Static and spherically symmetric vacuum spacetimes with non-expanding principal null directions in $f(R)$ gravity*

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*arXiv:2407.13262 [gr-qc]

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Motivation

The **Nariai solution** was presented back in the 1950's, and can be described in suitable coordinates by the line element

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right) dt^2 - \left(1 - \frac{r^2}{r_0^2}\right)^{-1} dr^2 - r_0^2 d\Omega^2,$$

with $r \in (-r_0, r_0)$, where r_0 is a non-null constant, and $d\Omega^2$ is the line element of the 2-sphere.

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- **Static** and **spherically symmetric** solution of GR (with positive cosmological constant $\lambda = 1/r_0^2$).
- Usually characterized as the limit of Schwarzschild-de Sitter when the two cosmological and event horizon coincides. $\leftarrow$ **not defined in a meaningful sense:**

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Other ways to characterize Nariai?

Weyl decomposition of the Riemann tensor

The Riemann tensor can be decomposed as

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}\left(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}\right)$$
$$- \frac{R}{6}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

where C is the Weyl tensor, with the following properties:

- Is **traceless**, so it is related with *pure* gravitational fields (curvature not due to matter content: Schwarzschild, Kerr, etc.)
- Is **invariant under conformal transformations** (volume changes, $\tilde{g} = e^{2\lambda(x^\mu)}g$).

Petrov classification of the Weyl tensor

Petrov developed a **classification of spacetimes** studying the algebraic structure of the Weyl tensor.

This study can be carried out by studying the eigenvibectors of the Weyl tensor, which are associated to four null vectors which determine the so called **principal null directions** (PNDs) of the Weyl tensor (at most 4 *different*).

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In particular, **static and spherically symmetric** spacetimes are of **type D**: there are two (double) principal null directions, which we denote $\mathbf{l}$ and $\mathbf{n}$.

This classification induces the use of **Newman and Penrose formalism** (NP).

Characterization of Nariai spacetime

The main idea is to construct a null tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$, i.e. a new basis of the tangent space, formed by

- Two real null vectors, $\mathbf{l}$ and $\mathbf{n}$, which we take as the principal null directions of the Weyl tensor.
- Two complex null vectors, $\mathbf{m}$ and $\overline{\mathbf{m}}$, are constructed by combining a pair of real orthogonal spacelike unit vectors.

They must satisfy the relations $l^\mu n_\mu = 1$, $m^\mu \bar{m}_\mu = -1$ and others are zero.

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The **principal null directions** in the **Nariai** solution are **non-expanding**. In terms of NP this is $\rho = \mu = 0$. In fact, is the only static and spherically symmetric vacuum solution with this property.

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Question

Is the Nariai solution the only static and spherically symmetric spacetime with non-expanding principal null directions for $\mathbf{f(R)}$ **theories**?

Brief introduction to $f(R)$ gravity

The action in $f(R)$ formalism is given by

$$S = \int d^4x\sqrt{-g}f(R).$$

The field equations are

$$-R_{\mu\nu} = \Delta t_{\mu\nu},$$

where we have defined the effective stress-energy tensor

$$\Delta t_{\mu\nu} \equiv F(R)^{-1}\Big(-\frac{1}{2}f(R)g_{\mu\nu} + [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]F(R)\Big),$$

with $F(R) = df(R)/dR$. The associated trace equation is

$$R = F(R)^{-1}\left(2f(R) + 3\Box F(R)\right).$$

Initial Ansatz

Let (M, g) be a static and spherically symmetric spacetime with

$$ds^2 = p(r)dt^2 - s(r)dr^2 - q(r)d\Omega^2,$$

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In this case, the principal null directions are

$$\begin{aligned} l_\mu &= \frac{1}{\sqrt{2}}\left(p(r)^{1/2}, s(r)^{1/2}, 0, 0\right), \\ n_\mu &= \frac{1}{\sqrt{2}}\left(p(r)^{1/2}, -s(r)^{1/2}, 0, 0\right), \end{aligned}$$

and we complete the null tetrad with $m_\mu = \frac{q(r)}{\sqrt{2}}\left(0, 0, 1, -i\sin\theta\right),$

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Imposing the **non-expanding condition** on $\mathbf{l}$, *i.e.* $\rho = 0$, then $q(r) = r_0^2$ (and $\mathbf{n}$ is non-expanding).

Field equations

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The so called **Ricci scalars**, Φ_{ab} with $a, b \in \{0, 1, 2\}$, are defined as the contractions of the Ricci tensor with the null tetrad vectors. The only non-vanishing Ricci scalar is

$$\Phi_{11} = -\frac{1}{2} R_{\mu\nu} l^{\mu} n^{\nu} + 3\Lambda,$$

where $\Lambda = R/24$, with R being the Ricci scalar curvature.

Field equations

We define the ***physical contractions*** of the effective stress-energy tensor in an analogous way as the Ricci scalars. The only non-vanishing scalars are

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\Phi_{00}^{ph} &= \frac{1}{2}\Delta t_{\mu\nu} l^{\mu} l^{\mu}, \\
\Phi_{11}^{ph} &= \frac{1}{2}\Delta t_{\mu\nu} l^{\mu} n^{\nu} + 3\Lambda^{ph}, \\
\Phi_{22}^{ph} &= \frac{1}{2}\Delta t_{\mu\nu} n^{\mu} n^{\nu},
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with $\Lambda^{ph} = R/24$, where R is now obtained in terms of $f(R)$ by using the trace equation.

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with $\Lambda^{ph} = R/24$, where R is now obtained in terms of $f(R)$ by using the trace equation.

At this point, **the only field equations** which are not identically zero in the new basis **are** given by

$$\Phi_{00}^{ph} = \Phi_{22}^{ph} = 0,\ \Phi_{11}^{ph} = \Phi_{11} \text{ and } \Lambda^{ph} = \Lambda.$$

Field equations

Equation $\Phi^{ph}_{00} = 0$ implies

$$F(r) \equiv F(R(r)) = a(1 + br),$$

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We now solve the remaining field equations.

Non-constant Ricci scalar solutions

Assuming $b \neq 0$, the solution is given by

$$p(r) = c_1 - \frac{r}{br_0^2} - \frac{r^2}{2r_0^2} + \frac{\gamma}{b^2 r_0^2} \log|1 + br|,$$

where $\gamma = 1 + c_2 b r_0^2$, being c_1 and c_2 two integration constants which **cannot be removed under changes of coordinates**.

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The Ricci scalar is given by

$$R(r) = \frac{1}{r_0^2}\left(3 + \frac{\gamma}{(1 + br)^2}\right),$$

and it shows a **curvature singularity** at $r = -1/b$.

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The solution(s) come from a only $f(R)$ theory which has the form

$$f(R) = \frac{1}{r_0}\left|R - \frac{3}{r_0^2}\right|^{1/2}.$$

Constant Ricci scalar solutions

For completeness, assuming $b = 0$, we reobtain the Nariai solution.

The set of **compatible** $f(R)$ **theories** are those functions fulfilling the *one point* differential equation

$$R_0 \frac{df}{dR}(R_0) = 2f(R_0),$$

which is actually the trace equation for constant Ricci scalar.

These theories are **not necessarily GR** although the corresponding field equations can be interpreted, for constant $R = R_0$, in terms of GR with a cosmological constant given by $\lambda = \frac{R_0}{2} - \frac{f(R_0)}{2F(R_0)}$.

Thanks!