The self dual action: Ashtekar variables without gauge fixing

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Summary

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- The **self dual action**.
- A few words on the **GNH method** for **singular Hamiltonian Systems**.
- Ashtekar formulation **without the time gauge**.

The self-dual action for Euclidean GR back

- Basic fields: $\mathbf{e}^i \in \Omega^1(\mathcal{M})$, $\boldsymbol{\omega}^i \in \Omega^1(\mathcal{M})$, $\boldsymbol{\alpha} \in \Omega^1(\mathcal{M})$; $i = 1, 2, 3$.
- $\boldsymbol{\alpha}$ and $\mathbf{e}^i$ chosen so that $\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + \mathbf{e}_i \otimes \mathbf{e}^i$ is a Euclidean metric. As a consequence $(\boldsymbol{\alpha}, \mathbf{e}^i)$ defines a non-degenerate tetrad.
- Covariant exterior differential $\mathbf{D}$ acting on $\mathbf{e}_i$ as

$$\mathbf{D}\mathbf{e}_i := \mathbf{d}\mathbf{e}_i + \varepsilon_{ijk}\boldsymbol{\omega}^j \wedge \mathbf{e}^k \,,$$

Curvature 2-form

$$\mathbf{F}^i := \mathbf{d}\boldsymbol{\omega}^i + \frac{1}{2}\varepsilon^i{}_{jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k \,.$$

- The Euclidean self-dual action for General Relativity is

$$S(\mathbf{e}, \boldsymbol{\omega}, \boldsymbol{\alpha}) := \int_{\mathcal{M}} \left(\frac{1}{2}\varepsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{F}^k - \boldsymbol{\alpha} \wedge \mathbf{e}_i \wedge \mathbf{F}^i \right) \,.$$

- $\mathcal{M} = \mathbb{R} \times \Sigma$.

The self-dual action for Euclidean GR back

- The first term is the Husain-Kuchař action.
- The indices $i, j, k = 1, 2, 3$ are "$SO(3)$ indices" because the action is invariant under the infinitesimal gauge transformations $[\mathbf{\Lambda}^k \in C^\infty(\mathcal{M})]$

$$\begin{aligned} \delta_1 \boldsymbol{\omega}^i &= \mathbf{D}\mathbf{\Lambda}^i\,, \\ \delta_1 \boldsymbol{\alpha} &= 0\,, \\ \delta_1 \mathbf{e}^i &= \varepsilon^i{}_{jk}\mathbf{e}^j \mathbf{\Lambda}^k\,. \end{aligned}$$

- The action is also invariant under the transformations $[\mathbf{\Upsilon}^k \in C^\infty(\mathcal{M})]$

$$\begin{aligned} \delta_2 \boldsymbol{\omega}^i &= 0\,, \\ \delta_2 \boldsymbol{\alpha} &= \mathbf{\Upsilon}_i \mathbf{e}^i\,, \\ \delta_2 \mathbf{e}^i &= -\mathbf{\Upsilon}^i \boldsymbol{\alpha} + \varepsilon^i{}_{jk}\mathbf{e}^j \mathbf{\Upsilon}^k\,,. \end{aligned}$$

- δ_1 and δ_2 are *independent* but do not commute. Some linear combinations of them do commute. Full symmetry: $SO(4) = SO(3) \otimes SO(3)$.

The self-dual action for Euclidean GR back

- The field equations are

$$
\begin{aligned}
&\mathbf{D}(\boldsymbol{\alpha}\wedge\mathbf{e}_k)+\epsilon_{ijk}\mathbf{e}^i\wedge\mathbf{D}\mathbf{e}^j=0\,,\\
&\epsilon_{ijk}\mathbf{e}^j\wedge\mathbf{F}^k+\boldsymbol{\alpha}\wedge\mathbf{F}_i=0\,,\\
&\mathbf{e}^i\wedge\mathbf{F}_i=0\,.
\end{aligned}
$$

They are equivalent to the Euclidean Einstein equations in vacuum.

The GNH method

back

- An **alternative to Dirac's** "algorithm".
- If we are interested only in the dynamics in Hamiltonian form we can look for vector fields **on the primary constraint hypersurface** where the Hamiltonian is **uniquely defined** (quantization is a different issue).
- We have to work with the **pull-back** ω **of the canonical symplectic form** Ω to the primary constraint submanifold. This ω is generically **degenerate:** the solutions to $\imath_X\omega - \mathbb{d}H = 0$ are not unique. Some components of X may be **arbitrary** (gauge symmetries!).
- The equation $\imath_X\omega - \mathbb{d}H = 0$ is **inhomogeneous**, hence, extra conditions may be necessary to guarantee its solvability (**secondary constraints**).
- **Consistency** of the dynamics is enforced through **tangency conditions**.
- From a practical point of view an advantage of this method is that **we can avoid using Poisson brackets** (boundaries).
- Another beneficial side-effect is that **computations are shorter**.

Hamiltonian description of the self-dual action back

- In a "phase space" M **spanned by the fields** $(e_{\mathrm{t}}, e^i, \omega_{\mathrm{t}}, \omega^i, \alpha_{\mathrm{t}}, \alpha)$.
- Vector fields in M have components $\mathbb{Y}_0 = (Y^i_{e_{\mathrm{t}}}, Y^i_e, Y^i_{\omega_{\mathrm{t}}}, Y^i_\omega, Y_{\alpha_{\mathrm{t}}}, Y_\alpha)$.
- The **presymplectic 2-form** on M can be written as

$$\omega = \int_\Sigma \mathbb{d}\omega^i \mathbin{\wedge\!\!\!\wedge} \mathbb{d}\left(\frac{1}{2}\epsilon_{ijk} e^j \wedge e^k + e_i \wedge \alpha\right),$$

or, acting on vector fields $\mathbb{Y}, \mathbb{Z}$ in M

$$\omega(\mathbb{Z}_0, \mathbb{Y}_0) = \int_\Sigma \Big(Y^i_e \wedge (\epsilon_{ijk} e^j \wedge Z^k_\omega + \alpha \wedge Z_{\omega i}) + Y^i_\omega \wedge (Z_\alpha \wedge e_i - \epsilon_{ijk} Z^j_e \wedge e^k - Z^i_e \wedge \alpha) + Y_\alpha \wedge Z^i_\omega \wedge e_i \Big).$$

Hamiltonian description of the self-dual action back

- **Secondary constraints**

$$\epsilon_{ijk}e^j \wedge F^k + \alpha \wedge F_i = 0\,,$$
$$D(\frac{1}{2}\epsilon_{ijk}e^j \wedge e^k + e_i \wedge \alpha) = 0\,,$$
$$e_i \wedge F^i = 0\,.$$

- **Equations for the components** of the Hamiltonian vector field $\mathbb{Z}_0$

$$(\epsilon_{ijk}e^j + \delta_{ik}\alpha) \wedge (Z_\omega^k - D\omega_{\mathrm{t}}^k) = (\delta_{ik}\alpha_{\mathrm{t}} + \epsilon_{ijk}e_{\mathrm{t}}^j)F^k\,,$$
$$(\epsilon_{ijk}e^j - \delta_{ik}\alpha) \wedge (Z_e^k - De_{\mathrm{t}}^k - \epsilon^k{}_{\ell m}e^\ell \omega_{\mathrm{t}}^m) + e_i \wedge (Z_\alpha - \mathrm{d}\alpha_{\mathrm{t}})$$
$$= e_{\mathrm{t}}^i \mathrm{d}\alpha + (\epsilon_{ijk}e_{\mathrm{t}}^j - \alpha_{\mathrm{t}}\delta_{ik})De^k$$
$$e_i \wedge (Z_\omega^i - D\omega_{\mathrm{t}}^i) = e_{\mathrm{t}}^i F_i\,.$$

- There are **no conditions** on $Z^i_{e_{\mathrm{t}}}$, $Z^i_{\omega_{\mathrm{t}}}$ and $Z_{\alpha_{\mathrm{t}}}$. They are **arbitrary** and, hence, the dynamics of e_{t}^i, ω_{t}^i and α_{t} is also arbitrary.

Hamiltonian description of the self-dual action back

- **Tangency conditions**

$$(\epsilon_{ijk} Z_e^j + \delta_{ik} Z_\alpha) \wedge F^k + (\epsilon_{ijk} e^j + \delta_{ik}\alpha) \wedge DZ_\omega^k = 0\,,$$
$$D(\epsilon_{ijk} e^j \wedge Z_e^k - Z_\alpha \wedge e_i - \alpha \wedge Z_{ei}) + Z_\omega^k \wedge (e^i \wedge e_k - \epsilon^i{}_{km}\alpha \wedge e^m) = 0\,,$$
$$Z_e^i \wedge F_i + e_i \wedge DZ_\omega^i = 0\,.$$

- One has to **solve for the vector field** in the equations written above. It is important to find the simplest way to write down the solutions to these equations in order to check that the solutions **satisfy the tangency conditions**.

- This last step is highly non-trivial, but it is **a crucial consistency condition** that has been neglected in previous work on this subject).

Ashtekar formulation without gauge fixing

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- The form of the pullback of the pre-symplectic form ω suggests to **introduce the object**

$$H_i := \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k + e_i \wedge \alpha\,,$$

which would be **canonically conjugate** to ω_i in the sense that:

$$\omega = \int_\Sigma \mathbb{d}\omega^i \mathbin{\wedge\!\!\!\wedge} \mathbb{d}H_i\,.$$

- The **number of independent components** in H_i and e_i are the same, hence it makes sense to write e_i in terms of H^i and α to get a cleaner Hamiltonian description of Euclidean gravity.
- By proceeding in this way one arrives at the Ashtekar formulation for Euclidean gravity **without having to introduce any gauge fixing**. Let us see how.

Ashtekar formulation without gauge fixing

back

- To get usual Ashtekar variables we introduce a fiducial, non-dynamical (i.e. field independent) volume form vol_0 and define the "vector field"

$$\widetilde{H}_i := \left(\frac{\cdot \wedge H_i}{\mathrm{vol}_0}\right) .$$

- Now the pre-symplectic form can be written as

$$\omega = \int_\Sigma \mathbb{d}\omega^i \wedge\!\!\!\wedge \mathbb{d}H_i = \int_\Sigma \big(\mathbb{d}\omega^i \wedge\!\!\!\wedge \mathbb{d}\widetilde{H}_i\big)\mathrm{vol}_0 .$$

- In terms of $\widetilde{H}_i$ the constraints are equivalent to

$$\begin{aligned}
&\mathrm{div}_0 \widetilde{H}_i + \epsilon_{ijk}\imath_{\widetilde{H}^k}\omega^j = 0\\
&\imath_{\widetilde{H}_i} F^i = 0 ,\\
&\epsilon^{ijk}\imath_{\widetilde{H}_i}\imath_{\widetilde{H}_j} F_k = 0 ,
\end{aligned}$$

which are the **Gauss law**, the **vector** and the **Hamiltonian constraint**. The Gauss law is, actually, independent of the fiducial vol_0.

Ashtekar formulation without gauge fixing back

- In order to discuss the Hamiltonian vector fields it is useful to introduce a non-degenerate triad h_i naturally associated with $\widetilde{H}_i$ $[\alpha =: \alpha_i e^i]$.

$$h^i := \frac{1}{\sqrt{1+\alpha^2}}\left(e^i + \alpha^i \alpha + \epsilon^{ijk}\alpha_j e_k\right),$$

- The non-arbitrary components of the Hamiltonian vector fields are

$$Z^k_\omega = D\omega^k_{\mathrm{t}} - \widehat{\alpha}_t \; {}^h\mathbb{F}^k{}_\ell e^\ell - \epsilon_{\ell mn}\widehat{e}_{\mathrm{t}}{}^m \; {}^h\mathbb{F}^{nk} h^\ell ,$$

$$Z^k_h = D\widehat{e}_{\mathrm{t}}{}^k + \epsilon^k{}_{\ell m} h^\ell \omega^m_{\mathrm{t}} - \frac{1}{2}\widehat{\alpha}_{\mathrm{t}} \; {}^h\mathbb{B}\, h^k - \epsilon_{\ell mn}\widehat{e}_{\mathrm{t}}{}^m \; {}^h\mathbb{B}^{nk} h^\ell + \epsilon^k{}_{\ell m}\widehat{X}^m h^\ell + \widehat{\alpha}_{\mathrm{t}} \; {}^h\mathbb{B}^{k\ell} h_\ell ,$$

with $\widehat{\alpha}_{\mathrm{t}} := \dfrac{\alpha_{\mathrm{t}} - (e_{\mathrm{t}}\cdot\alpha)}{\sqrt{1+\alpha^2}}$, $\widehat{e}_{\mathrm{t}}{}^i := \dfrac{e^i_{\mathrm{t}} + \alpha_{\mathrm{t}}\alpha^i - \epsilon^{ijk} e_{\mathrm{t}j}\alpha_k}{\sqrt{1+\alpha^2}}$,

$${}^h\mathbb{F}_{ij} := \left(\frac{F_i \wedge h_j}{\mathrm{vol}_h}\right), {}^h\mathbb{B}_{ij} := \left(\frac{Dh_i \wedge h_j}{\mathrm{vol}_h}\right), \widehat{X}_i := -\frac{1}{2}\epsilon_{ijk}\left(\frac{\mathrm{d}\widehat{\alpha}_{\mathrm{t}} \wedge h^j \wedge h^k}{\mathrm{vol}_h}\right)$$

The time gauge

back

- It is very interesting to compare the formulation that we have obtained in terms of the h_i with the original one in the **time gauge** $\alpha = 0$. This can be immediately obtained by substituting $\alpha = 0$ in the pre-symplectic form ω, the constraints and the Hamiltonian vector fields Z_ω^k, Z_e^k.
- By doing this **one immediately gets the standard Ashtekar formulation** in terms of ω_i and $\widetilde{E}_i$ defined in the obvious way from the triads e_i.
- The non-arbitrary components of the Hamiltonian vector fields are now

$$
\begin{aligned}
Z_\omega^k &= D\omega_{\mathrm{t}}^k - \alpha_t\, \mathbb{F}^k{}_\ell e^\ell - \epsilon_{\ell mn} e_{\mathrm{t}}{}^m\, \mathbb{F}^{nk} e^\ell\,,\\
Z_e^k &= De_{\mathrm{t}}{}^k + \epsilon^k{}_{\ell m} e^\ell \omega_{\mathrm{t}}^m - \frac{1}{2}\alpha_{\mathrm{t}}\, \mathbb{B}\, e^k - \epsilon_{\ell mn} e_{\mathrm{t}}{}^m\, \mathbb{B}^{nk} e^\ell \\
&\qquad + \epsilon^k{}_{\ell m} X^m e^\ell + \alpha_{\mathrm{t}}\, \mathbb{B}^{k\ell} e_\ell\,,
\end{aligned}
$$

with

$$
\mathbb{F}_{ij} := \left(\frac{F_i \wedge e_j}{\mathrm{vol}_e}\right), \mathbb{B}_{ij} := \left(\frac{De_i \wedge e_j}{\mathrm{vol}_e}\right), X_i := -\frac{1}{2}\epsilon_{ijk}\left(\frac{\mathrm{d}\alpha_{\mathrm{t}} \wedge e^j \wedge e^k}{\mathrm{vol}_e}\right).
$$

Comments

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- A remarkable thing happens: the form of the **presymplectic form**, the **constraints** and the **Hamiltonian vector fields** obtained either by working with the h_i variables or going to the time gauge in the original formulation **is exactly the same** once we replace the arbitrary objects α_{t} and e_{t}^i by the, also arbitrary, $\widehat{\alpha}_{\mathrm{t}}$ and $\widehat{e}_{\mathrm{t}}{}^i$.
- An interesting observation regarding this replacement of parameters is the fact that this comes from one of the $SO(3)$ factors of the $SO(4)$ symmetry of the action.

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- An interesting observation regarding this replacement of parameters is the fact that this comes from one of the $SO(3)$ factors of the $SO(4)$ symmetry of the action.

Thank you