SPANISH AND PORTUGUESE RELATIVITY MEETING 2024

New approach to conserved charges of generic gravity in AdS

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Phys. Rev. D 99, 044016 (2019)

23 July, 2024

Overview

Motivation

- Starting from a divergence-free rank-4 tensor of which the trace is the cosmological Einstein tensor, we give a construction of conserved charges in Einstein's gravity and its higher derivative extensions for asymptotically anti-de Sitter spacetimes. The current yielding the charge is explicitly gauge-invariant, and the charge expression involves the linearized Riemann tensor at the boundary.

$\mathcal{P}$- tensor

Let us start with a seemingly innocent question. Given the Riemann tensor $R^{\nu}{}_{\mu\beta\sigma}$, its single trace yields the Ricci tensor $R_{\mu\sigma}$; is there a rank-4 tensor whose single trace is not the Ricci tensor but the cosmological Einstein tensor,

$$\mathcal{G}_{\mu\sigma} = R_{\mu\sigma} - \frac{1}{2} R g_{\mu\sigma} + \Lambda g_{\mu\sigma}, \tag{1}$$

with the condition that this four-index tensor has the symmetries of the Riemann tensor and it is divergence-free just like the Einstein tensor? The answer is yes!

$\mathcal{P}$- tensor

The tensor

$$\mathcal{P}^{\nu\mu\beta\sigma} := R^{\nu\mu\beta\sigma} + g^{\sigma\nu} R^{\beta\mu} - g^{\beta\nu} R^{\sigma\mu} + g^{\beta\mu} R^{\sigma\nu} - g^{\sigma\mu} R^{\beta\nu} + \left(\frac{R}{2} - \frac{\Lambda\,(n-3))}{n-1} \right) \tag{2}$$

whose construction will be given below does the job. It is divergence-free for all smooth metrics, *i.e.* without the use of any field equations

$$\nabla_\nu \mathcal{P}^{\nu}{}_{\mu\beta\sigma} = 0, \tag{3}$$

and its trace is the cosmological Einstein tensor as desired

$$\mathcal{P}^{\nu}{}_{\mu\nu\sigma} = (3-n)\mathcal{G}_{\mu\sigma}. \tag{4}$$

$\mathcal{P}$- tensor

Clearly, the interesting exception is that one cannot do this construction in three dimensions. What happens for $n = 3$ is that the $\mathcal{P}$-tensor vanishes *identically* since, due to the vanishing of the Weyl tensor, the Riemann and the Ricci tensors carry the same amount of information and the Riemann tensor can be expressed in terms of the Ricci tensor as

$$R^{\nu\mu\beta\sigma} = R^{\nu\beta} g^{\mu\sigma} + R^{\mu\sigma} g^{\nu\beta} - R^{\nu\sigma} g^{\mu\beta} - R^{\mu\beta} g^{\nu\sigma} - \frac{R}{2}\left(g^{\nu\beta} g^{\mu\sigma} - g^{\mu\beta} g^{\nu\sigma}\right). \tag{5}$$

$\mathcal{P}$- tensor

The natural question is how one arrives at the $\mathcal{P}$-tensor (2). We have found the $\mathcal{P}$-tensor from the following construction: starting from the Bianchi identity

$$\nabla_{\nu} R_{\sigma\beta\mu\rho} + \nabla_{\sigma} R_{\beta\nu\mu\rho} + \nabla_{\beta} R_{\nu\sigma\mu\rho} = 0, \quad (6)$$

and carrying out the $g^{\nu\rho}$ multiplication, one arrives at the $\mathcal{P}^{\nu}{}_{\mu\beta\sigma}$ as given in (2) after making use of $\nabla_{\mu}\mathcal{G}^{\mu\nu} = 0$ and $\nabla_{\mu} g_{\alpha\beta} = 0$. Note that this still leaves an ambiguity in the $\mathcal{P}$-tensor, since one can add an arbitrary constant times $g^{\mu\sigma} g^{\beta\nu}$, but that part can be fixed by demanding that the $\mathcal{P}$-tensor has the symmetries of the Riemann tensor and also vanishes for constant curvature backgrounds, which we assumed.

$\mathcal{P}$- tensor

This tensor turns out to be extremely useful in finding conserved charges of Einstein's gravity for asymptotically AdS spacetimes for $n > 3$ dimensions. Recently we gave a brief account of this formulation in Einstein's theory, and in the current work, we shall extend this formulation to quadratic and generic gravity theories.[1]

[1] Conserved charges in AdS: A new formula, E. Altas, B. Tekin, Phys. Rev. D 99, 044026 (2019).

Conserved charges

Conserved Deser-Tekin charges[2] of generic gravity theory in asymptotically *AdS* spacetimes is an extension of the Abbott-Deser charges[3] of the cosmological Einstein theory. The latter is a generalization of the ADM charges[4] which are valid for asymptotically flat spacetimes.
Let us summarize this construction. Consider a generic gravity theory defined by the field equations

$$\mathcal{E}_{\mu\nu}\left(g,\mathcal{R},\nabla\mathcal{R},\mathcal{R}^2,...\right)=\kappa\tau_{\mu\nu}, \tag{7}$$

where $\nabla_\mu\mathcal{E}^{\mu\nu}=0$, and κ is the *n*-dimensional Newton constant while $\tau_{\mu\nu}$ represents a localized conserved source.

[2] S. Deser and B. Tekin, Energy in generic higher curvature gravity theories, Phys. Rev. D 67, 084009 (2003)

[3] L.F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. B 195, 76 (1982)

[4] R. Arnowitt, S. Deser and C.W. Misner, Canonical variables for general relativity, Phys. Rev. 117, 1595 (1960); The dynamics of general relativity, Gen. Rel. Grav. 40, 1997 (2008).

Conserved charges

A nontrivial, partially conserved current arises after one splits the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \tag{8}$$

which yields a splitting of the field equations as

$$\kappa(\mathcal{E}_{\mu\nu})^{(1)}(h) = \kappa\tau_{\mu\nu} - \kappa^2(\mathcal{E}_{\mu\nu})^{(2)}(h) + \mathcal{O}\left(\kappa^3\right), \tag{9}$$

where we assumed that $\bar{g}$ solves the field equations, $\mathcal{E}_{\mu\nu}(\bar{g}) = 0$, exactly in the absence of any source $\tau_{\mu\nu}$ and $(\mathcal{E}_{\mu\nu})^{(1)}(h) := \frac{d}{d\kappa}\mathcal{E}_{\mu\nu}(\bar{g} + \kappa h)\,|_{\kappa=0}$. Hence defining $(\mathcal{E}_{\mu\nu})^{(1)} := T_{\mu\nu}$, one has the desired partially conserved current, if the background admits a Killing vector $\bar{\xi}$:

$$\mathcal{J}^{\mu} := \sqrt{-\bar{g}}\,\bar{\xi}_{\nu}\,(\mathcal{E}^{\mu\nu})^{(1)}. \tag{10}$$

Conserved charges

Making use of the Stokes theorem, given a spacelike hypersurface $\bar{\Sigma}$, one has the conserved charge for each background Killing vector

$$Q\left(\bar{\xi}\right) := \int_{\bar{\Sigma}} d^{n-1}y\,\sqrt{\bar{\gamma}}\,\bar{n}_{\mu}\,\bar{\xi}_{\nu}\left(\mathcal{E}^{\mu\nu}\right)^{(1)}, \tag{11}$$

where we assumed the that $\mathcal{J}^{\mu}$ vanishes at spacelike infinity. To proceed further one must know the field equations and express $\bar{\xi}_{\nu}\left(\mathcal{E}^{\mu\nu}\right)^{(1)}$ as a divergence of an antisymmetric two tensor. We have shown that[5] one can reformulate this problem in the cosmological Einstein theory in *AdS* spacetimes *without* using the explicit form of the linearized cosmological Einstein tensor. This is possible because in Einstein spaces one has

$$\bar{\xi}^{\mu} = \bar{\nabla}_{\nu}\bar{\mathcal{F}}^{\nu\mu}, \tag{12}$$

where $\bar{\mathcal{F}}^{\nu\mu} = -\frac{2}{\bar{R}}\bar{\nabla}^{\nu}\bar{\xi}^{\mu}$ with $\bar{R}$ being the constant scalar curvature.

[5] Conserved charges in AdS: A new formula, E. Altas, B. Tekin, Phys. Rev. D 99, 044026 (2019)

Conserved charges

In the *AdS* background, we have

$$\bar{R}_{\mu\alpha\nu\beta} = \frac{2\Lambda}{(n-2)(n-1)}\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\alpha\nu}\right), \tag{13}$$

$$\bar{R}_{\mu\nu} = \frac{2\Lambda}{n-2}\bar{g}_{\mu\nu}, \qquad \bar{R} = \frac{2n\Lambda}{n-2}.$$

To find the conserved charges of a gravity theory defined on an asymptotically *AdS* spacetime $\mathscr{M}$, let us assume that there is an antisymmetric two form, $\mathcal{F}_{\mu\nu}$, on the manifold. Then one has the *exact* equation for any smooth metric

$$\nabla_\nu(\mathcal{F}_{\beta\sigma}\mathcal{P}^{\nu\mu\beta\sigma}) - \mathcal{P}^{\nu\mu\beta\sigma}\nabla_\nu\mathcal{F}_{\beta\sigma} = 0. \tag{14}$$

Conserved charges

Linearization of (14) about the *AdS* background yields

$$\bar{\nabla}_\nu\left((\mathcal{P}^{\nu\mu\beta\sigma})^{(1)}\bar{\mathcal{F}}_{\beta\sigma}\right) - (\mathcal{P}^{\nu\mu\beta\sigma})^{(1)}\bar{\nabla}_\nu\bar{\mathcal{F}}_{\beta\sigma} = 0, \tag{15}$$

which is the main equation from which we will read the conserved current.

Einstein's theory

Using the following equivalent form of the $\mathcal{P}$-tensor, written in terms of the cosmological Einstein tensor,

$$\mathcal{P}^{\nu}{}_{\mu\beta\sigma} := R^{\nu}{}_{\mu\beta\sigma} + 2\delta^{\nu}_{[\sigma}\mathcal{G}_{\beta]\mu} + 2\mathcal{G}^{\nu}_{[\sigma}g_{\beta]\mu} + (R - \frac{2\Lambda(n+1)}{n-1})\delta^{\nu}_{[\sigma}g_{\beta]\mu}, \quad (16)$$

one arrives at its linearized form

$$(\mathcal{P}^{\nu\mu\beta\sigma})^{(1)} = (R^{\nu\mu\beta\sigma})^{1} + 2(\mathcal{G}^{\mu[\beta})^{(1)}\bar{g}^{\sigma]\nu} + 2(\mathcal{G}^{\nu[\sigma})^{(1)}\bar{g}^{\beta]\mu}$$
$$+(R)^{(1)}\bar{g}^{\mu[\beta}\bar{g}^{\sigma]\nu} + \frac{4\Lambda}{(n-1)(n-2)}(h^{\mu[\sigma}\bar{g}^{\beta]\nu} + \bar{g}^{\mu[\sigma}h^{\beta]\nu}). \quad (17)$$

Einstein's theory

For the particular antisymmetric background tensor

$$\bar{\mathcal{F}}_{\alpha\beta} := \bar{\nabla}_\alpha \bar{\xi}_\beta, \tag{18}$$

where $\bar{\xi}_\beta$ is an *AdS* Killing vector, one finds from (15) the following conserved current:

$$\bar{\xi}_\lambda (\mathcal{G}^{\lambda\mu})^{(1)} = \frac{(n-1)(n-2)}{4\Lambda\,(n-3)} \bar{\nabla}_\nu \left((\mathcal{P}^{\nu\mu\beta\sigma})^{(1)} \bar{\mathcal{F}}_{\beta\sigma} \right). \tag{19}$$

Einstein's theory

Comparing this with the integrand of (11), and using the Stokes theorem one more time, we find the desired result

$$Q\left(\bar{\xi}\right) = \frac{(n-1)(n-2)}{8(n-3)\Lambda G\Omega_{n-2}} \int_{\partial\bar{\Sigma}} d^{n-2}x \sqrt{\bar{\gamma}}\, \bar{\epsilon}_{\mu\nu} \left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)} \bar{\mathcal{F}}^{\beta\sigma}, \qquad (20)$$

where $\left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)}$ is the linearized part of the Riemann tensor about the *AdS* background. Observe that on the boundary $\left(\mathcal{P}^{\nu\mu}{}_{\beta\sigma}\right)^{(1)} = \left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)}$, since the linearized Einstein tensor and the linearized scalar curvature vanish. The barred quantities refer to the background spacetime $\bar{\mathscr{M}}$ with the boundary $\partial\bar{\mathscr{M}}$.

Generic theory

Consider a generic gravity theory which starts with the Einsteinian part as

$$\mathcal{E}_{\mu\nu} = \frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_0 g_{\mu\nu}\right) + \sigma E_{\mu\nu} = \tau_{\mu\nu}, \qquad (21)$$

where at this stage all we know about the $E_{\mu\nu}$-tensor is that it is a symmetric divergence-free tensor (which can come from an action) and σ is a dimensionful parameter. To proceed further, it is better to recast the equation as

$$\mathcal{E}_{\mu\nu} = \frac{1}{\kappa}\mathcal{G}_{\mu\nu} + \frac{\Lambda_0 - \Lambda}{\kappa}g_{\mu\nu} + \sigma E_{\mu\nu} = \tau_{\mu\nu}, \qquad (22)$$

whose $(A)dS$ vacua are determined by

$$\bar{\mathcal{E}}_{\mu\nu} = \frac{\Lambda_0 - \Lambda}{\kappa}\bar{g}_{\mu\nu} + \sigma \bar{E}_{\mu\nu} = 0. \qquad (23)$$

Generic theory

We shall assume that Λ represents any one of the viable vacua. To find the conserved charges in this theory, we use the same procedure as the one in the previous section and define

$$(\mathcal{E}_{\mu\nu})^{(1)} = T_{\mu\nu}, \tag{24}$$

where the right-hand side has all the higher order terms

$$T_{\mu\nu} = \tau_{\mu\nu} - \kappa(\mathcal{E}_{\mu\nu})^{(2)} - \kappa^2(\mathcal{E}_{\mu\nu})^{(3)} - \dots . \tag{25}$$

So we have the background conserved current

$$\bar{\nabla}_\nu\left(\bar{\xi}_\mu(\mathcal{E}^{\mu\nu})^{(1)}\right) = 0, \tag{26}$$

and the partially conserved current is $\mathcal{J}^\nu = \sqrt{-\bar{g}}\,\bar{\xi}_\mu\,(\mathcal{E}^{\mu\nu})^{(1)}$. Hence we must compute

$$\bar{\xi}_\mu(\mathcal{E}^{\mu\nu})^{(1)} = \frac{1}{\kappa}\bar{\xi}_\mu(\mathcal{G}^{\mu\nu})^{(1)} - \frac{\Lambda_0 - \Lambda}{\kappa}\bar{\xi}_\mu h^{\mu\nu} + \sigma\bar{\xi}_\mu(E^{\mu\nu})^{(1)}. \tag{27}$$

Generic theory

We have already computed the first part in the previous subsection, and hence, the new parts are the second and the third terms. But when the theory is not given, one cannot proceed further from this point. For this reason, let us consider the quadratic theory as an example which also covers all the $f(Riemann)$ type theories. The action of the quadratic theory is

$$I = \int d^n x \sqrt{-g} \Big(\frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \\ + \gamma \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \Big). \tag{28}$$

Generic theory

Inserting (13) in the last equation, one finds the equation satisfied by Λ:

$$\frac{\Lambda-\Lambda_0}{2\kappa}+\left((n\alpha+\beta)\frac{(n-4)}{(n-2)^2}+\gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}\right)\Lambda^2\bigg)=0. \qquad (29)$$

Defining the constant

$$c:=\frac{1}{\kappa}+\frac{4\Lambda n}{n-2}\alpha+\frac{4\Lambda}{n-1}\beta+\frac{4\Lambda\,(n-3)\,(n-4)}{(n-1)\,(n-2)}\gamma, \qquad (30)$$

Generic theory

one obtains

$$\bar{\xi}_\nu (\mathcal{E}^{\mu\nu})^{(1)} = \left(c + \frac{4\Lambda}{(n-1)(n-2)}\beta \right) \bar{\xi}_\nu (\mathcal{G}^{\mu\nu})^{(1)}$$
$$+ 2\beta \bar{\nabla}_\alpha \left(\bar{\xi}_\nu \bar{\nabla}^{[\alpha} (\mathcal{G}^{\mu]\nu})^{(1)} + (\mathcal{G}^{\nu[\alpha})^{(1)} \bar{\nabla}^{\mu]} \bar{\xi}_\nu \right)$$
$$+ (2\alpha + \beta) \bar{\nabla}_\alpha \left(2\bar{\xi}^{[\mu} \bar{\nabla}^{\alpha]} (R)^{(1)} + (R)^{(1)} \bar{\nabla}^\mu \bar{\xi}^\alpha \right). \quad (31)$$

Generic theory

Therefore the conserved charges in quadratic gravity in $(A)dS$ read as

$$Q\left(\bar{\xi}\right)=\lambda\int_{\partial\bar{\Sigma}}d^{n-2}x\,\sqrt{\bar{\gamma}}\,\bar{\epsilon}_{\mu\nu}\left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)}\bar{\mathcal{F}}^{\beta\sigma}, \tag{32}$$

where

$$\lambda=\frac{(n-1)\,(n-2)}{4\Lambda\,(n-3)}\left(c+\frac{4\Lambda\beta}{(n-1)\,(n-2)}\right). \tag{33}$$

Observe that for asymptotically AdS spacetimes, the second and third lines in do not contribute. Therefore, for asymptotically AdS spacetimes, the only difference between the conserved charges in Einstein's theory (20) and the quadratic theory is the numerical factor in (32).

Gauge invariance issue

The problem of the gauge transformations of the charge and the current that yields the charge is important. Clearly, one expects the charge to be gauge-invariant in any valid formulation, but the current need not be. In fact Abbott-Deser conserved charges [6] used gauge-variant currents which yield gauge-invariant charges. Let us show this in the expression of Deser-Tekin charges [7] for the cosmological Einstein theory:

$$2\bar{\xi}_\nu (\mathcal{G}^{\mu\nu})^{(1)} = \bar{\nabla}_\alpha \mathcal{J}^{\alpha\mu}, \tag{34}$$

where the antisymmetric current is

$$\begin{aligned} \mathcal{J}^{\alpha\mu} :=& \bar{\xi}^\alpha \bar{\nabla}_\beta h^{\mu\beta} - \bar{\xi}^\mu \bar{\nabla}_\beta h^{\alpha\beta} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{\alpha\nu} - \bar{\xi}_\nu \bar{\nabla}^\alpha h^{\mu\nu} + \bar{\xi}^\mu \bar{\nabla}^\alpha h - \bar{\xi}^\alpha \bar{\nabla}^\mu h \\ & + h^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu - h^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu - h \bar{\nabla}^\alpha \bar{\xi}^\mu. \end{aligned} \tag{35}$$

[6] L.F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. B 195, 76 (1982).

[7] S. Deser and B. Tekin, Energy in generic higher curvature gravity theories, Phys. Rev. D 67, 084009 (2003); Gravitational energy in quadratic curvature gravities, Phys. Rev. Lett. 89, 101101 (2002).

Gauge invariance issue

Consider an infinitesimal coordinate transformation generated by a vector field ζ (not to be confused with the Killing field ξ); one has

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu = \mathscr{L}_\zeta \bar{g}_{\mu\nu}, \tag{36}$$

where $\mathscr{L}_\zeta$ denotes the Lie derivative and hence $\delta_\zeta h^{\mu\nu} = -\mathscr{L}_\zeta \bar{g}^{\mu\nu}$. It is easy to see that $\delta_\zeta (\mathcal{G}^{\mu\nu})^{(1)} = \mathscr{L}_\zeta \bar{\mathcal{G}}^{\mu\nu} = 0$. But this only implies from (34) that one has the divergence of the gauge-transformed current to vanish

$$\bar{\nabla}_\alpha \delta_\zeta \mathcal{J}^{\alpha\mu} = 0, \tag{37}$$

and hence $\mathcal{J}^{\alpha\mu}$ is not necessarily gauge invariant.

Gauge invariance issue

In fact one can show that $\mathcal{J}^{\alpha\mu}$ varies, under the gauge transformations (36), as

$$\delta_\zeta \mathcal{J}^{\alpha\mu} = \bar{\nabla}_\nu \Big(\bar{\xi}^\alpha \bar{\nabla}^\nu \zeta^\mu + \bar{\xi}^\mu \bar{\nabla}^\alpha \zeta^\nu + \bar{\xi}^\nu \bar{\nabla}^\mu \zeta^\alpha + 2\zeta^\alpha \bar{\nabla}^\nu \bar{\xi}^\mu + \zeta^\nu \bar{\nabla}^\mu \bar{\xi}^\alpha - (\mu \leftrightarrow \alpha) \Big). \tag{38}$$

Clearly, since the variation is a boundary term and since $\mathcal{J}^{\alpha\mu}$ is the integrand on the boundary of the spatial slice, the boundary term does not contribute to the charges (as $\partial\partial\bar{\Sigma} = 0$) and hence the charge is gauge invariant. But this exercise shows us that the current is only gauge invariant up to a boundary term.

Gauge invariance issue

On the other hand, since $\delta_\zeta (R^{\nu\mu}{}_{\beta\sigma})^{(1)}$ is gauge invariant, our charge expression (20) is explicitly gauge invariant without an additional boundary term. Let us show this. One has

$$\delta_\zeta (R^{\nu\mu}{}_{\beta\sigma})^{(1)} = \mathscr{L}_\zeta \bar{R}^{\nu\mu}{}_{\beta\sigma}. \tag{39}$$

For the AdS background, one clearly has $\mathscr{L}_\zeta \bar{R}^{\nu\mu}{}_{\beta\sigma} = 0$ and hence $\delta_\zeta (R^{\nu\mu}{}_{\beta\sigma})^{(1)} = 0$, and so $\delta_\zeta Q = 0$ as expected. So in our formalism, not only is the charge explicitly gauge-invariant, but also the current is explicitly gauge invariant.

Conclusions

- In a gauge or gravity theory, the conserved charges make sense if they are gauge or coordinate invariant. The ADM and AD charges and their generalizations to higher order gravity, are all gauge invariant. However, the current terms is gauge invariant *only* up to a divergence term which vanishes on the boundary.
- The obvious question is to try to understand if the gauge-invariant charges can be written in an *explicitly* gauge-invariant way with the help of the Riemann tensor.
- There is a stronger motivation for such a search: outside the sources, the Riemann tensor carries all the information about gravity. Naturally, it must carry the information about the conserved charges.
- The construction is somewhat nontrivial and is valid only for asymptotically AdS spacetimes.
- For generic gravity theories, the construction is analogous, but there arise many more terms in the final expressions.

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