt{2}} \gamma_m \psi$, with $\lambda \in C^\infty(M, \mathbb{C})$.
2. **Gravitational dofs** correspond to **waves**. Associated **matter** sources to superposition of **pure radiation** and **spacelike** perfect **fluid**.
3. **Physically relevant case**: **absent spacelike fluid**, corresponding to standard **Siklos space-times**.

# Conclusions

Space-times with **spinors satisfying** a given **differential condition**:

1. **Prescribed form** of **Ricci curvature**.
2. **GW interpretation**.

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**Muito obrigado!**