Reference metrics on hyperboloidal slices for free evolution

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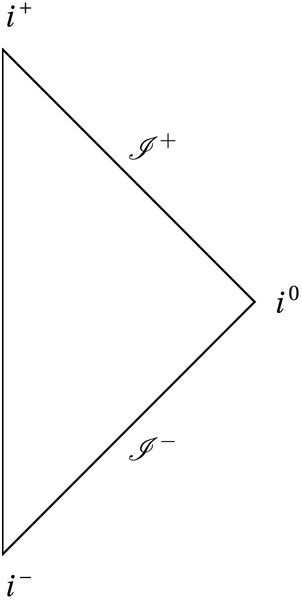
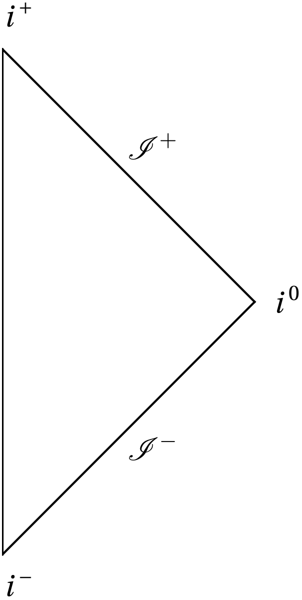
Hyperboloidal slices



Standard slicing options for the initial value formulation of the Einstein equations:

Hyperboloidal slices



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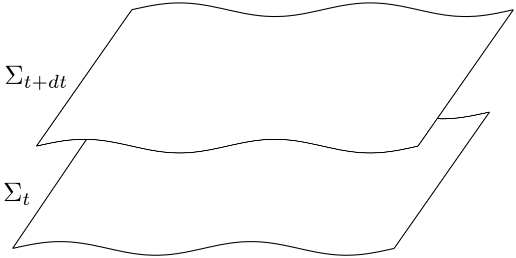

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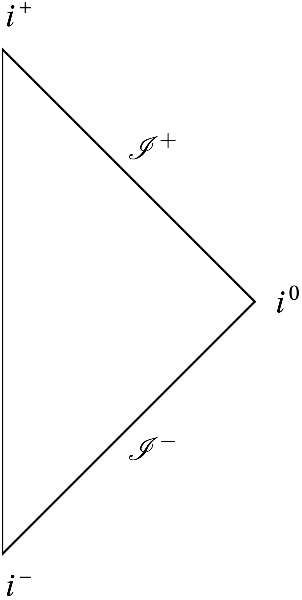

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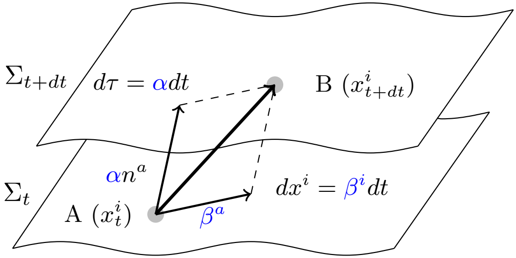

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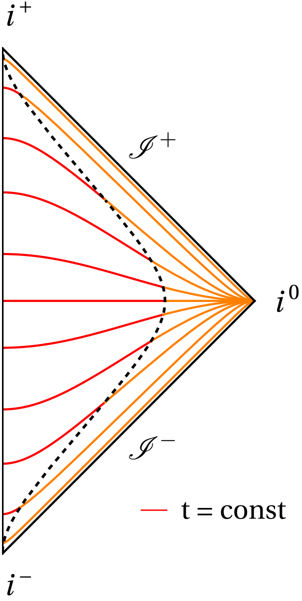

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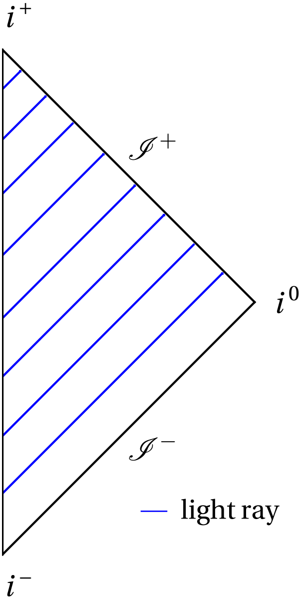
Hyperboloidal slices



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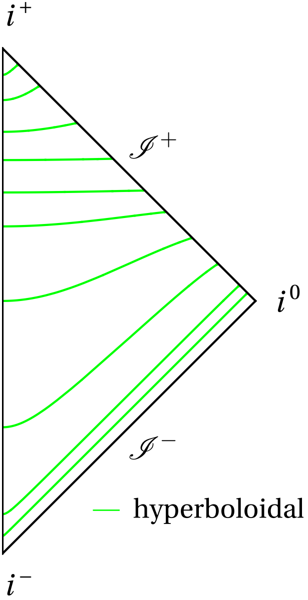

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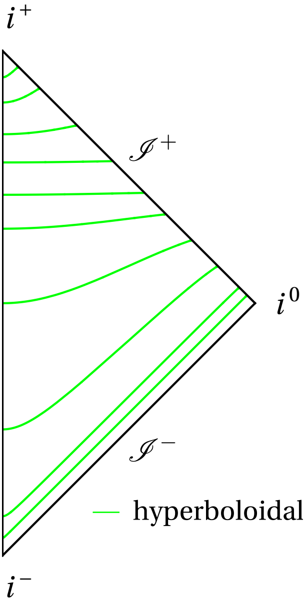

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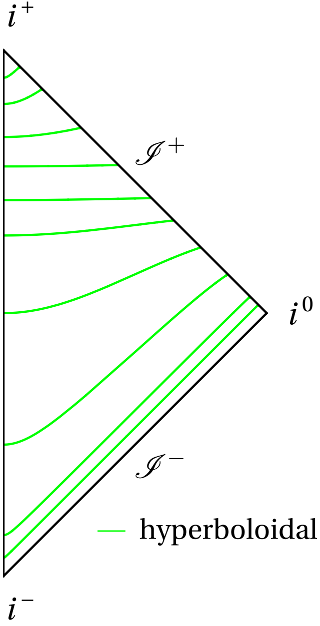
Advantages of the hyperboloidal approach:

- Slices are spacelike.
- Reach $\mathscr{I}^+$ without approximations.

Aim: hyperboloidal for compact binary coalescence



Future null infinity ($\mathscr{I}^+$) is a region of spacetime of interest

- for the study of global properties of spacetimes and
- for the extraction of gravitational waves (only well described at $\mathscr{I}^+$, where observers are located; GW memory effect).

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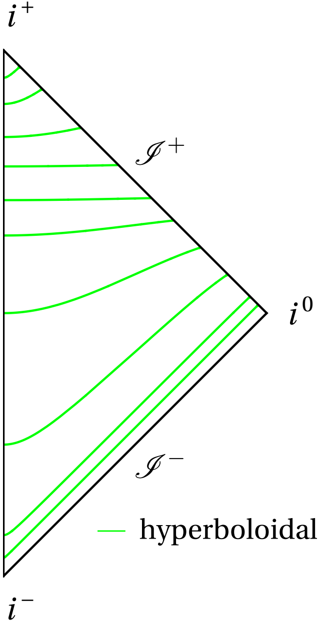

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How to include future null infinity in a numerical integration domain?

$\Rightarrow$ Balance the blowup of the coordinates with the decay of the fields.

Example: $g_{ab} = \Omega^2 \tilde{g}_{ab}$
Penrose's conformal compactification (60s).

Reference metric

Asymptotic behaviour of evolution quantities is different ($\tilde{t} = t + h$):

Cauchy $\tilde{t} = const$

- $\tilde{K} = 0$
- $\tilde{\alpha} = 1$
- $\beta^{\tilde{r}} = 0$

Hyperboloidal $t = const$

- $\tilde{K} \neq 0$
- $\alpha = O(1) > 0$
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Spatial reference metric $\hat{\gamma}_{ij}$ in 3+1 decomposed Einstein equations:

$$\Lambda^i = \Gamma^i - \hat{\Gamma}^i = \gamma^{jk}\left(\Gamma^i_{jk} - \hat{\Gamma}^i_{jk}\right) \tag{1}$$

Formulation: Brown. *Phys. Rev.* D79 (2009)
Implementation in spherical coordinates with PIRK:
Baumgarte et al. *Phys.Rev.* D87.4 (2013) (BSSN)
Sanchis-Gual et al. *Phys. Rev.* D89.10 (2014) (Z4)

Construction of gauge

$$\text{Evolved metric:} \quad g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$\text{Reference (time-independent) metric:} \quad \hat{g}_{\mu\nu} = \begin{pmatrix} -\hat{\alpha}^2 + \hat{\beta}_k\hat{\beta}^k & \hat{\beta}_j \\ \hat{\beta}_i & \hat{\gamma}_{ij} \end{pmatrix}$$

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Lapse equation:

$$\partial_t\alpha = \beta^i\partial_i\alpha - \left(n_{cK}(r_{\mathscr{I}}^2 - r^2)^2 + \alpha^2\right)\frac{\tilde{K}}{\Omega} + f(\alpha, \beta^i, \gamma_{ij}; \hat{\alpha}, \hat{\beta}^i, \hat{\gamma}_{ij}; x^i)$$

Shift equation:

$$\partial_t\beta^i = \beta^j\partial_j\beta^j + \left(\lambda(r_{\mathscr{I}}^2 - r^2)^2 + \frac{3}{4}\alpha^2\right)\Lambda^i + g(\alpha, \beta^i, \gamma_{ij}; \hat{\alpha}, \hat{\beta}^i, \hat{\gamma}_{ij}; x^i)$$

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Lapse equation:

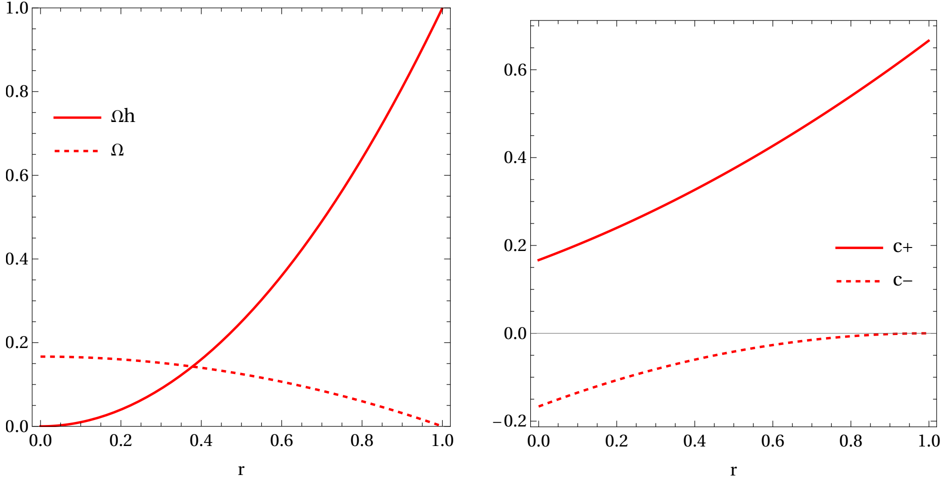
$$\dot{\alpha} = \beta^r\alpha' - \hat{\beta}^r\hat{\alpha}' - (n_{cK}(r_{\mathscr{I}}^2 - r^2)^2 + \alpha^2)\frac{\Delta\tilde{K}}{\Omega} + \frac{\Omega'}{\Omega}(\hat{\beta}^r\hat{\alpha} - \beta^r\alpha) + \frac{\xi_{cK}(\hat{\alpha} - \alpha)}{\Omega}$$

Shift equation:

$$\dot{\beta}^r = \beta^r\beta^{r\prime} - \hat{\beta}^r\hat{\beta}^{r\prime} + \left(\lambda(r_{\mathscr{I}}^2 - r^2)^2 + \frac{3}{4}\alpha^2\right)\Lambda^r + \eta(\hat{\beta}^r - \beta^r) + \xi_{\beta^r}\left(\frac{\hat{\beta}^r}{\Omega} - \frac{\beta^r}{\Omega}\right)$$

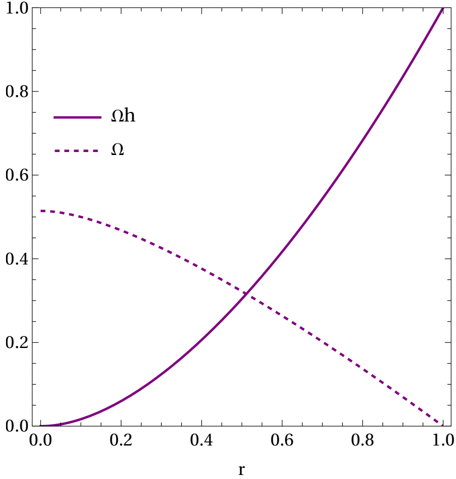
CMC – default choice, with $\hat{\gamma}_{rr} = 1$

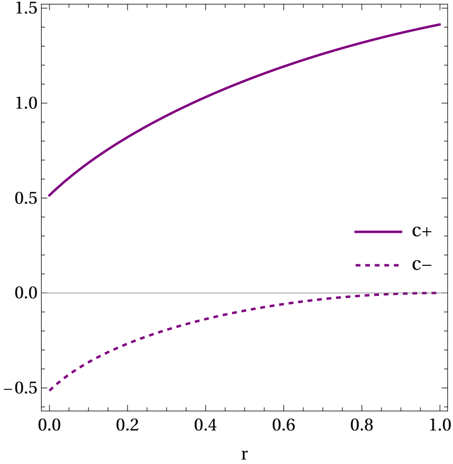
$$h(\tilde{r}) = \sqrt{\left(\frac{3}{K_C}\right)^2 + \tilde{r}^2} + \frac{3}{K_C} \quad \rightarrow \quad h(r) = \sqrt{\left(\frac{3}{K_C}\right)^2 + \left(\frac{r}{\Omega}\right)^2} + \frac{3}{K_C}$$



Similar to CMC

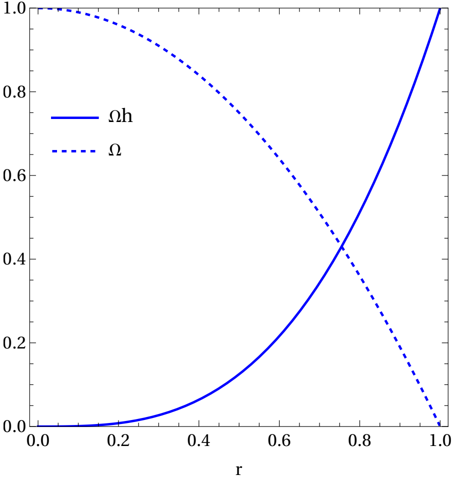
$$h(r) = \frac{r}{\Omega} + \frac{C}{1 + r/\Omega} - C, \qquad \hat{\gamma}_{rr} = 1 \qquad \to \qquad \Omega \text{ numerical}$$

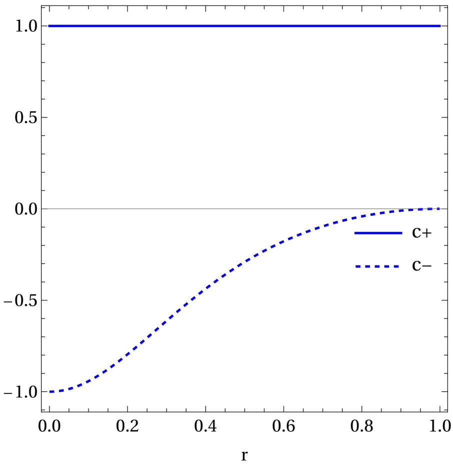




$c_+ = 1$

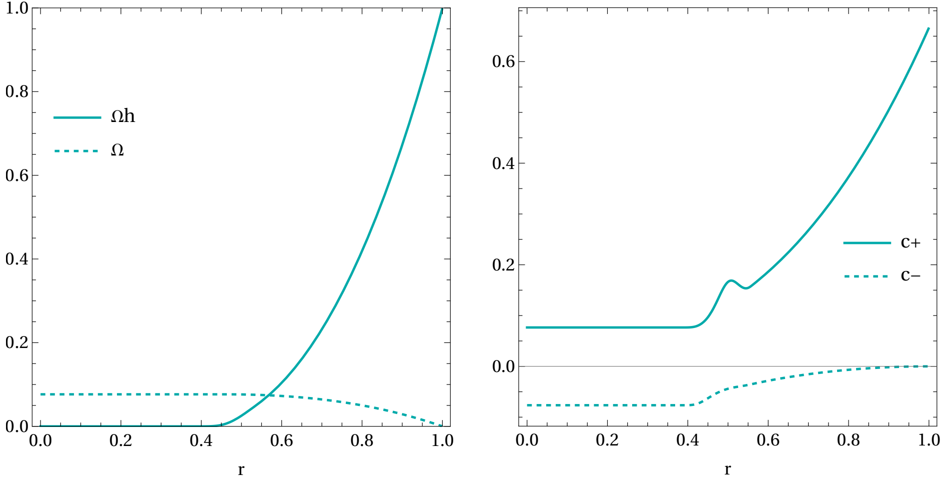
$$h(r) = \frac{r}{\Omega} - r, \qquad \Omega = 1 - r^2, \qquad \rightarrow \qquad \hat{\gamma}_{rr} \neq 1$$



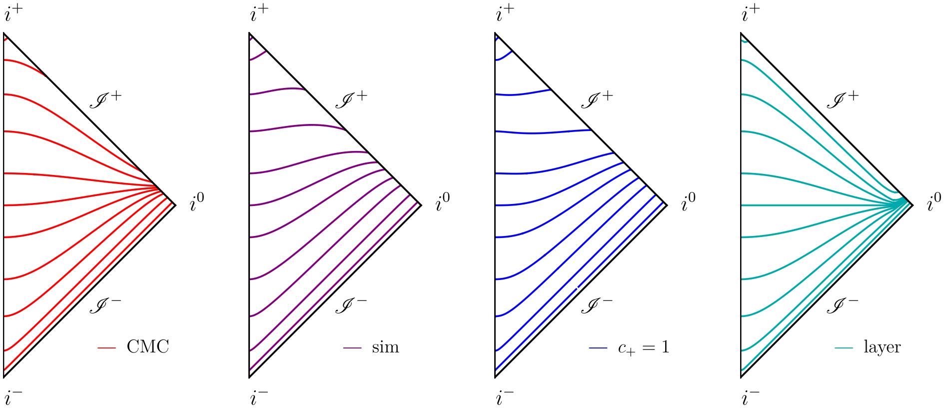


Hyperboloidal layer (matched)

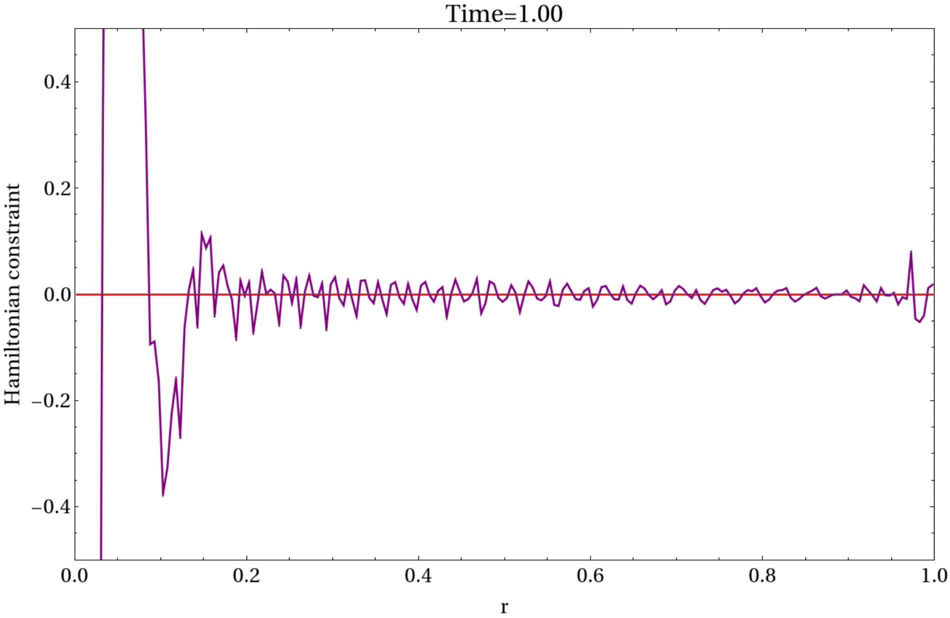
$$h(r<\tilde{r}\Omega)=0, \quad h(r>\tilde{r}\Omega)=\frac{3}{K_C}+\sqrt{\left(\frac{3}{K_C}\right)^2+\left(\frac{r}{\Omega}-\tilde{r}_{match}\right)^2}, \qquad \hat{\gamma}_{rr}=1$$



Carter-Penrose diagrams



Long-term stability - ok x



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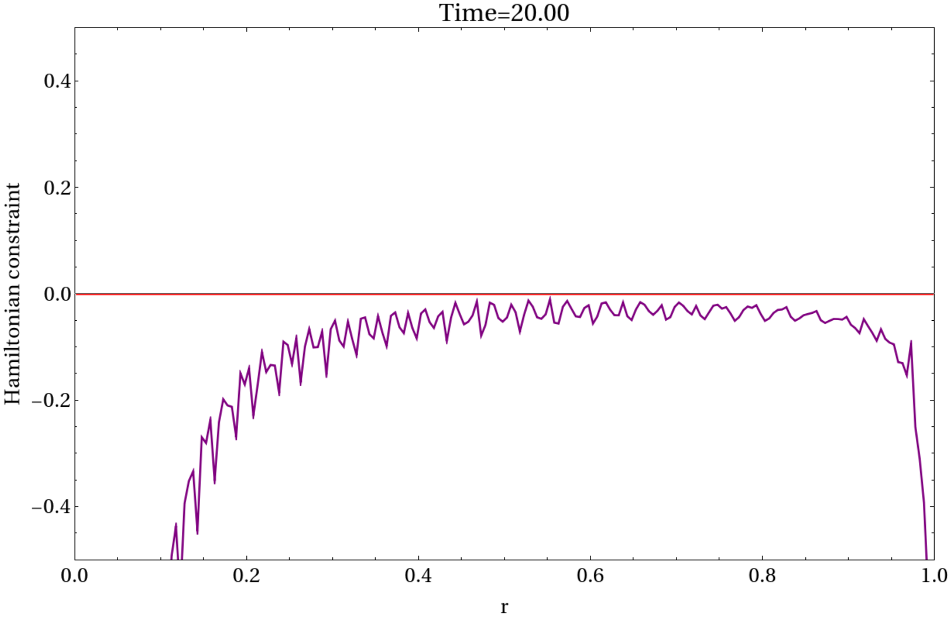

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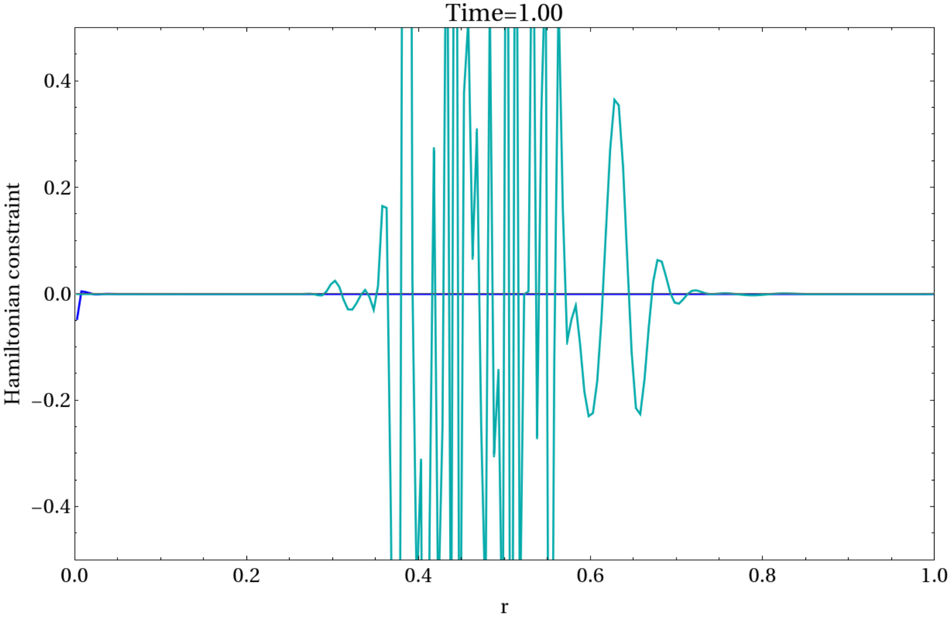

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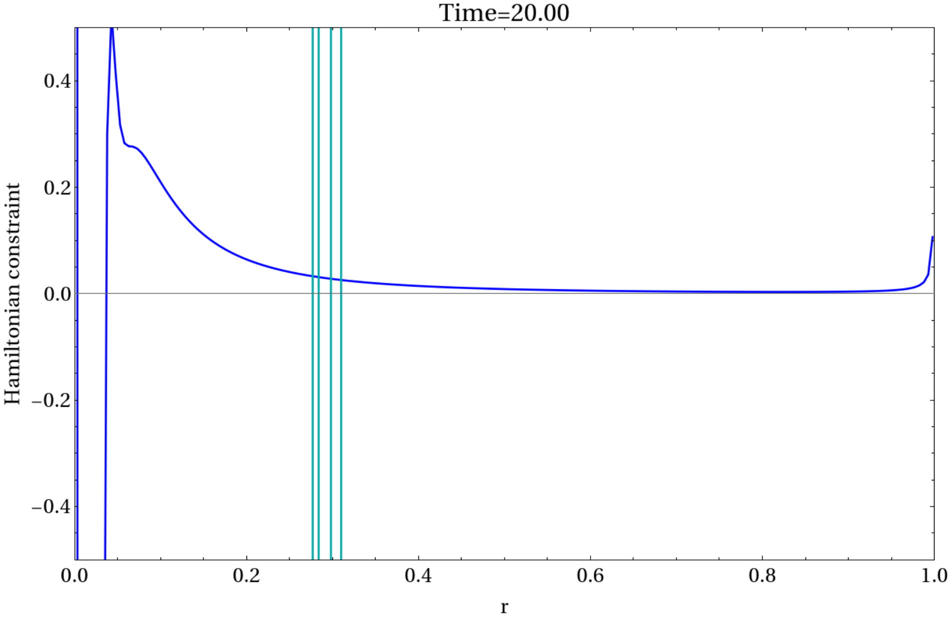

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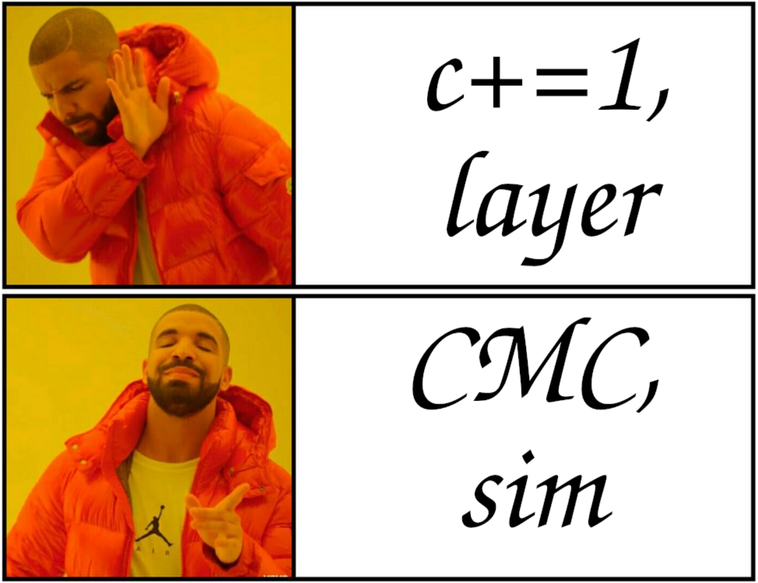

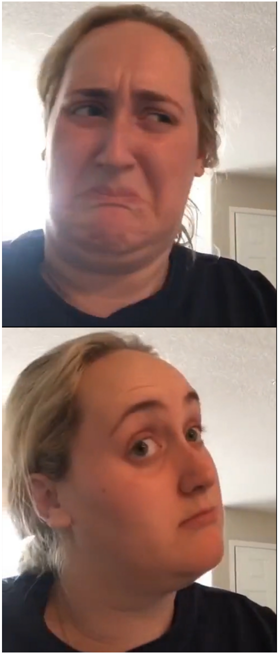

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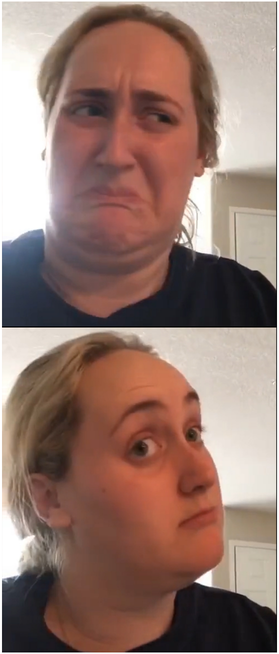
- Physics is unaffected by it.
- Why bother?



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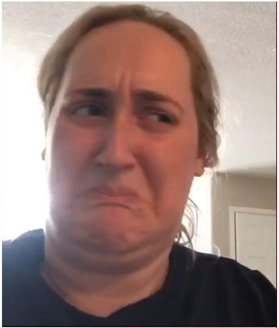

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Thank you!

Backup slides

Preferred conformal gauge and Bondi time

The preferred conformal gauge,

$$\Box\Omega|_{\mathscr{I}^+} = 0, \tag{2}$$

makes the null tangent to at $\mathscr{I}^+$ be affinely parametrized and simplifies how the divergent terms in the Einstein equations cancel.

It will hold for $\dot{\alpha}$ and $\dot{\beta}^r$ derived from

$$\tilde{\Lambda}^c = \tilde{g}^{ab}\left(\tilde{\Gamma}^c_{ab} - \hat{\tilde{\Gamma}}^c_{ab}\right) = \tilde{F}^c. \tag{3}$$

The Bondi time at $\mathscr{I}^+$ is related to our code time t via

$$dt_{Bondi} = \frac{\alpha^2\omega}{\beta^r\Omega'}dt, \tag{4}$$

where $\omega = 1$ if the preferred conformal gauge holds. Otherwise, ω is to be determined by solving an ODE at $\mathscr{I}^+$.

CCE/M vs. hyperbol. layer vs. CMC hyperboloidal