On Perturbative Constraints for Vacuum High Order Theories of Gravity

EREP Spanish and Portuguese Relativity Meeting

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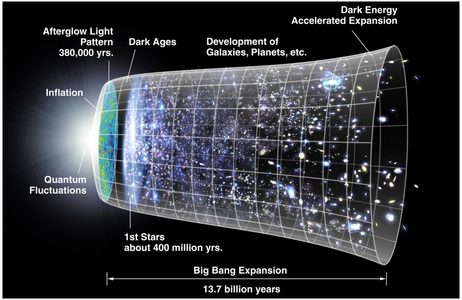
Motivation

Alternative theories, theory and observations



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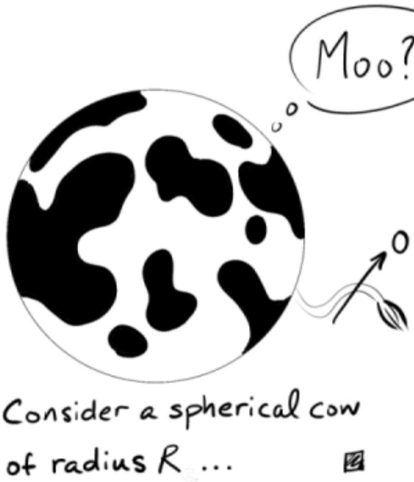
Perturbation theory



General relativity and modified theories of gravity $f(R)$, $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$

Einstein field equations

The action:

$$S = \frac{1}{2k^2}\int d^4x\sqrt{-g}R + \int d^4x\mathcal{L}_M(g_{\mu\nu}, \Psi_M). \tag{1}$$

The Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}.$$

Field equations in $f(R)$ modified theories of gravity

The action in the metric formalism in $f(R)$ modified theories of gravity is:

$$S = \frac{1}{2k^2}\int d^4x\sqrt{-g}f(R) + \int d^4x\mathcal{L}_M(g_{\mu\nu}, \Psi_M). \qquad (2)$$

The field equations in $f(R)$ modified theories of gravity

$$\Sigma_{\mu\nu} \equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \kappa^2 T_{\mu\nu}.$$

We have the Einstein gravity when $f(R) = R$. For the boundary term see *Guarnizo, Castaneda, Tejeiro (2010)*

We consider the general Lagrangian

$$S = \frac{1}{2k^2} \int d^4x \sqrt{-g} f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M) \tag{3}$$

The field equations

$$\begin{aligned} P^{\mu\nu} \equiv &- \frac{1}{2} f g^{\mu\nu} + f_X R^{\mu\nu} + 2 f_Y R^{\rho(\mu} R^{\nu)}{}_{\rho} + 2 f_Z R^{\delta\sigma\rho(\mu} R^{\mu)}{}_{\rho\sigma\delta} \\ &+ f_{X;\rho\sigma}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) + \Box(f_Y R^{\mu\nu}) + g^{\mu\nu}(f_Y R^{\rho\sigma})_{;\rho\sigma} \\ &- 2(f_Y R^{\rho(\mu})_{;}{}^{\nu)}{}_{\rho} - 4(f_Z R^{\sigma(\mu\nu)\rho})_{;\rho\sigma} = \kappa^2 T_{\mu\nu} \end{aligned}$$

General relativity and modified theories of gravity $f(R)$, $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$

Constant Ricci scalar

Constant Ricci scalar

Theorem

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f \in \mathcal{C}^r$, $r \geq 1$, and $f'(R) \neq 0$. Also, suppose a constant Ricci scalar with constant value $R = R_0$. Then, the field equations for $f(R)$ MTG in vacuum are reduced to two possibilities:

1. *GR field equations without cosmological constant if $R_0 = 0$.*
2. *GR field equations with non-vanishing cosmological constant if $R_0 \neq 0$.*

General relativity and modified theories of gravity $f(R)$, $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$

Spherical symmetry

$f(R) = R + \alpha R^2$

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$$f^{\rm comp}(r) = \left(1-\frac{2M}{r}\right)^{-1} - \sqrt{\frac{2C_0^3\alpha}{3}}K_0R_*\exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right]$$
$$+(1-C_0)\frac{K_0}{12}\left[C_0^2(r-R_*)^2-7\sqrt{6C_0^3\alpha}(r-R_*)\right]\exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right]$$
$$+\alpha\left[\frac{C_0K_0}{4}(1-5C_0)-\frac{\sqrt{6}}{3}R_*C_0^{3/2}K_1\right]\exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right]$$
$$+\alpha\frac{5}{6}(R_*K_0C_0)^2\exp\left[-2\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right], \quad (39)$$

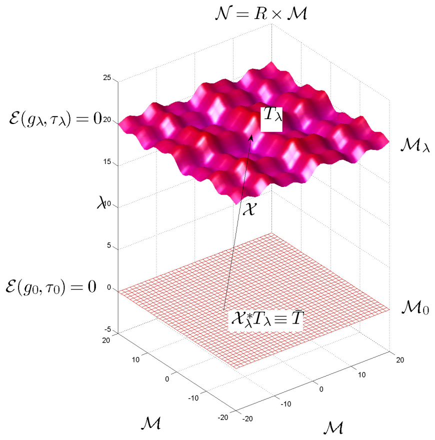
$$h^{\rm comp}(r) = 1-\frac{2M}{r}+\alpha(C_0-3)\frac{K_0}{C_0}\exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right], \quad (40)$$

$f(R) = R + \alpha R^2$

$$
\begin{aligned}
\mathcal{R}^{\text{comp}}(r) =& K_0 \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] + \sqrt{\alpha}K_1 \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&+ \frac{K_1 R_* \sqrt{6C_0}(C_0-1) + 3K_0(C_0^2+C_0+6)}{24R_*^2}(r-R_*)^2 \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&- \sqrt{\alpha}\frac{2\sqrt{6C_0}R_*K_1(C_0+3) - 3(C_0^2-2C_0+9)K_0}{8R_*^2\sqrt{6C_0}}(r-R_*) \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&- \sqrt{\alpha}\frac{K_0}{24R_*}\left[\sqrt{6C_0}(1-C_0)\frac{(r-R_*)^2}{\alpha} + (6C_0+18)\frac{r-R_*}{\sqrt{\alpha}} + 3\sqrt{6C_0} + 9\sqrt{\frac{6}{C_0}}\right] \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&- \sqrt{\frac{\alpha C_0}{6}}R_*K_0^2 \exp\left[-2\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] + \alpha K_2 \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] + \alpha\frac{K_0^3R_*^2C_0}{4}\exp\left[-3\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&+ \alpha\frac{K_0^2}{12}\left[-C_0(C_0-1)\frac{(r-R_*)^2}{\alpha} + 2\sqrt{6C_0^3}\frac{r-R_*}{\sqrt{\alpha}} - \frac{4K_1}{K_0}R_*\sqrt{6C_0}\right] \exp\left[-2\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right] \\
&+ \alpha\frac{K_0}{R_*^2}\left[\frac{C_0}{192}(C_0-1)^2\frac{(r-R_*)^4}{\alpha^2} - \sqrt{6C_0}\frac{(9C_0+11)}{288}(C_0-1)\frac{(r-R_*)^3}{\alpha^{3/2}}\right] \exp\left[-\sqrt{\frac{C_0}{6\alpha}}(r-R_*)\right], \qquad (41)
\end{aligned}
$$

Perturbation theory

Definition of perturbation

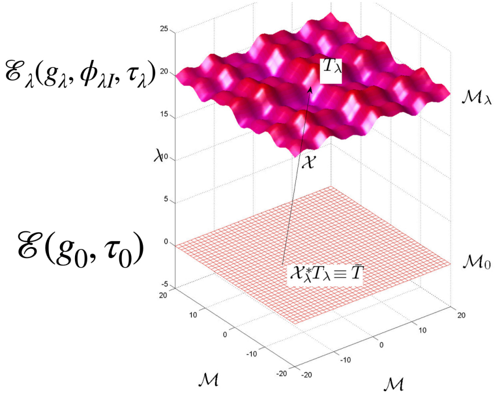


$$
\begin{aligned}
\mathcal{X}_\lambda^* T_\lambda &= T_0 + \lambda \mathcal{L}_X T + \frac{\lambda^2}{2!} \mathcal{L}_X^2 T + \cdots \\
&= \overset{(0)}{T} + \lambda \overset{(1)}{T} + \frac{\lambda^2}{2!} \overset{(2)}{T} + \cdots \\
&= T_0 + {}^{\mathcal{X}}\Delta T_\lambda
\end{aligned}
$$

J. A. Schouten(1954), Bruni et al (1997)

The scenario

$$\mathscr{N} = \mathbb{R} \times \mathscr{M}$$



Perturbation theory

Third scenario

Let $\bar{P}$ and $\bar{Q}$ be two tensors

$$\bar{P} = \overset{(0)}{P} + \lambda \overset{(1)}{P} + \frac{\lambda^2}{2!} \overset{(2)}{P} + \cdots \tag{4}$$

$$\bar{Q} = \overset{(0)}{Q} + \lambda \overset{(1)}{Q} + \frac{\lambda^2}{2!} \overset{(2)}{Q} + \cdots \tag{5}$$

thus, it can be see

$$\bar{P}\bar{Q} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{P} \overset{(n-i)}{Q} \tag{6}$$

then, the n-th term of the perturbation of the multiplication of $\bar{P}$ and $\bar{Q}$ is

$$\overset{(n)}{PQ} = \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{P} \overset{(n-i)}{Q} . \tag{7}$$

Procedure

$$\bar{\Sigma}_{\mu\nu} = \bar{f}'\bar{R}_{\mu\nu} - \frac{1}{2}\bar{f}\bar{g}_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu\bar{f}' + \bar{g}_{\mu\nu}\bar{\Box}\bar{f}' \tag{8}$$

where

$$\bar{f} = \overset{(0)}{f} + \lambda\overset{(1)}{f} + \frac{\lambda^2}{2!}\overset{(2)}{f} + \cdots \tag{9}$$

$$\bar{f}' = \overset{(0)}{f'} + \lambda\overset{(1)}{f'} + \frac{\lambda^2}{2!}\overset{(2)}{f'} + \cdots \tag{10}$$

$$\bar{g}_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + \lambda\overset{(1)}{g}_{\mu\nu} + \frac{\lambda^2}{2!}\overset{(2)}{g}_{\mu\nu} + \cdots \tag{11}$$

$$\bar{R}_{\mu\nu} = \overset{(0)}{R}_{\mu\nu} + \lambda\overset{(1)}{R}_{\mu\nu} + \frac{\lambda^2}{2!}\overset{(2)}{R}_{\mu\nu} + \cdots \tag{12}$$

Field equation to n order

$$
\begin{aligned}
\overset{(n)}{\Sigma}_{\mu\nu} = & \sum_{i=0}^{n} \left[\binom{n}{i} \overset{(i)}{f'} \overset{(n-i)}{R}_{\mu\nu} - \frac{1}{2} \binom{n}{i} \overset{(i)}{f} \overset{(n-i)}{g}_{\mu\nu} \right] \\
& - \nabla_\mu \nabla_\nu \overset{(n)}{f'} + \sum_{i=0}^{n} \binom{n}{i} \overset{(n-i)}{C}_{\mu\nu}{}^{\alpha} \nabla_\alpha \overset{(i)}{f'} \\
& + \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{n}{i} \binom{i}{k} \overset{(n-i)}{g}_{\mu\nu} \overset{(i-k)}{g}{}^{\alpha\beta} \\
& \cdot \left[\nabla_\alpha \nabla_\beta \overset{(k)}{f'} - \sum_{l=0}^{k} \binom{k}{l} \overset{(k-l)}{C}_{\alpha\beta}{}^{\delta} \nabla_\delta \overset{(l)}{f'} \right]
\end{aligned}
\tag{13}
$$

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Starobinsky

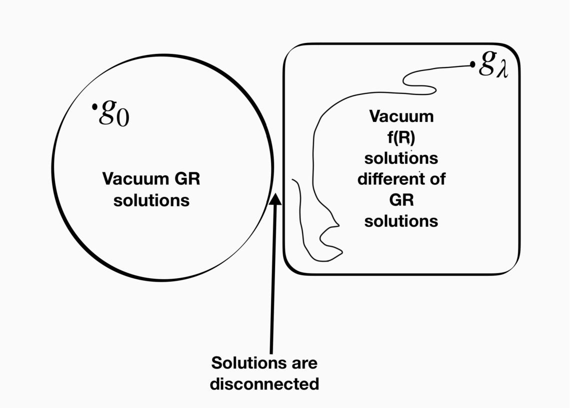
Theorem

Let $\bar{\Sigma}_{ab} = 0$ *be* $f(R)$ *MTG field equations in vacuum for the model* $f(\bar{R}) = \bar{R} + \lambda \bar{R}^2$*, then* $\bar{\Sigma}_{ab} = \bar{G}_{ab}$.

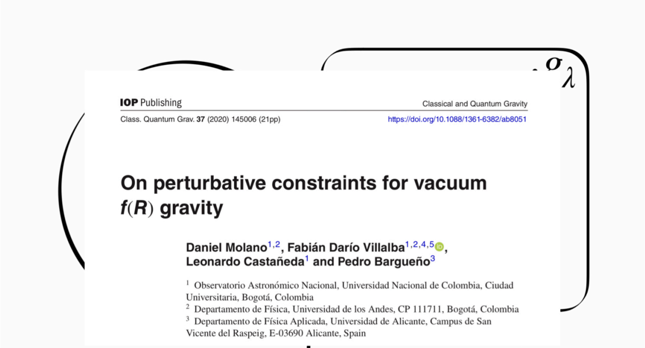
Theorem

Let $\bar{\Sigma}_{ab} = 0$ *be the* $f(R)$ *MTG field equations in vacuum for the model* $f(\bar{R}) = \bar{R} + \lambda\Psi(\bar{R})$*, where* $\Psi(R)$ *is analytic in a vicinity of* $\lambda = 0$*, and* $\Psi(0) = 0$*. Then,* $\bar{\Sigma}_{ab} = \bar{G}_{ab}$ *in vacuum.*

Discussion


$\cdot g_0$
Vacuum GR solutions
$\cdot g_\lambda$
Vacuum f(R) solutions different of GR solutions
Solutions are disconnected

Paper



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On perturbative constraints for vacuum *f*(*R*) gravity

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Actual work

General combinations of Ricci and Riemann curvature

Theorem

Let $\bar{P}_{ab} = 0$ be the vacuum field quations in $f(R, R_{\mu\nu}R^{\mu\nu})$ MTG for the model $f(\bar{R}, \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}) = \bar{R} + \lambda\Psi(\bar{R}, \bar{R}_{\mu\nu}\bar{R}^{\mu\nu})$, where $\Psi(R, R_{\mu\nu}R^{\mu\nu})$ is analytic in a vicinity of $\lambda = 0$, and $\Psi(0,0) = 0$. Then, $\bar{P}_{ab} = \bar{G}_{ab}$ in vacuum.

$$f(R, R_{\mu\nu}R^{\mu\nu}) = R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu}) \longrightarrow \text{disconnected}$$

$$f(R, R_{\mu\nu}R^{\mu\nu}, R_{\sigma\rho\mu\nu}R^{\sigma\rho\mu\nu}) = R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu}, R_{\sigma\rho\mu\nu}R^{\sigma\rho\mu\nu}) \longrightarrow \text{connected}$$

f(R) and Scalar-Tensor as a particular case

Corollary

Let $\bar{\Sigma}_{ab} = 0$ be the $f(R)$ vacuum field equations for the model $f(\bar{R}) = \bar{R} + \lambda\Psi(\bar{R})$. Then, $\bar{\Sigma}_{ab} = \bar{G}_{ab}$ in vacuum.

Corollary

Let $\Phi_{\mu\nu}$ the vacuum field equation in the scalar-tensor theories of gravity with density Lagrangian of the form $\mathcal{L} = R + \lambda\psi R - V(\psi)$ with $V(\psi) = \chi(\psi)(1 + \lambda\psi) - f(\chi(\psi))$ where we assume ψ' invertible and $\psi'(\chi) = \psi$, then $\bar{\Psi}_{\mu\nu} = \bar{G}_{\mu\nu}$

Weak-field limit

$$\mathcal{L} = R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{14}$$

where α, β, and γ are constants. Utilizing the well-known result that the Gauss-Bonnet is a total divergence, we can rewrite (14) as

$$\mathcal{L} = R + \alpha_1 R^2 + \beta_1 R_{\mu\nu}R^{\mu\nu} \tag{15}$$

with a redefinition of constants α_1, β_1.

Corollary

Let $\tilde{P}_{\mu\nu}$ be the field equation of the Lagrangian density given in (14), then $\bar{P}_{\mu\nu} \equiv \bar{G}_{\mu\nu}$

Conjecture

Conjecture

If the geometric part of the family of the Lagrangian $\mathcal{L}_\lambda(g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\delta}, \ldots) = 0$ in the background, where $\mathcal{L}_0 = R$, then the class of solutions is either GR or another solution that is disconnected from GR or not analytic with respect to λ in vacuum.

First order

For the case of a nonlinear form of $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$, a new system of differential equations arises for the perturbed quantities. For instance, at first order in vacuum, we have

$$\overset{(1)}{P^{\mu\nu}} = \overset{(1)}{R^{\mu\nu}} - \frac{1}{2}\overset{(1)}{R}\,\overset{(0)}{g^{\mu\nu}} + 4\,\overset{(0)}{\Psi_Z}\,\overset{(0)}{R^{\delta\sigma\rho(\mu}}\,\overset{(0)}{R^{\nu)}}{}_{\rho\sigma\delta} - 8\nabla_\rho\nabla_\sigma\,\overset{(0)}{\Psi_Z}\,\overset{(0)}{R^{\sigma(\mu\nu)\rho}} = 0 \qquad (16)$$

for the case $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}) = \frac{1}{2}R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$. Notably, the last two terms in (16) exhibit characteristics resembling a non vanishing energy-momentum tensor.

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$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left[\frac{1}{2}R + \lambda f(\mathcal{G})\right] + S_M. \qquad (17)$$



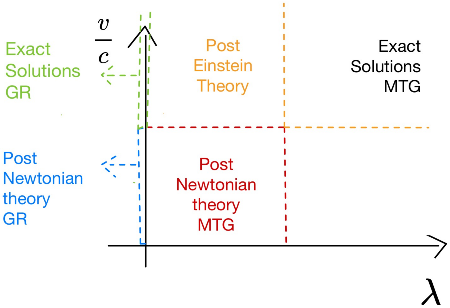
where $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.

Consider $f(\mathcal{G}) = (\mathcal{G})^\eta$ then the first order solution

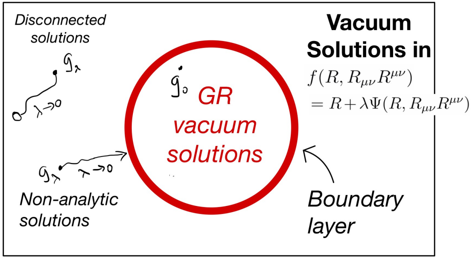
$$g_{tt} = -1 + \frac{2GM}{r} + \lambda\left[\frac{3^{\eta-1}16^\eta(\eta-1)GM\left(\frac{G^2M^2}{r^6}\right)^{\eta-1}((6\eta-1)GM - 4\eta r)}{(2\eta-1)r^4}\right] + O(\lambda^2)$$

$$g_{rr} = 1 - \frac{2GM}{r} + \lambda\left[\frac{3^{\eta-1}16^\eta(\eta-1)GM\left(\frac{G^2M^2}{r^6}\right)^{\eta-1}((2\eta(7-12\eta)+1)GM + 6\eta(2\eta-1)r)}{(2\eta-1)r^2(r-2GM)^2}\right] + \cdots$$

Summarize



$\frac{v}{c}$
Exact Solutions GR
Post Einstein Theory
Exact Solutions MTG
Post Newtonian theory GR
Post Newtonian theory MTG
λ



Disconnected solutions
Vacuum Solutions in
$f(R, R_{\mu\nu}R^{\mu\nu})$
$= R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu})$
g_λ
$\lambda \to 0$
g_0
GR vacuum solutions
Non-analytic solutions
Boundary layer

Scan me

Thank you!

Questions?