STEGR, numerical relativity and f(Q) cosmology

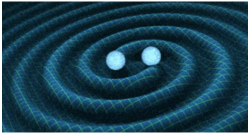
María José Guzmán M.

University of Tartu

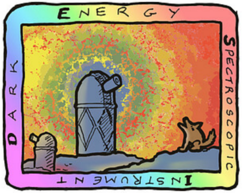
July 24, 2024

Outline

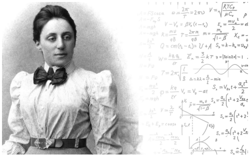
Motivation



Strong gravity is nonlinear, needs numerical relativity to be accurately described



Cosmological tensions H_0 and DESI variable dark energy results suggest something is missing in GR



Theoretical understanding of the metric affine framework gives insights about GR itself

Teleparallel framework

- Metric and affine connection are two separate entities. The metric can be defined from the tetrad θ and cotetrad e

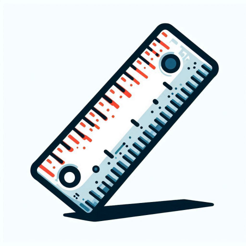
$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu \eta_{ab}, \qquad \eta_{ab} = g_{\mu\nu} e_a{}^\mu e_b{}^\nu. \tag{1}$$

- The most general linear connection is related to the spin connection ω and the tetrad as

$$\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho \partial_\nu \theta^a{}_\mu + e_a{}^\rho \omega^a{}_{b\nu} \theta^b{}_\mu \tag{2}$$

- In the metric affine framework $\Gamma^\rho{}_{\mu\nu}$ is free but in teleparallelism it satisfies some conditions.

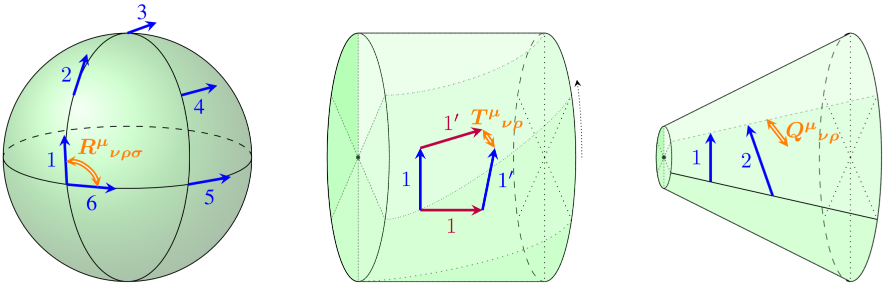
Metric and affine connection



$$\longrightarrow \qquad g_{\mu\nu}$$

$$\longrightarrow \qquad \Gamma^{\rho}{}_{\mu\nu}$$

Curvature, Torsion, Nonmetricity



https://www.akadeemia.ee/wp-content/uploads/2020/08/ev_preemaid_2020_veebi1.pdf

Teleparallel framework

The condition of vanishing curvature

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \equiv 0 \qquad (3)$$

is satisfied by the most general connection parameterised by an element $L^{\mu}{}_{\nu}$ of the general linear group $GL(4, \mathbb{R})$ as

$$\Gamma^{\alpha}{}_{\mu\nu} = (L^{-1})^{\alpha}{}_{\lambda}\partial_{\mu}L^{\lambda}{}_{\nu}$$

$$\omega^{a}{}_{b\mu} = \Lambda^{a}{}_{c}\,\partial_{\mu}(\Lambda^{-1})^{c}{}_{b}$$

metric tele

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial \xi^{\sigma}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \xi^{\sigma}}{\partial x^{\nu}}\right)$$

symmetric tele

Teleparallel framework

$$R^{\rho}{}_{\sigma\mu\nu} = 0$$

↓

and

↙ ↘

$$Q_{\rho\mu\nu} = 0 \quad \text{or} \quad T^{\rho}{}_{\mu\nu} = 0$$

↓ ↓

metric tele symm tele

Symmetric teleparallel

The nonmetricity scalar

$$\mathbb{Q} = \frac{1}{4}Q_{\rho\mu\nu}Q^{\rho\mu\nu} - \frac{1}{2}Q_{\rho\mu\nu}Q^{\mu\nu\rho} - \frac{1}{4}Q_{\mu}Q^{\mu} + \frac{1}{2}Q_{\mu}\tilde{Q}^{\mu} \quad (4)$$

is a special quadratic combination of the nonmetricity tensor $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$, since it is equivalent to the Ricci scalar up to a boundary term

$$\mathring{R} = -\mathbb{Q} - \mathring{\nabla}_\mu(Q^\mu - \tilde{Q}^\mu) \quad (5)$$

where it has been defined the following two (independent) traces of the nonmetricity tensor

$$Q^\mu = Q^{\mu\nu}{}_{\nu}, \quad \tilde{Q}^\mu = Q_\nu{}^{\mu\nu} = Q_\nu{}^{\nu\mu}.$$

Boundary term

After setting the coincident gauge and writing the STEGR Lagrangian as the $\Gamma\Gamma$ part of the Ricci scalar, we obtain

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{\gamma}\alpha\left[-{}^{(3)}\mathbb{Q} - D_i(Q^i - \tilde{Q}^i) + K^2 - K^{ij}K_{ij}\right], \qquad (6)$$

where it has been defined the 3 dimensional nonmetricity scalar as

$${}^{(3)}\mathbb{Q} = {}^{(3)}\Gamma^i{}_{ij}{}^{(3)}\Gamma^{jk}{}_k - {}^{(3)}\Gamma^i{}_{jk}{}^{(3)}\Gamma^j{}_i{}^k \qquad (7)$$

Without any further action, this action is equivalent to the Einstein-Hilbert action of GR, since it can be proved that

$$\boxed{{}^{(3)}\mathbb{R} = -{}^{(3)}\mathbb{Q} - D_i(Q^i - \tilde{Q}^i).} \qquad (8)$$

[F. D'Ambrosio, M. Garg, L. Heisenberg, S. Zentarra, 2007.03261]

Modified Hamiltonian constraint

After integration by parts, it can be obtained

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{\gamma}\alpha\left[-{}^{(3)}\mathbb{Q} + K^2 - K^{ij}K_{ij} + \frac{\partial_i\alpha}{\alpha}(Q^i - \tilde{Q}^i)\right]. \qquad (9)$$

Evidently, the Hamiltonian constraint is modified as

$$\boxed{H = -{}^{(3)}\mathbb{Q} + K^2 - K^{ij}K_{ij} + \frac{\partial_i\alpha}{\alpha}(Q^i - \tilde{Q}^i)} \qquad (10)$$

[M.J. Guzman, 2311.01424]

Modified Hamiltonian constraint

For an intrinsic metric with spherical symmetry

$$dl^2 = A(t,r)dr^2 + r^2 B(t,r)\left[d\theta^2 + \sin^2(\theta)d\varphi^2\right] = \gamma_{ij}dx^i dx^j, \tag{11}$$

the Hamiltonian constraint in STEGR is

$$\mathcal{H} = AK_B(2K_A + K_B) + \frac{\partial_r \alpha}{\alpha}\left(\frac{4}{Ar} + \frac{2}{AB}\partial_r B\right) + {}^{(3)}\mathbb{Q} \tag{12}$$

which is actually simpler than the GR one [Alcubierre (2008)]

$$\begin{aligned} \mathcal{H} = {} & AK_B(2K_A + K_B) + \frac{1}{r^2 B}(A - B) - \partial_r D_B \\ & + \frac{1}{r}(D_A - 3D_B) + \frac{D_A D_B}{2} - \frac{3D_B^2}{4}. \end{aligned} \tag{13}$$

($D_A := \partial_r \ln A, \qquad D_B := \partial_r \ln B$)

[M.J.Guzman, 2311.01424]

f(Q) gravity

The nonlinear modification of STEGR is $f(Q)$ gravity

$$S = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}f(\mathbb{Q}) + S_{\mathrm{m}}(g)\,, \tag{14}$$

with metric equations of motion

$$f_Q\mathring{G}_{\mu\nu} + 2f_{QQ}P^{\lambda}{}_{\mu\nu}\partial_\lambda\mathbb{Q} + \frac{1}{2}g_{\mu\nu}\left(\mathbb{Q}f_Q - f\right) = \kappa^2\mathcal{T}_{\mu\nu}, \tag{15}$$

and connection equation (dependent on the previous ones)

$$\nabla_\mu\nabla_\nu\left(\sqrt{-g}f_QP^{\mu\nu}{}_\alpha\right) = 0. \tag{16}$$

Traditional cosmological solutions in f(Q)

Known cosmological solutions in f(Q) for the metric

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{17}$$

require the presence of a nonvanishing connection

$$\Gamma^\rho{}_{\mu\nu} = \left[\begin{bmatrix} \gamma(t), \gamma(t) + \dot{\gamma}/\gamma, -\frac{\dot{\gamma}(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & \gamma(t) & 0 & 0 \\ 0 & 0 & r^2\gamma(t) & 0 \\ 0 & 0 & 0 & r^2\gamma(t)\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & \gamma(t) & 0 & 0 \\ \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \right.$$
$$\left. \begin{bmatrix} 0 & 0 & \gamma(t) & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \gamma(t) & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \gamma(t) \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \gamma(t) & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right]. \tag{18}$$

Symmetry assumptions

One common approach to solve the eom of f(T)/f(Q) is to impose the Lie derivative along the Killing vectors of the metric

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu \xi^\sigma g_{\sigma\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma} = 0, \tag{19}$$

also on the connection

$$\mathcal{L}_\xi \Gamma^\mu{}_{\nu\rho} = \xi^\sigma \partial_\sigma \Gamma^\mu{}_{\nu\rho} - \partial_\sigma \xi^\mu \Gamma^\sigma{}_{\nu\rho} + \partial_\nu \xi^\sigma \Gamma^\mu{}_{\sigma\rho} + \partial_\rho \xi^\sigma \Gamma^\mu{}_{\nu\sigma} + \partial_\nu \partial\rho \xi^\mu. \tag{20}$$

The procedure works, but what if we are missing other families of solutions?

Cosmological stability of f(Q) with different connections

- Stability of cosmological background evolution can be studied near the general relativity regime across radiation, matter, dark energy, and geometric dark energy dominated eras.
- In connection set 1 the general relativity regime can be realized in two ways and both exhibit stable behavior throughout all evolutionary epochs.
- However, in connection set 2, trivial GR limit is stable. Nontrivial limit exhibits stability during radiation era and marginal stability during matter era. For dark energy and geometric dark energy, results are inconclusive.
- For a generic $f(Q)$ the connection sets 2 and 3 are prone to trigger a sudden singularity.

[M.J.Guzman, L. Järv, L. Pati, arxiv:gr-qc/2406.11621.]

Letting go of connection symmetries: work in progress!

- Firstly, we can generate flat symmetric connections from

$$\tilde{\Gamma}^{\alpha}{}_{\mu\beta} = \frac{\partial x^{\alpha}}{\partial \xi^{\rho}} \partial_{\mu} \partial_{\beta} \xi^{\rho}, \tag{21}$$

  where $\xi^{\rho} = \xi^{\rho}(x^{\mu})$.
- A connection will be called nontrivial if the Lie derivative along the Killing vectors of a certain metric does not vanishes [B. Siimon, BSc thesis].
- Another premise to be tested is to work with a nondiagonal metric, trying to hit the presumably existing remnant symmetries in $f(Q)$ [Blixt, Golovnev, Guzman, Maksyutov 2306.09289] that also exist in $f(T)$.

[M.J.Guzman, B. Siimon, work in progress.]

Conclusions

- The geometrical trinity of gravity could imply fruitful reformulations of the numerical relativity formalism [Guzman, Jaarma, work in progress.]
- $f(Q)$ gravity is a popular modified gravity model, with cosmological equations prone to sudden singularities (even before the ghosts!) [Guzman, Järv, Pati, 2406.11621].
- Therefore, new methods for finding novel background cosmologies are interesting and under investigation.[Guzman, Siimon, work in progress.]

Suur tänan! Muito obrigada!

Thank you!

mjguzman(at)ut.ee
cosmologa.xyz

Extra slides

Numerical relativity: learning by examples

- Numerical implementation of a self-gravitating scalar field

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left[\phi^{,\alpha}\phi_{,\alpha} + 2V(\phi)\right] \tag{22}$$

in spherical symmetry:

$$ds^2 = -\alpha^2(t,r)dt^2 + a^2(t,r)dr^2 + r^2 d\Omega^2 \tag{23}$$

with equations of motion

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \Box\phi(t,r) - \phi(t,r) = 0. \tag{24}$$

- The setup will be $\kappa = 8\pi G$ therefore $c = 1$, $V(\phi) = m^2\phi^2/2$, m mass of a spinless boson (scalar field).
- In order to eliminate constants from eom it was used the scaled variables $\phi \to \sqrt{\kappa}\phi$, $r \to mr$ and $t \to mt$.
- F.S.Guzmán (2006), Introduction to numerical relativity through examples

3+1 splitted eom

A convenient choice of first order variables is $\psi = \partial_r \phi$ and $\pi = a\partial_t\phi/\alpha$. Using those, Einstein's equations become

$$\frac{\partial_r a}{a} = \frac{1-a^2}{2r} + \frac{r}{4}\left[\psi^2 + \pi^2 + a^2\phi^2\right], \tag{25}$$

$$\frac{\partial_r \alpha}{\alpha} = \frac{\partial_r a}{a} + \frac{a^2-1}{r} - \frac{1}{2} r a^2 \phi^2, \tag{26}$$

$$\partial_t a = \frac{1}{2} r\alpha\phi\pi, \tag{27}$$

and the Klein-Gordon equations are

$$\partial_t \phi = \frac{\alpha}{a}\pi, \tag{28}$$

$$\partial_t \pi = \frac{1}{r^2}\partial_r\left(\frac{r^2\alpha\psi}{a}\right) - a\alpha\phi, \tag{29}$$

$$\partial_t \psi = \partial_r\left(\frac{\alpha\pi}{a}\right). \tag{30}$$

Initial conditions and parameters

- We provide a Gaussian profile as initial condition for the scalar field
$$\psi(0,r) = Ae^{-r^2/\sigma^2}, \tag{31}$$
which determines the spatial derivative $\psi(0,r)$.
- Time symmetry at the initial time implies $\pi = 0$.
- These initial conditions are used to integrate the Hamiltonian constraint (t,t) assuming spatial flatness at the origin ($a = 1$, $\partial_r a = 0$) up to the edge of the radial domain r_N,
- there it is assumed Schwarzschild-like spacetime, therefore $\alpha(0,r_N) = 1/a(0,r_N)$, $\partial_r\alpha(0,r_N) = 0$, and integrate the slicing condition (r,r) for α inwards. A fourth-order Runge-Kutta integrator was used.

Evolution

- The initial data obtained previously is evolved in time through the Klein-Gordon equations.
- Boundary conditions are enforced for π and ψ.
- The Hamiltonian constraints is solved outwards assuming spatial flatness at the origin $a(r_0) = 1$.
- At the outer boundary the spacetime is again Schwarzschild lile, therefore $\alpha(r_N) = 1/a(r_N)$ and the slicing condition is integrated inwards up to the origin.
- The new values of α and a are used to calculate new values for the scalar field variables using Klein-Gordon equations,
- then boundary conditions are applied.

Results

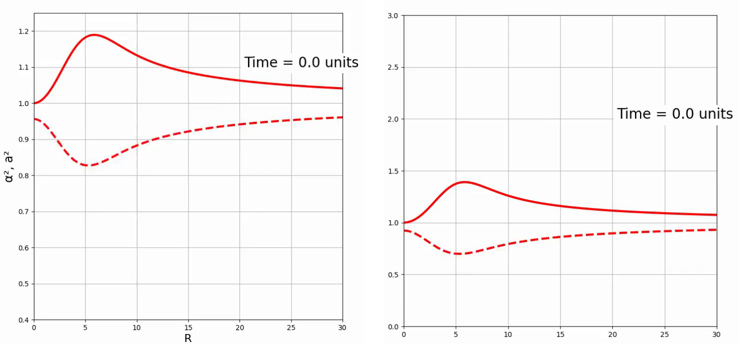


Figure 1: Time evolution of α^2 (upper) and a^2 (lower) for $A = 0.3$, $\sigma = 5.35$ (left) and $A = 0.4$, $\sigma = 0.35$ (right).

[full Python simulation and gif creation by M.F. Jaarma, BSc thesis]

Some attempts

We attempted to find solutions for a connection that does not satisfy the symmetries of the metric. The set of functions

$$\xi^\mu = (f_1(t), f_2(R), f_3(\theta), f_4(\phi)) \tag{32}$$

generates the following flat torsionless connection

$$\begin{aligned}
\tilde{\Gamma}^0{}_{00} = \gamma(t) \equiv \frac{\ddot{f}_1(t)}{\dot{f}_1(t)},\ \tilde{\Gamma}^1{}_{11} = \alpha(R) \equiv \frac{\ddot{f}_2(R)}{\dot{f}_2(R)}, \\
\tilde{\Gamma}^2{}_{22} = \beta(\theta) \equiv \frac{\ddot{f}_3(\theta)}{\dot{f}_3(\theta)},\ \tilde{\Gamma}^3{}_{33} = \delta(\phi) \equiv \frac{\ddot{f}_4(\phi)}{\dot{f}_4(\phi)}.
\end{aligned} \tag{33}$$

while the remaining components vanish. It has a simple form without imposing symmetries, but in combination with a diagonal metric gives inhomogeneous cosmology.

[M.J.Guzman, B. Siimon, work in progress.]

Another attempt

For nondiagonal metric, we have chosen the McVittie metric (black hole in an expanding universe) without the black hole

$$ds^2 = \left(1 - H(t)^2 R^2\right) dt^2 + 2H(t) R dR\, dt - dR^2 - R^2 d\Omega^2. \qquad (34)$$

This is in coordinates (t, R, θ, ϕ) that imitate better the static form of Scharzschild metric, proposed by Kaloper, Kleban and Martin (2010). One set of equations found with connection set 1

$$-f - 4 f_Q \dot{H} = 2\kappa(p + \rho), \qquad 6 f_Q H^2 - f_Q \boldsymbol{Q} = 2\kappa\rho, \qquad (35)$$

does not have the correct GR limit.

[M.J.Guzman, B. Siimon, work in progress.]